

Research Article

The Levenberg-Marquardt-Type Methods for a Kind of Vertical Complementarity Problem

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Two kinds of the Levenberg-Marquardt-type methods for the solution of vertical complementarity problem are introduced. The methods are based on a nonsmooth equation reformulation of the vertical complementarity problem for its solution. Local and global convergence results and some remarks about the two kinds of the Levenberg-Marquardt-type methods are also given. Finally, numerical experiments are reported.

1. Introduction

In this paper, we consider the following kind of vertical complementarity problem:

$$\begin{aligned} F_i(x) &\geq 0, \quad i = 1, \dots, m, \\ \prod_{i=1}^m F_i^j(x) &= 0, \quad j = 1, \dots, n, \end{aligned} \tag{1.1}$$

where the functions $F_i(x) : R^n \rightarrow R^m$ are assumed to be continuously differentiable and $F_i^j(x)$ denotes the j th component of the function $F_i(x)$. The above vertical complementarity problem is of concrete background, for instance, the Karush-Kuhn-Tucker system of nonlinear programs, complementarity problems, and many problems in mechanics and

engineering. When $m = 2$, (1.1) is the generalized complementarity problem, which has been considered in [1]. When x satisfied

$$\begin{aligned} \min_{j \in J_1} \{F_j^1(x)\} &= 0, \\ &\vdots \\ \min_{j \in J_n} \{F_j^n(x)\} &= 0, \end{aligned} \tag{1.2}$$

x solves (1.1), where “min” denotes the componentwise minimum operator, $F_j^i : R^n \rightarrow R$, for, $j \in J_i, i = 1, 2, \dots, n$, are also continuously differentiable, and $J_i, i = 1, 2, \dots, n$, are finite index sets. Throughout this section, we denote

$$\begin{aligned} g_i(x) &= \min_{j \in J_i} F_j^i(x), \quad x \in R^n, \quad i = 1, 2, \dots, n, \\ G(x) &= (g_1(x), \dots, g_n(x))^T, \quad x \in R^n, \\ J_i &= \{j \in J_i \mid F_j^i(x) = g_i(x)\}, \quad x \in R^n, \quad i = 1, 2, \dots, n. \end{aligned} \tag{1.3}$$

Thus, (1.2) can be briefly rewritten as

$$G(x) = (g_1(x), \dots, g_n(x))^T = 0, \tag{1.4}$$

which is nonsmooth equation. Nonsmooth equations are much more difficult than smooth equations. Solving the above equations is a classical problem of numerical analysis, see [1–11]. Many existing classical methods for smooth equations cannot be extended to nonsmooth equations directly. Those difficult motivate us to seek higher-quality methods for nonsmooth equations. One of the classical iterative methods for solving (1.4) is the Levenberg-Marquardt-type method, which is based on the generalized Jacobian. The Levenberg-Marquardt method and its variants are of particular importance also because of their locally fast convergent rates. Levenberg-Marquardt method is also a famous method for nonlinear equations, which can be regarded as a modification of Newton method [12–18]. In the smoothing-type methods, some conditions are needed to ensure that the Jacobian matrix is nonsingular; the Levenberg-Marquardt method does not need such conditions [12–19].

By the above analysis, the purpose of this paper is to consider using the Levenberg-Marquardt-type methods for solving (1.1). We reformulate (1.1) as a system of nonsmooth equations and present two kinds of the Levenberg-Marquardt-type methods for solving it. The outline of the paper is as follows. In Section 2, we recall some background concepts and propositions which will play an important role in the subsequent analysis of convergence results. In Section 3, we give the two kinds of Levenberg-Marquardt-type methods and present their local and global convergence results and some remarks. Finally, numerical experiments and some final remarks about the numerical experiments are listed.

Throughout this paper, R^n denotes the space of n -dimensional real column vectors and T denotes transpose. For any differentiable function $f : R^n \rightarrow R$, $\nabla f(x)$ denotes the gradient of f at x .

2. Preliminaries

A locally Lipschitz function $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be semismooth at x provided that the following limit

$$\lim_{\substack{V \in \partial H(x+th') \\ h' \rightarrow h, t \rightarrow 0^+}} Vh' \quad (2.1)$$

exists for any $h \in \mathbb{R}^n$. The class of semismooth functions is very broad; it includes smooth functions, convex functions, and piecewise smooth functions. Moreover, the sums, differences, products, and composites of semismooth functions are also semismooth. Maximum of a finite number of smooth functions are semismooth too.

Proposition 2.1. *Function G in (1.4) is semismooth.*

Proposition 2.2. *The following statements are equivalent:*

- (i) $H(x)$ is semismooth at x .
- (ii) For $V \in \partial H(x+h)$, $h \rightarrow 0$.

$$Vh - H'(x; h) = o(\|h\|). \quad (2.2)$$

(iii) *One has*

$$\lim_{x+h \in D_H, h \rightarrow 0} \frac{H'(x+h; h) - H'(x; h)}{\|h\|} = 0. \quad (2.3)$$

If for any $V \in \partial H(x+h)$, $h \rightarrow 0$,

$$Vh - H'(x; h) = o(\|h\|^{1+p}), \quad (2.4)$$

where $0 < p \leq 1$, then one says $H(x)$ is p -order semismooth at x . Obviously, p -order semismoothness implies semismoothness. From [8], one knows that if $H(x)$ is semismooth at x , then, for any $h \rightarrow 0$

$$H(x+h) - H(x) - H'(x; h) = o(\|h\|). \quad (2.5)$$

In [6], Gao considered a system of equations of max-type functions, which is the system of equations of smooth compositions of max-type functions. For solving the systems of equations, Gao take ∂_* as a tool instead of the Clarke generalized Jacobian, B -differential, and b -differential. Based on [6], we give the following $\partial_* G(x)$ for G in (1.4):

$$\partial_* G(x) = \left\{ \left(\nabla F_{j_1}^1(x), \dots, \nabla F_{j_n}^n(x) \right)^T \mid j_1 \in J_1(x), \dots, j_n \in J_n(x), x \in \mathbb{R}^n \right\}. \quad (2.6)$$

Obviously, $\partial_* G(x)$ is a finite set of points and can be easily calculated by determining $J_i(x)$, $i = 1, \dots, n$, and $\nabla F_{j_i}^i(x)$, $j_i \in J_i(x)$, $i = 1, \dots, n$. In what follows, we use (2.6) as a tool instead of the Clarke generalized Jacobian and b -differential in the Levenberg-Marquardt method (I).

Proposition 2.3. *The $\partial_*G(x)$ defined as (2.6) is nonempty and compact set for any x , and the point to set map is also upper semicontinuous.*

Proposition 2.4. *Suppose that $G(x)$ and $\partial_*G(x)$ are defined by (1.4) and by (2.6), and all $V \in \partial_*G(x)$ are nonsingular. Then there exists a scalar $\beta > 0$ such that*

$$\begin{aligned} \|V^{-1}\| &\leq \beta, \quad \forall V \in \partial_*G(x), \\ \|V\| &\leq \gamma, \quad \forall V \in \partial_*G(x), \quad x \in N(x, \epsilon), \end{aligned} \quad (2.7)$$

holds for some constants $\gamma > 0$, $\epsilon > 0$, and $N(x, \epsilon)$ is a neighbor of x .

By the continuously differentiable property of G in (1.1), the above Proposition 2.4 can be easily obtained.

Definition 2.5. One says that G is BD-regular at x if all elements in $\partial G(x)$ are nonsingular.

Definition 2.6. $M \in R^{n \times n}$ is a P_0 matrix if and only if there exists an index $i \in \{1, 2, \dots, n\}$ such that $x_i \neq 0$ and $x_i[Mx]_i \geq 0$, for all $x \in R^n$, $x \neq 0$.

3. The Levenberg-Marquardt-type Methods and Their Convergence Results

In this section, we present two kinds of the Levenberg-Marquardt-type methods for solving vertical complementarity problem (1.1). Firstly, we briefly recall some results on the Levenberg-Marquardt-type method for the solution of nonsmooth equations and their convergence results, see, for example, [7] and also [9, 10]. And we also give the new kinds of the Levenberg-Marquardt methods for solving vertical complementarity problem (1.1) and analyze their convergence results. We are now in the position to consider exact and inexact versions of the Levenberg-Marquardt-type method.

Given a starting vector $x_0 \in R^n$, let

$$x_{k+1} = x_k + d_k, \quad (3.1)$$

where d_k is the solution of the system

$$\left((V_k)^T V_k + \sigma_k I \right) d = -(V_k)^T G(x_k), \quad V_k \in \partial_*G(x_k), \quad \sigma_k \geq 0. \quad (3.2)$$

In the inexact versions of this method, d_k can be given by the solution of the system

$$\left((V_k)^T V_k + \sigma_k I \right) d = -(V_k)^T G(x_k) + r_k, \quad V_k \in \partial_*G(x_k), \quad \sigma_k \geq 0, \quad (3.3)$$

where r_k is the vector of residuals, and we can assume $\|r_k\| \leq \alpha_k \|(V_k)^T G(x_k)\|$ for some $\alpha_k \geq 0$.

We now give a local convergence Levenberg-Marquardt-type method (I) for (1.1) as follows.

3.1. The Levenberg-Marquardt Method (I)

Step 1. Given $x_0, \epsilon > 0, \lambda_i^k \in R^n, 0 < |\lambda_i^k| < +\infty$.

Step 2. Solve the system to get d_k ,

$$\left((V_k)^T V_k + \text{diag} \left(\lambda_i^{(k)} g_i(x_k) \right) \right) d = -(V_k)^T G(x_k) + r_k, \quad V_k \in \partial_x G(x), \quad (3.4)$$

for $i = 1, \dots, n$, and r_k is the vector of residuals

$$\|r_k\| \leq \alpha_k \left\| (V_k)^T G(x_k) \right\|, \quad \alpha_k \geq 0. \quad (3.5)$$

Step 3. Set $x_{k+1} = x_k + d_k$; if $\|G(x_k)\| \leq \epsilon$, terminate. Otherwise, let $k := k + 1$, and go to Step 2.

Based upon the above analysis, we give the following local convergence result. Similar results have also been mentioned in [9].

Theorem 3.1. *Suppose that $\{x_k\}$ is a sequence generated by the above method and there exist constants $a > 0, \alpha_k \leq a$ for all k , and there exist constant $M > 0, \|\text{diag}(\lambda_i^{(k)} g_i(x))\| \leq M < +\infty$. Let x^* be a BD-regular solution of $G(x) = 0$. Then the sequence $\{x_k\}$ converge Q-linearly to x^* for $\|x_0 - x^*\| \leq \epsilon$.*

Theorem 3.2. *Suppose that $\{x_k\}$ is a sequence generated by the above method and there exist constants $a > 0, \alpha_k \leq a$ for all k . $\|r_k\| \leq \alpha_k \|G(x_k)\|, \alpha_k \geq 0$. And suppose that there exist constants $M > 0, \|\text{diag}(\lambda_i^{(k)} g_i(x))\| \leq M < +\infty$. Then the sequence $\{x_k\}$ converge Q-linearly to x^* for $\|x_0 - x^*\| \leq \epsilon$.*

By Propositions 2.2, 2.3, and 2.4, we can get the proof of Theorem 3.2 similarly to [9, Theorem 3.1], so we omit it.

Remark 3.3. Theorems 3.1 and 3.2 hold with $\|r_k\| = 0$.

Remark 3.4. In the Levenberg-Marquardt method (I), if d_k is computed by (3.3), Theorems 3.1, 3.2, and Remark 3.3 can also be obtained.

When $m = 2$ in (1.1), the vertical complementarity problem reduces to the well-known generalized complementarity problem (GCP) in [1]. The GCP is to find $x_* \in R^n$ satisfying

$$F_1(x) \geq 0, \quad F_2 \geq 0, \quad F_1(x)^T F_2(x) = 0, \quad (3.6)$$

where $F_i : R^n \rightarrow R^n, i = 1, 2$ and are continuously differentiable functions. If $F_1(x) = x$, then the generalized complementarity problem (GCP) is the nonlinear complementarity problem (NCP). In the following, we give the Levenberg-Marquardt-type method (II) for the generalized complementarity problem (GCP). The different merit function is based on the well-known F - B function

$$\varphi(a, b) = \sqrt{a^2 + b^2} - a - b. \quad (3.7)$$

Then $G(x) = \varphi(F_1, F_2)$, and we denote the corresponding merit function as

$$\Psi(x) = \frac{1}{2} \|G(x)\|^2, \quad (3.8)$$

where Ψ is a continuously differentiable function. Now, we give the following Levenberg-Marquardt-type method (II).

3.2. The Levenberg-Marquardt Method (II)

Step 1. Given a starting vector $x_0 \in R^n$, $\rho > 0$, $p > 2$, $\beta \in (0, 1/2)$, $\epsilon \geq 0$, $\lambda_i^k \in R^n$, $0 < |\lambda_i^k| < +\infty$.

Step 2. If $\Psi(x_k) \leq \epsilon$, stop.

Step 3. Select an element $V_k \in \partial G(x_k)$, and find an approximate solution $d_k \in R^n$ of the system

$$\left((V_k)^T V_k + \text{diag}(\lambda_i^{(k)} g_i(x_k)) \right) d = -(V_k)^T G(x_k), \quad V_k \in \partial G(x), \quad (3.9)$$

where $\lambda_i^{(k)}$ are the Levenberg-Marquardt parameters. If the condition

$$\nabla \Psi(x_k)^T d_k \leq -\rho \|d_k\|^p \quad (3.10)$$

is not satisfied, set $d_k = -(V_k)^T G(x_k)$.

Step 4. Find the smallest $i^k \in \{0, 1, 2, \dots\}$ such that

$$\Psi(x_k + 2^{-i^k} d_k) \leq \Psi(x_k) + \beta 2^{-i^k} \nabla \Psi(x_k)^T d_k. \quad (3.11)$$

Set $x_{k+1} = x_k + 2^{-i^k} d_k$, let $k := k + 1$, and go to Step 2.

Notice that if V_k is nonsingular in (3.9), the choice of $\lambda_i^k = 0$, at each step, is allowed by the above algorithm. Then (3.9) is equivalent to the generalized Newton equation in [4]. In what follows, as usual in analyzing the behavior of algorithms, we also assume that the above Levenberg-Marquardt method (II) produces an infinite sequence of points. Based upon the above analysis, we can get the following global convergence results about vertical complementarity problem (1.1). The main proof of the following theorem is similar to [7, Theorem 12]. But the system (3.9), which is used for the solution of d_k , is different from [7, Theorem 12].

Theorem 3.5. *Suppose that there exist constants $M > 0$, $\|\text{diag}(\lambda_i^{(k)} g_i(x))\| \leq M < +\infty$. Then each accumulation point of the sequence $\{x_k\}$ generated by the above Levenberg-Marquardt method (II) is a stationary point of Ψ .*

Proof. Assume that $\{x_k\}_K \rightarrow x^*$. If there are infinitely many $k \in K$ such that $d_k = -\nabla \Psi(x_k)$, then the assertion follows immediately from [5, Proposition 1.16]. Hence we can assume

without loss of generality that if $\{x_k\}_K$ is a convergent subsequence of $\{x_k\}$, then d_k is always given by (3.9). We show that for every convergent subsequence $\{x_k\}_K$ for which

$$\lim_{k \in K, k \rightarrow \infty} \nabla \Psi(x_k) \neq 0, \quad (3.12)$$

there holds

$$\begin{aligned} \limsup_{k \in K, k \rightarrow \infty} \|d_k\| &< \infty, \\ \limsup_{k \in K, k \rightarrow \infty} \left| \nabla \Psi(x_k)^T d_k \right| &> 0 \end{aligned} \quad (3.13)$$

In the following, we assume that $x_k \rightarrow x^*$. Suppose that x^* is not a stationary point of Ψ . By (3.9), we have

$$\begin{aligned} \|\nabla \Psi(x_k)\| &= \left\| \left((V_k)^T V_k + \text{diag}(\lambda_i^{(k)} g_i(x_k)) \right) d_k \right\| \\ &\leq \left\| (V_k)^T V_k + \text{diag}(\lambda_i^{(k)} g_i(x_k)) \right\| \|d_k\|, \end{aligned} \quad (3.14)$$

so

$$\|d_k\| \geq \frac{\|\nabla \Psi(x_k)\|}{\left\| (V_k)^T V_k + \text{diag}(\lambda_i^{(k)} g_i(x_k)) \right\|}. \quad (3.15)$$

Note that the denominator in the above inequality is nonzero; otherwise by (3.14), we have $\|\nabla \Psi(x_k)\| = 0$. x_k would be a stationary point and the algorithm would have stopped. By assumption $\|\text{diag}(\lambda_i^{(k)} g_i(x))\| \leq M < +\infty$ and Proposition 2.4, there exists a constant $k_1 > 0$ such that

$$\left\| (V_k)^T V_k + \text{diag}(\lambda_i^{(k)} g_i(x_k)) \right\| \leq k_1. \quad (3.16)$$

From the above inequality and (3.13), we get

$$\|d_k\| \geq \frac{1 - \alpha_k}{k_1} \|\nabla \Psi(x_k)\| \geq \frac{1 - a}{k_1} \|\nabla \Psi(x_k)\|. \quad (3.17)$$

Formula (3.13) now readily follows from the fact that we are assuming that the direction satisfies (3.10) with $p > 2$, while the gradient $\nabla \Psi(x_k)$ is bounded on the convergent sequence $\{x_k\}$. If (12) is not satisfied, there exists a subsequence $\{x_k\}_{K'}$ of $\{x_k\}_K$

$$\lim_{k \in K', k \rightarrow \infty} \left| \nabla \Psi(x_k)^T d_k \right| = 0. \quad (3.18)$$

This implies, by (3.10), that $\lim_{k \in K', k \rightarrow \infty} \|d_k\| = 0$. Together with (3.17) one implies

$$\lim_{k \in K', k \rightarrow \infty} \|\nabla \Psi(x_k)\| = 0 \quad (3.19)$$

contradicting (3.12). The sequence $\{d_k\}$ is uniformly gradient related to $\{x_k\}$ according to the definition given in [5], and the assertion of the theorem also follows from [5, Proposition 1.16]. We completed the proof. \square

Remark 3.6. In the Levenberg-Marquardt method (II), (3.11) can be replaced by the following nonmonotone line search:

$$\Psi(x_k + 2^{-ik} d_k) - \max_{0 \leq j \leq m(k)} \Psi(x_{k-j}) \leq \beta 2^{-ik} \nabla \Psi(x_k)^T d_k, \quad (3.20)$$

where $m(0) = 0$, $m(k) = \min\{m(k-1) + 1, M_0\}$, and M_0 is a nonnegative integer.

Remark 3.7. Let the assumptions of Theorem 3.5 hold. Equation (3.9) is replaced by (3.3). Then each accumulation point of the sequence $\{x_k\}$ generated by the above Levenberg-Marquardt method (II) is a stationary point of Ψ .

Remark 3.8. The assumption that there exist constants $M > 0$, $\|\text{diag}(\lambda_i^{(k)} g_i(x))\| \leq M < +\infty$ in Theorems 3.1, 3.5, and Remark 3.7 can be easily obtained, for the continuously differentiable of F in (1.1).

Remark 3.9. Let x_* be a stationary point of Ψ , $F'_2(x_*)$ nonsingular, and $F'_1(x_*)F'_2(x_*)^{-1}$ a P_0 matrix. Then x_* is a solution of GCP(F_1, F_2).

Remark 3.10. We can use φ_{MS} in [20] to construct merit function Ψ in the Levenberg-Marquardt method (II).

Remark 3.11. We can use a family of new NCP functions based on the Fischer-Burmeister function to construct merit function in the Levenberg-Marquardt method (II), which is defined by

$$\phi_p(a, b) = \|(a, b)\|_p - (a + b), \quad (3.21)$$

where p is any fixed real number in the interval $(1, +\infty)$ and $\|(a, b)\|_p$ denotes the p -norm of (a, b) . Numerical results based on the function ϕ_p for the test problems from MCPLIB indicated that the algorithm has better performance in practice [21].

4. Numerical Tests and Final Remarks

In this section, in order to show the performance of the above Levenberg-Marquardt type methods, we present some numerical results for the Levenberg-Marquardt method (I) and the Levenberg-Marquardt method (II). The results indicate that the Levenberg-Marquardt algorithms work quite well in practice. We coded the algorithms in Matlab 7.0. Some remarks are also attached.

Table 1

Step	x_k	$G(x)$
1	$(0.1000000, 0.7000000)^T$	$(0.0100000, 0.0050000)^T$
2	$(0.0500998, 0.7000000)^T$	$(0.0025099, 0.0012549)^T$
3	$(0.0250999, 0.7000000)^T$	$1.0e - 003 * (0.630004, 0.315002)^T$
4	$(0.0125749, 0.7000000)^T$	$1.0e - 003 * (0.158130, 0.079065)^T$
5	$(0.0063000, 0.7000000)^T$	$1.0e - 004 * (0.396906, 0.198453)^T$
6	$(0.0031563, 0.7000000)^T$	$1.0e - 005 * (0.996230, 0.498115)^T$
7	$(0.0015813, 0.7000000)^T$	$1.0e - 005 * (0.250052, 0.125026)^T$
8	$(0.0007922, 0.7000000)^T$	$1.0e - 006 * (0.627630, 0.313815)^T$
9	$(0.0003969, 0.7000000)^T$	$1.0e - 006 * (0.157534, 0.078767)^T$
10	$(0.0001988, 0.7000000)^T$	$1.0e - 007 * (0.395410, 0.197705)^T$
11	$(0.0000996, 0.7000000)^T$	$1.0e - 008 * (0.992475, 0.496237)^T$

Example 4.1. We consider the following vertical complementarity problem:

$$\begin{aligned} F_i(x) &\geq 0, \quad i = 1, \dots, m, \\ \prod_{i=1}^m F_i^j(x) &= 0, \quad j = 1, \dots, n, \end{aligned} \quad (4.1)$$

where the functions

$$F_1(x_1, x_2) = \left(x_1^2, x_1^2 + x_2^2\right)^T, \quad F_2(x_1, x_2) = \left(2x_1^2 + 6, \frac{1}{2}x_1^2\right)^T. \quad (4.2)$$

Both F_1 and F_2 are $R^2 \rightarrow R^2$ continuously differentiable functions.

We use the Levenberg-Marquardt method (I) to compute Example 4.1. Results for Example 4.1 with initial point $x_0 = (0.1, 0.7)^T$, $\alpha_k \equiv 1$ and $\lambda_1 = 0.01$, $\lambda_2 = 0.01$ are presented in Table 1.

We use the Levenberg-Marquardt method (I) to compute Example 4.1. Results for Example 4.1 with initial point $x_0 = (10, 10)^T$, and $x_0 = (100, 100)^T$, $\alpha_k \equiv 1$ and $\lambda_1 = 0.01$, $\lambda_2 = 0.01$ are presented in Table 2.

When we compute d_k by (3.3) in the Levenberg-Marquardt method (I) to compute Example 4.1. Results for Example 4.1 with initial point $x_0 = (0.1, 0.7)^T$, $x_0 = (10, 10)^T$, and $x_0 = (100, 100)^T$, $\alpha_k \equiv 1$ and $\lambda_1 = 0.01$, $\lambda_2 = 0.01$ are presented in Table 3.

Remark 4.2. From the numerical results for the Levenberg-Marquardt method (I) in Table 1, Table 2, and Table 3, we can see that the modification of (3.4) in the Levenberg-Marquardt method (I) works quite better in practice than (3.3) which has been used in [7].

Example 4.3. We consider the following vertical complementarity problem:

$$\begin{aligned} F_i(x) &\geq 0, \quad i = 1, \dots, m, \\ \prod_{i=1}^m F_i^j(x) &= 0, \quad j = 1, \dots, n, \end{aligned} \quad (4.3)$$

Table 2

Step	x_k	$G(x)$
	$x_0 = (10, 10)^T$	
1	$(10.000000, 10.000000)^T$	$(100.000000, 25.000000)^T$
2	$(5.009980, 10.000000)^T$	$(25.099900, 12.549950)^T$
3	$(2.509990, 10.000000)^T$	$(6.300049; 3.150024)^T$
4	$(1.257499, 10.000000)^T$	$(1.581306, 0.7906530)^T$
5	$(0.630004; 10.000000)^T$	$(0.396906, 0.198453)^T$
6	$(0.315631, 10.000000)^T$	$(0.099623, 0.049811)^T$
7	$(0.158130, 10.000000)^T$	$(0.025005, 0.012502)^T$
8	$(0.079223, 10.000000)^T$	$(0.006276, 0.003138)^T$
9	$(0.039690, 10.000000)^T$	$(0.001575, 0.000787)^T$
10	$(0.019884, 10.000000)^T$	$1.0e - 003 * (0.395410, 0.197705)^T$
11	$(0.009962, 10.000000)^T$	$1.0e - 004 * (0.992475, 0.496237)^T$
12	$(0.004991, 10.000000)^T$	$1.0e - 004 * (0.249110, 0.124555)^T$
13	$(0.002500, 10.000000)^T$	$1.0e - 005 * (0.625264, 0.312632)^T$
14	$(0.001252, 10.000000)^T$	$1.0e - 005 * (0.156940, 0.078470)^T$
15	$(0.000627, 10.000000)^T$	$1.0e - 006 * (0.393919, 0.196959)^T$
16	$(0.000314, 10.000000)^T$	$1.0e - 007 * (0.988734, 0.494367)^T$
17	$(0.000157, 10.000000)^T$	$1.0e - 007 * (0.248171, 0.124085)^T$
18	$(0.0000789, 10.000000)^T$	$1.0e - 008 * (0.6229079, 0.3114539)^T$
	$x_0 = (100, 100)^T$	
21	$1.0e + 002 * (0.0000009, 1.0000000)^T$	$1.0e - 008 * (0.985008, 0.492504)^T$

Table 3

Step	x_k	$G(x)$
	$x_0 = (0.1, 0.7)^T$	
40	$(0.007326, 0.700000)^T$	$1.0e - 004 * (0.536774, 0.268387)^T$
	$x_0 = (10, 10)^T$	
47	$(0.007301, 10.000000)^T$	$1.0e - 004 * (0.533144, 0.266572)^T$
	$x_0 = (100, 100)^T$	
50	$1.0e + 002 * (0.000073, 1.000000)^T$	$1.0e - 004 * (0.537728, 0.268864)^T$

where the functions

$$F_1(x_1, x_2) = \left(x_1^2 + x_2^2, x_2^2 \right)^T, \quad F_2(x_1, x_2) = \left(x_1^2, x_1^2 + \frac{1}{2}x_2^2 + 1 \right)^T. \quad (4.4)$$

Both F_1 and F_2 are $R^2 \rightarrow R^2$ continuously differentiable functions.

Now, we use the Levenberg-Marquardt method (II) to compute Example 4.3. For the special construction of Example 4.3, we can use (2.6) to get the Clarke generalized Jacobian in the Levenberg-Marquardt method (II). Results for Example 4.3 with initial point $x_0 = (1, 1)^T$, $x_0 = (10, 0.1)^T$, and $x_0 = (100, 10)^T$ and $\lambda_1 = 0.01$, $\lambda_2 = 1$ are presented in Table 4.

Then we compute d_k by (3.3) in the Levenberg-Marquardt method (II) to compute Example 4.3. Results for Example 4.3 with initial point $x_0 = (1, 1)^T$, $x_0 = (10, 0.1)^T$, and $x_0 = (100, 10)^T$ and $\lambda_1 = 0.01$, $\lambda_2 = 1$ are presented in Table 5.

Table 4

Step	x_k	$G(x)$
	$x_0 = (1, 1)^T$	
3	$(-0.001250, 1.000000)^T$	$1.0e - 005 * (0.156444, 0.156444)^T$
	$x_0 = (10, 0.1)^T$	
15	$(0.099976, 0.100000)^T$	$(0.009995, 0.009995)^T$
	$x_0 = (100, 10)^T$	
17	$(0.096868, 10.000000)^T$	$(0.009383, 0.009383)^T$

Table 5

Step	x_k	$G(x)$
	$x_0 = (1, 1)^T$	
3	$(-0.001250, 1.000000)^T$	$1.0e - 005 * (0.156444, 0.156444)^T$
	$x_0 = (10, 0.1)^T$	
15	$(0.099989, 0.100000)^T$	$(0.009997, 0.009997)^T$
	$x_0 = (100, 10)^T$	
17	$(0.097467, 10.000000)^T$	$(0.009499, 0.009499)^T$

Remark 4.4. From the numerical results for in the Levenberg-Marquardt method (II) in Table 4 and Table 5, we can see that the modification of (3.9) in the Levenberg-Marquardt method (II) works quite as well as (3.3) which has been used in [7].

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References

- [1] C. Kanzow and M. Fukushima, "Equivalence of the generalized complementarity problem to differentiable unconstrained minimization," *Journal of Optimization Theory and Applications*, vol. 90, no. 3, pp. 581–603, 1996.
- [2] J.-S. Pang and L. Q. Qi, "Nonsmooth equations: motivation and algorithms," *SIAM Journal on Optimization*, vol. 3, no. 3, pp. 443–465, 1993.
- [3] L. Qi and P. Tseng, "On almost smooth functions and piecewise smooth functions," *Nonlinear Analysis, Theory, Methods & Applications*, vol. 67, no. 3, pp. 773–794, 2007.
- [4] L. Q. Qi and J. Sun, "A nonsmooth version of Newton's method," *Mathematical Programming*, vol. 58, no. 3, pp. 353–367, 1993.
- [5] D. P. Bertsekas, *Constrained Optimization and Lagrange Multiplier Methods*, Computer Science and Applied Mathematics, Academic Press, New York, NY, USA, 1982.
- [6] Y. Gao, "Newton methods for solving two classes of nonsmooth equations," *Applications of Mathematics*, vol. 46, no. 3, pp. 215–229, 2001.
- [7] F. Facchinei and C. Kanzow, "A nonsmooth inexact Newton method for the solution of large-scale nonlinear complementarity problems," *Mathematical Programming*, vol. 76, no. 3, pp. 493–512, 1997.

- [8] A. Fischer, V. Jeyakumar, and D. T. Luc, "Solution point characterizations and convergence analysis of a descent algorithm for nonsmooth continuous complementarity problems," *Journal of Optimization Theory and Applications*, vol. 110, no. 3, pp. 493–513, 2001.
- [9] S. Q. Du and Y. Gao, "A modified Levenberg-Marquardt method for nonsmooth equations with finitely many maximum functions," *Mathematical Problems in Engineering*, vol. 2008, Article ID 942391, 10 pages, 2008.
- [10] N. Yamashita and M. Fukushima, "Modified Newton methods for solving a semismooth reformulation of monotone complementarity problems," *Mathematical Programming*, vol. 76, no. 3, pp. 469–491, 1997.
- [11] M. J. Śmiałowski, "An approximate Newton method for non-smooth equations with finite max functions," *Numerical Algorithms*, vol. 41, no. 3, pp. 219–238, 2006.
- [12] C. Ma and J. Tang, "The quadratic convergence of a smoothing Levenberg-Marquardt method for nonlinear complementarity problem," *Applied Mathematics and Computation*, vol. 197, no. 2, pp. 566–581, 2008.
- [13] C. Kanzow, N. Yamashita, and M. Fukushima, "Levenberg-Marquardt methods with strong local convergence properties for solving nonlinear equations with convex constraints," *Journal of Computational and Applied Mathematics*, vol. 172, no. 2, pp. 375–397, 2004.
- [14] N. Yamashita and M. Fukushima, "On the rate of convergence of the Levenberg-Marquardt method," *Computing*, vol. 15, pp. 227–238, 2001.
- [15] J. Fan and J. Pan, "Convergence properties of a self-adaptive Levenberg-Marquardt algorithm under local error bound condition," *Computational Optimization and Applications*, vol. 34, no. 1, pp. 47–62, 2006.
- [16] K. Ueda and N. Yamashita, "On a Global Complexity Bound of the Levenberg-Marquardt Method," *Journal of Optimization Theory and Applications*, vol. 147, no. 3, pp. 443–453, 2010.
- [17] R. A. Polyak, "Regularized Newton method for unconstrained convex optimization," *Mathematical Programming*, vol. 120, no. 1, pp. 125–145, 2009.
- [18] K. Ueda and N. Yamashita, "Convergence properties of the regularized Newton method for the unconstrained nonconvex optimization," *Applied Mathematics and Optimization*, vol. 62, no. 1, pp. 27–46, 2010.
- [19] H. Yin, Z.-H. Huang, and L. Qi, "The convergence of a Levenberg-Marquardt method for nonlinear inequalities," *Numerical Functional Analysis and Optimization*, vol. 29, no. 5-6, pp. 687–716, 2008.
- [20] O. L. Mangasarian and M. V. Solodov, "Nonlinear complementarity as unconstrained and constrained minimization," *Mathematical Programming*, vol. 62, no. 2, pp. 277–297, 1993.
- [21] J. S. Chen and S. Pan, "A family of NCP functions and a descent method for the nonlinear complementarity problem," *Computational Optimization and Applications*, vol. 40, no. 3, pp. 389–404, 2008.