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Research Article

The Stability of a Quadratic Functional Equation with the Fixed Point Alternative

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Lee, An and Park introduced the quadratic functional equation $f(2x+y)+f(2x-y) = 8f(x)+2f(y)$ and proved the stability of the quadratic functional equation in the spirit of Hyers, Ulam and Th. M. Rassias. Using the fixed point method, we prove the generalized Hyers-Ulam stability of the quadratic functional equation in Banach spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1 (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\left\| f(x+y) - f(x) - f(y) \right\| \le \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.2}$$

exists for all $x \in E$, *and* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.3}$$

for all $x \in E$. *Also, if for each* $x \in E$ *the function* $f(tx)$ *is continuous in* $t \in \mathbb{R}$, *then* L *is* $\mathbb{R}$*-linear.*

The above inequality (1.1) has provided a lot of influence in the development of what is now known as a *generalized Hyers-Ulam stability* of functional equations. Beginning around the year 1980, the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [5] generalized the Rassias' result.

Theorem 1.2 (see [6–8]). *Let* X *be a real normed linear space and* Y *a real complete normed linear space. Assume that* $f : X \to Y$ *is an approximately additive mapping for which there exist constants* $\theta \ge 0$ *and* $p \in \mathbb{R} - \{1\}$ *such that* f *satisfies inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta \cdot \|x\|^{p/2} \cdot \|y\|^{p/2} \tag{1.4}$$

for all $x, y \in X$. *Then there exists a unique additive mapping* $L : X \to Y$ *satisfying*

$$\|f(x) - L(x)\| \le \frac{\theta}{|2^p - 2|}\|x\|^p \tag{1.5}$$

for all $x \in X$. *If, in addition,* $f : X \to Y$ *is a mapping such that the transformation* $t \to f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then* L *is an* $\mathbb{R}$*-linear mapping.*

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.6}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic function*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [9] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [10] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [11] proved the generalized Hyers-Ulam stability of the quadratic functional equation. Several functional equations have been investigated in [12–25].

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.3 (see [26–28]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{1.7}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$, *for all* $n \geq n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq (1/(1-L)) d(y, Jy)$ *for all* $y \in Y$.

Lee et al. [29] proved that a mapping $f : X \to Y$ satisfies

$$f(2x + y) + f(2x - y) = 8f(x) + 2f(y) \tag{1.8}$$

for all $x, y \in X$ if and only if the mapping $f : X \to Y$ satisfies

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.9}$$

for all $x, y \in X$.

Using the fixed point method, Park [14] proved the generalized Hyers-Ulam stability of the quadratic functional equation

$$f(2x + y) = 4f(x) + f(y) + f(x + y) - f(x - y) \tag{1.10}$$

in Banach spaces.

In this paper, using the fixed point method, we prove the generalized Hyers-Ulam stability of the quadratic functional equation (1.8) in Banach spaces.

Throughout this paper, assume that X is a normed vector space with norm $|| \cdot ||$ and that Y is a Banach space with norm $\| \cdot \|$.

2. Fixed Points and Generalized Hyers-Ulam Stability of a Quadratic Functional Equation

For a given mapping $f : X \to Y$, we define

$$Cf(x, y) := f(2x + y) + f(2x - y) - 8f(x) - 2f(y) \tag{2.1}$$

for all $x, y \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the quadratic functional equation $Cf(x, y) = 0$.

Theorem 2.1. *Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X^2 \to [0,\infty)$ with $f(0) = 0$ such that*

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.2}$$

for all $x, y \in X$. If there exists an $L < 1$ such that $\varphi(x,y) \le 4L\varphi(x/2, y/2)$ for all $x, y \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.8) *and*

$$\|f(x) - Q(x)\| \le \frac{1}{8-8L}\varphi(x,0) \tag{2.3}$$

for all $x \in X$.

Proof. Consider the set

$$S := \{g : X \longrightarrow Y\}, \tag{2.4}$$

and introduce the *generalized metric* on S:

$$d(g,h) = \inf\{K \in \mathbb{R}_+ : \|g(x) - h(x)\| \le K\varphi(x,0),\ \forall x \in X\}. \tag{2.5}$$

It is easy to show that (S, d) is complete.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x) \tag{2.6}$$

for all $x \in X$.

By [30, Theorem 3.1],

$$d(Jg, Jh) \le Ld(g,h) \tag{2.7}$$

for all $g, h \in S$.

Letting $y = 0$ in (2.2), we get

$$\|2f(2x) - 8f(x)\| \le \varphi(x,0) \tag{2.8}$$

for all $x \in X$. So

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{8}\varphi(x,0) \tag{2.9}$$

for all $x \in X$. Hence $d(f, Jf) \le 1/8$.

By Theorem 1.3, there exists a mapping $Q : X \to Y$ such that

(1) Q is a fixed point of J, that is,

$$Q(2x) = 4Q(x) \tag{2.10}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}. \tag{2.11}$$

This implies that Q is a unique mapping satisfying (2.10) such that there exists $K \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\| \le K\varphi(x, 0) \tag{2.12}$$

for all $x \in X$.

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{4^n} = Q(x) \tag{2.13}$$

for all $x \in X$.

(3) $d(f, Q) \le (1/(1-L))d(f, Jf)$, which implies the inequality

$$d(f, Q) \le \frac{1}{8 - 8L}. \tag{2.14}$$

This implies that the inequality (2.3) holds.

It follows from (2.2) and (2.13) that

$$\|CQ(x, y)\| = \lim_{n\to\infty} \frac{1}{4^n} \|Cf(2^n x, 2^n y)\| \le \lim_{n\to\infty} \frac{1}{4^n} \varphi(2^n x, 2^n y) \le \lim_{n\to\infty} L^n \varphi(x, y) = 0 \tag{2.15}$$

for all $x, y \in X$. So $CQ(x, y) = 0$ for all $x, y \in X$.

By [29, Proposition 2.1], the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 2.2. *Let $0 < p < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|Cf(x, y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{2.16}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.8) *and*

$$\|f(x) - Q(x)\| \le \frac{\theta}{8 - 2^{p+1}} \|x\|^p \tag{2.17}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x,y) := \theta\left(\|x\|^p + \|y\|^p\right) \tag{2.18}$$

for all $x, y \in X$. Then $L = 2^{p-2}$, and we get the desired result. □

Theorem 2.3. *Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X^2 \to [0,\infty)$ satisfying* (2.2) *and $f(0) = 0$. If there exists an $L < 1$ such that $\varphi(x,y) \leq (L/4)\varphi(2x,2y)$ for all $x, y \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (1.8) *and*

$$\|f(x) - Q(x)\| \leq \frac{L}{8-8L}\varphi(x,0) \tag{2.19}$$

for all $x \in X$.

Proof. We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right) \tag{2.20}$$

for all $x \in X$.

It follows from (2.8) that

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq \frac{1}{2}\varphi\left(\frac{x}{2},0\right) \leq \frac{L}{8}\varphi(x,0) \tag{2.21}$$

for all $x \in X$. Hence $d(f, Jf) \leq L/8$.

By Theorem 1.3, there exists a mapping $Q : X \to Y$ such that

(1) Q is a fixed point of J, that is,

$$Q(2x) = 4Q(x) \tag{2.22}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}. \tag{2.23}$$

This implies that Q is a unique mapping satisfying (2.22) such that there exists $K \in (0,\infty)$ satisfying

$$\|f(x) - Q(x)\| \leq K\varphi(x,0) \tag{2.24}$$

for all $x \in X$.

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x) \tag{2.25}$$

for all $x \in X$.

(3) $d(f,Q) \leq (1/(1-L))d(f,Jf)$, which implies the inequality

$$d(f,Q) \leq \frac{L}{8-8L}, \tag{2.26}$$

which implies that the inequality (2.19) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $p > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.16)*. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (1.8) *and*

$$\|f(x) - Q(x)\| \leq \frac{\theta}{2^{p+1}-8}\|x\|^p \tag{2.27}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x,y) := \theta\left(\|x\|^p + \|y\|^p\right) \tag{2.28}$$

for all $x, y \in X$. Then $L = 2^{2-p}$ and, we get the desired result. □

Theorem 2.5. *Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X^2 \to [0,\infty)$ satisfying* (2.2)*. If there exists an $L < 1$ such that $\varphi(x,y) \leq 9L\varphi(x/3, y/3)$ for all $x, y \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (1.8) *and*

$$\|f(x) - Q(x)\| \leq \frac{1}{9-9L}\varphi(x,x) \tag{2.29}$$

for all $x \in X$.

Proof. Consider the set

$$S := \{g : X \longrightarrow Y\}, \tag{2.30}$$

and introduce the *generalized metric* on S:

$$d(g,h) = \inf\{K \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq K\varphi(x,x),\ \forall x \in X\}. \tag{2.31}$$

It is easy to show that (S,d) is complete.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{9}g(3x) \tag{2.32}$$

for all $x \in X$.

By [30, Theorem 3.1],

$$d(Jg, Jh) \le Ld(g, h) \tag{2.33}$$

for all $g, h \in S$.

Letting $y = x$ in (2.2), we get

$$\|f(3x) - 9f(x)\| \le \varphi(x, x) \tag{2.34}$$

for all $x \in X$. So

$$\left\|f(x) - \frac{1}{9}f(3x)\right\| \le \frac{1}{9}\varphi(x, x) \tag{2.35}$$

for all $x \in X$. Hence $d(f, Jf) \le 1/9$.

By Theorem 1.3, there exists a mapping $Q : X \to Y$ such that

(1) Q is a fixed point of J, that is,

$$Q(3x) = 9Q(x) \tag{2.36}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}. \tag{2.37}$$

This implies that Q is a unique mapping satisfying (2.36) such that there exists $K \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\| \le K\varphi(x, x) \tag{2.38}$$

for all $x \in X$.

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(3^n x)}{9^n} = Q(x) \tag{2.39}$$

for all $x \in X$.

(3) $d(f, Q) \le (1/(1-L))d(f, Jf)$, which implies the inequality

$$d(f, Q) \le \frac{1}{9 - 9L}. \tag{2.40}$$

This implies that the inequality (2.29) holds.

It follows from (2.2) and (2.39) that

$$\|CQ(x, y)\| = \lim_{n\to\infty} \frac{1}{9^n}\|Cf(3^n x, 3^n y)\| \le \lim_{n\to\infty} \frac{1}{9^n}\varphi(3^n x, 3^n y) \le \lim_{n\to\infty} L^n \varphi(x, y) = 0 \tag{2.41}$$

for all $x, y \in X$. So $CQ(x, y) = 0$ for all $x, y \in X$.

By [29, Proposition 2.1], the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 2.6. *Let* $0 < p < 2$ *and* θ *be positive real numbers, and let* $f : X \to Y$ *be a mapping satisfying* (2.16). *Then there exists a unique quadratic mapping* $Q : X \to Y$ *satisfying* (1.8) *and*

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{9 - 3^p}\|x\|^p \tag{2.42}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \tag{2.43}$$

for all $x, y \in X$. Then $L = 3^{p-2}$ and, we get the desired result. □

Corollary 2.7. *Let* $0 < p < 1$ *and* θ *be positive real numbers, and let* $f : X \to Y$ *be a mapping such that*

$$\|Df(x, y)\| \leq \theta \cdot \|x\|^p \cdot \|y\|^p \tag{2.44}$$

for all $x, y \in X$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *satisfying* (1.8) *and*

$$\|f(x) - Q(x)\| \leq \frac{\theta}{9 - 9^p}\|x\|^{2p} \tag{2.45}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(x, y) := \theta \cdot \|x\|^p \cdot \|y\|^p \tag{2.46}$$

for all $x, y \in X$. Then $L = 9^{p-1}$ and, we get the desired result. □

Theorem 2.8. *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\varphi : X^2 \to [0, \infty)$ *satisfying* (2.2). *If there exists an* $L < 1$ *such that* $\varphi(x, y) \leq (L/9)\varphi(3x, 3y)$ *for all* $x, y \in X$, *then there exists a unique quadratic mapping* $Q : X \to Y$ *satisfying* (1.8) *and*

$$\|f(x) - Q(x)\| \leq \frac{L}{9 - 9L}\varphi(x, x) \tag{2.47}$$

for all $x \in X$.

Proof. We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 9g\left(\frac{x}{3}\right) \tag{2.48}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.9. *Let $p > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.16)*. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (1.8) *and*

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{3^p - 9}\|x\|^p \tag{2.49}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.8 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \tag{2.50}$$

for all $x, y \in X$. Then $L = 3^{2-p}$, and we get the desired result. □

Corollary 2.10. *Let $p > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.44)*. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (1.8) *and*

$$\|f(x) - Q(x)\| \leq \frac{\theta}{9^p - 9}\|x\|^{2p} \tag{2.51}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.8 by taking

$$\varphi(x, y) := \theta \cdot \|x\|^p \cdot \|y\|^p \tag{2.52}$$

for all $x, y \in X$. Then $L = 9^{1-p}$, and we get the desired result. □

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References

[1] S. M. Ulam, *Problems in Modern Mathematics*, John Wiley & Sons, New York, NY, USA, 1960.
[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.
[4] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
[5] P. Găvruta, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
[6] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.
[7] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Bulletin des Sciences Mathématiques*, vol. 108, no. 4, pp. 445–446, 1984.
[8] J. M. Rassias, "Solution of a problem of Ulam," *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.
[9] F. Skof, "Local properties and approximation of operators," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, pp. 113–129, 1983.

[10] P. W. Cholewa, "Remarks on the stability of functional equations," *Aequationes Mathematicae*, vol. 27, no. 1-2, pp. 76–86, 1984.
[11] S. Czerwik, "On the stability of the quadratic mapping in normed spaces," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, pp. 59–64, 1992.
[12] D. Miheţ and V. Radu, "On the stability of the additive Cauchy functional equation in random normed spaces," *Journal of Mathematical Analysis and Applications*, vol. 343, no. 1, pp. 567–572, 2008.
[13] C. Park, "Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras," *Fixed Point Theory and Applications*, vol. 2007, Article ID 50175, 15 pages, 2007.
[14] C. Park, "Generalized Hyers-Ulam stability of quadratic functional equations: a fixed point approach," *Fixed Point Theory and Applications*, vol. 2008, Article ID 493751, 9 pages, 2008.
[15] C. Park, Y. S. Cho, and M.-H. Han, "Functional inequalities associated with Jordan-von Neumann-type additive functional equations," *Journal of Inequalities and Applications*, vol. 2007, Article ID 41820, 13 pages, 2007.
[16] C. Park and J. Cui, "Generalized stability of C^*-ternary quadratic mappings," *Abstract and Applied Analysis*, vol. 2007, Article ID 23282, 6 pages, 2007.
[17] C. Park and J. Hou, "Homomorphisms between C^*-algebras associated with the Trif functional equation and linear derivations on C^*-algebras," *Journal of the Korean Mathematical Society*, vol. 41, no. 3, pp. 461–477, 2004.
[18] C. Park and A. Najati, "Homomorphisms and derivations in C^*-algebras," *Abstract and Applied Analysis*, vol. 2007, Article ID 80630, 12 pages, 2007.
[19] C. Park and J. Park, "Generalized Hyers-Ulam stability of an Euler-Lagrange type additive mapping," *Journal of Difference Equations and Applications*, vol. 12, no. 12, pp. 1277–1288, 2006.
[20] Th. M. Rassias, "On the stability of the quadratic functional equation and its applications," *Studia Universitatis Babeş-Bolyai Mathematica*, vol. 43, no. 3, pp. 89–124, 1998.
[21] Th. M. Rassias, "The problem of S. M. Ulam for approximately multiplicative mappings," *Journal of Mathematical Analysis and Applications*, vol. 246, no. 2, pp. 352–378, 2000.
[22] Th. M. Rassias, "On the stability of functional equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 264–284, 2000.
[23] Th. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.
[24] Th. M. Rassias and P. Šemrl, "On the Hyers-Ulam stability of linear mappings," *Journal of Mathematical Analysis and Applications*, vol. 173, no. 2, pp. 325–338, 1993.
[25] Th. M. Rassias and K. Shibata, "Variational problem of some quadratic functionals in complex analysis," *Journal of Mathematical Analysis and Applications*, vol. 228, no. 1, pp. 234–253, 1998.
[26] L. Cădariu and V. Radu, "On the stability of the Cauchy functional equation: a fixed point approach," in *Iteration Theory*, vol. 346 of *Grazer Mathematische Berichte*, pp. 43–52, Karl-Franzens-Universitaet Graz, Graz, Austria, 2004.
[27] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 74, pp. 305–309, 1968.
[28] V. Radu, "The fixed point alternative and the stability of functional equations," *Fixed Point Theory*, vol. 4, no. 1, pp. 91–96, 2003.
[29] J. Lee, J. S. An, and C. Park, "On the stability of quadratic functional equations," *Abstract and Applied Analysis*, vol. 2008, Article ID 628178, 8 pages, 2008.
[30] L. Cădariu and V. Radu, "Fixed points and the stability of Jensen's functional equation," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 1, article 4, 2003.