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Research Article

Fractional Evolution Equations Governed by Coercive Differential Operators

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This paper is concerned with evolution equations of fractional order $\mathbf{D}^{\alpha}u(t) = Au(t); u(0) = u_0, u'(0) = 0$, where A is a differential operator corresponding to a coercive polynomial taking values in a sector of angle less than π and $1 < \alpha < 2$. We show that such equations are well posed in the sense that there always exists an α-times resolvent family for the operator A.

1. Introduction

It is well known that the abstract Cauchy problem of first order

$$u'(t) = Au(t), \quad t > 0; \qquad u(0) = x \tag{1.1}$$

is well posed if and only if A is the generator of a C_0-semigroup. However, many partial differential operators (PDOs) such as the Schrödinger operator $i\Delta$ on $L^p(\mathbb{R}^n)$ $(p \neq 2)$ cannot generate C_0-semigroups. It was Kellermann and Hieber [1] who first showed that some elliptic differential operators on some function spaces generate integrated semigroups, and their results are improved and developed in [2, 3]. Because of the limitations of integrated semigroups, the results in [1–3] are confined to elliptic differential operators with constant coefficients. One of the limitations is that the resolvent sets of generators must contain a right half-plane; however, it is known that there are many nonelliptic operators whose resolvent sets are empty (see, e.g., [4]). On the other hand, the resolvent sets of the generators of regularized semigroups need not be nonempty; this makes it possible to apply the theory of regularized semigroups to nonelliptic operators, such as coercive operators and hypoelliptic operators (see [5–8]). Moreover, for second-order equations, Zheng [9] considered coercive differential operators with constant coefficients generating integrated cosine functions. The aim of this paper is to consider fractional evolution equations associated with coercive differential operators.

Let X be a Banach space, and let A be a closed linear unbounded operator with densely defined domain $D(A)$. A family of strongly continuous bounded linear operators on X, $\{R(t)\}_{t\geq 0}$, is called a *resolvent family* for A with kernel $a(t) \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ if $R(t)A \subset AR(t)$ and the *resolvent equation*

$$R(t)x = x + \int_0^t a(t-s)AR(s)x\,ds, \quad t \geq 0,\ x \in D(A) \tag{1.2}$$

holds. It is obvious that a C_0-semigroup is a resolvent family for its generator with kernel $a_1(t) \equiv 1$; a cosine function is a resolvent family for its generator with kernel $a_2(t) = t$. If we define the *α-times resolvent family* for A as being a resolvent family with kernel $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$, then such resolvent families interpolate C_0-semigroups and cosine functions.

Recently Bazhlekova studied classes of such resolvent families (see [10]). Let $0 < \alpha \leq 2$, and let m be the smallest integer greater than or equal to α. It was shown in [10] that the fractional evolution equation of order α,

$$\mathbf{D}^\alpha u(t) = Au(t), \quad t > 0; \qquad u^{(k)}(0) = x_k, \quad k = 0, 1, \dots, m-1, \tag{1.3}$$

is well posed if and only if there exists an α-times resolvent family for A. Here $\mathbf{D}^\alpha$ is the *Caputo fractional derivative* of order $\alpha > 0$ defined by

$$\mathbf{D}^\alpha f(t) := \int_0^t g_{m-\alpha}(t-s)\frac{d^m}{ds^m} f(s)\,ds, \tag{1.4}$$

where $f \in W^{m,1}(I)$ for every interval I. The hypothesis on f can be relaxed; see [10] for details. Fujita in [11] studied (1.3) for the case that $A = \Delta$, the Laplacian $(\partial/\partial x)^2$ on $\mathbb{R}$, which interpolates the heat equation and the wave equation. Since α-times resolvent families interpolate C_0-semigroups and cosine functions, this motivates us to consider the existence of fractional resolvent families for PDOs.

There are several examples of the existence of α-times resolvent families for concrete PDOs in [10], but Bazhlekova did not develop the theory of α-times resolvent families for general PDOs. The authors showed in [12] that there exist fractional resolvent families for elliptic operators. In this paper we will consider coercive operators. Since α-times resolvent families are not sufficient for applications we have in mind, we first extend, in Section 2, such a notion to the setting of C-regularized resolvent families which was introduced in [13]. To do this, we use methods of the Fourier multiplier theory.

This paper is organized as follows. Section 2 contains the definition and some basic properties of α-times regularized resolvent families. Section 3 prepares for the proof of the main result of this paper. Our main result, Theorem 4.1, shows that there are α-times regularized resolvent families for PDOs corresponding to coercive polynomials taking values in a sector of angle less than π. Some examples are also given in Section 4.

2. α-Times Regularized Resolvent Family

Throughout this paper, X is a complex Banach space, and we denote by $\mathbf{B}(X)$ the algebra of all bounded linear operators on X. Let A be a closed densely defined operator on X, let $D(A)$ and $R(A)$ be its domain and range, respectively, and let $\alpha \in (0,2]$, $C \in \mathbf{B}(X)$ be injective. Define $\rho_C(A) := \{\lambda \in \mathbb{C} : \lambda - A$ is injective and $R(C) \subset R(\lambda - A)\}$. Let $\Sigma_\theta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \theta\}$ be the open sector of angle 2θ in the complex plane, where arg is the branch of the argument between $-\pi$ and π.

Definition 2.1. A strongly continuous family $\{S_\alpha(t)\}_{t\geq 0} \subset \mathbf{B}(X)$ is called an *α-times C-regularized resolvent family for A* if

(a) $S_\alpha(0) = C$;

(b) $S_\alpha(t)A \subset AS_\alpha(t)$ for $t \geq 0$;

(c) $C^{-1}AC = A$;

(d) for $x \in D(A)$, $S_\alpha(t)x = Cx + \int_0^t ((t-s)^{\alpha-1}/\Gamma(\alpha))S_\alpha(s)Axds$.

$\{S_\alpha(t)\}_{t\geq 0}$ is called *analytic* if it can be extended analytically to some sector Σ_θ.

If $\|S_\alpha(t)\| \leq Me^{\omega t}$ $(t \geq 0)$ for some constants $M \geq 1$ and $\omega \in \mathbb{R}_+$, we will write $A \in \mathcal{C}_C^\alpha(M,\omega)$, and $\mathcal{C}_C^\alpha(\omega) := \cup\{\mathcal{C}_C^\alpha(M,\omega); M \geq 1\}$, $\mathcal{C}_C^\alpha := \cup\{\mathcal{C}_C^\alpha(\omega); \omega \geq 0\}$.

Define the operator $\widetilde{A}$ by

$$\widetilde{A}x = C^{-1}\left(\lim_{t\downarrow 0}\frac{\Gamma(\alpha+1)}{t^\alpha}(S_\alpha(t)x - Cx)\right), \quad x \in D\left(\widetilde{A}\right), \tag{2.1}$$

with

$$D\left(\widetilde{A}\right) = \left\{x \in X : \lim_{t\downarrow 0}\frac{S_\alpha(t)x - Cx}{t^\alpha} \text{ exists and is in } R(C)\right\}. \tag{2.2}$$

Proposition 2.2. *Suppose that there exists an α-times C-regularized resolvent family, $\{S_\alpha(t)\}_{t\geq 0}$, for the operator A, and let $\widetilde{A}$ be defined as above. Then $A = \widetilde{A}$.*

Proof. By the strong continuity of $S_\alpha(t)$, we have for every $x \in X$,

$$\begin{aligned}\left\|g_{\alpha+1}(t)^{-1}\int_0^t g_\alpha(t-s)S_\alpha(s)xds - Cx\right\| &\leq g_{\alpha+1}(t)^{-1}\int_0^t g_\alpha(t-s)\|S_\alpha(s)x - Cx\|ds \\ &\leq \sup_{0\leq s\leq t}\|S_\alpha(s)x - Cx\| \longrightarrow 0 \quad \text{as } t \longrightarrow 0.\end{aligned} \tag{2.3}$$

Thus for $x \in D(A)$, by Definition 2.1,

$$\begin{aligned}\lim_{t\downarrow 0} g_{\alpha+1}(t)^{-1}(S_\alpha(t)x - Cx) &= \lim_{t\downarrow 0} g_{\alpha+1}(t)^{-1}\int_0^t g_\alpha(t-s)S_\alpha(s)Axds \\ &= CAx,\end{aligned} \tag{2.4}$$

which means that $x \in D(\widetilde{A})$ and $\widetilde{A}x = Ax$. On the other hand, for $x \in D(\widetilde{A})$, by the definition of $\widetilde{A}$ and Definition 2.1,

$$\begin{aligned} C\widetilde{A}x &= \lim_{t\downarrow 0} g_{\alpha+1}(t)^{-1}(S_\alpha(t)x - Cx) \\ &= \lim_{t\downarrow 0} g_{\alpha+1}(t)^{-1} A\int_0^t g_\alpha(t-s)S_\alpha(s)x\,ds, \end{aligned} \tag{2.5}$$

but $\lim_{t\to 0} g_{\alpha+1}(t)^{-1}\int_0^t g_\alpha(t-s)S_\alpha(s)x\,ds = Cx$, by (d) of Definition 2.1. Thus it follows from the closedness of A that $Cx \in D(A)$ with $ACx = C\widetilde{A}x$. This implies that $x \in D(C^{-1}AC) = D(A)$, so we have $\widetilde{A} = A$. □

The following generation theorem and subordination principle for α-times C-regularized resolvent families can be proved similarly as those for α-times resolvent families (see [10]).

Theorem 2.3. *Let $\alpha \in (0,2]$. Then the following statements are equivalent:*

(a) $A \in \mathcal{C}_C^\alpha(M,\omega)$;

(b) $A = C^{-1}AC$, $(\omega^\alpha,\infty) \subseteq \rho_C(A)$ *and*

$$\left\| \frac{d^n}{d\lambda^n}\left(\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}C\right)\right\| \le \frac{Mn!}{(\lambda-\omega)^{n+1}}, \quad \lambda > \omega,\ n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}; \tag{2.6}$$

(c) $A = C^{-1}AC$, $(\omega^\alpha,\infty) \subseteq \rho_C(A)$ *and there exists a strongly continuous family* $\{S_\alpha(t)\}_{t\ge 0} \subset \mathbf{B}(X)$ *satisfying* $\|S_\alpha(t)\| \le Me^{\omega t}$ *such that*

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}Cx = \int_0^\infty e^{-\lambda t}S_\alpha(t)x\,dt, \quad \lambda > \omega,\ x \in X. \tag{2.7}$$

Theorem 2.4. *Suppose that $0 < \alpha < \beta \le 2$, $\gamma = \alpha/\beta$. If $A \in \mathcal{C}_C^\beta(\omega)$ then $A \in \mathcal{C}_C^\alpha(\omega^{1/\gamma})$ and the α-times C-regularized resolvent family for A, $\{S_\alpha(t)\}_{t\ge0}$, can be extended analytically to $\Sigma_{\min\{\theta(\gamma),\pi\}}$, where $\theta(\gamma) := (1/\gamma - 1)\pi/2$.*

3. Coercive Operators and Mittag-Leffler Functions

We now introduce a functional calculus for generators of bounded C_0-groups (cf. [14]), which will play a key role in our proof.

Let iA_j $(1 \le j \le n)$ be commuting generators of bounded C_0-groups on a Banach space X. Write $A = (A_1,\dots,A_n)$ and $A^\mu = A_1^{\mu_1}\cdots A_n^{\mu_n}$ for $\mu = (\mu_1,\dots,\mu_n) \in \mathbb{N}_0^n$. Similarly, write $D^\mu = D_1^{\mu_1}\cdots D_n^{\mu_n}$, where $D_j = -i\partial/\partial x_j$ for $j = 1,\dots,n$. For a polynomial $P(\xi) := \sum_{|\mu|\le m} a_\mu \xi^\mu$ $(\xi \in \mathbb{R}^n)$ $(|\mu| := \sum_{j=1}^n \mu_j)$ with constant coefficients, we define $P(A) = \sum_{|\mu|\le m} a_\mu A^\mu$ $(\xi \in \mathbb{R}^n)$ with maximal domain. Then $P(A)$ is closable. Let $\mathcal{F}$ be the Fourier transform, that is, $(\mathcal{F}u)(\eta) = \int_{\mathbb{R}^n} u(\xi)e^{-i(\xi,\eta)}d\xi$ for $u \in L^1(\mathbb{R}^n)$, where $(\xi,\eta) = \sum_{j=1}^n \xi_j\eta_j$. If $u \in \mathcal{F}L^1(\mathbb{R}^n) := \{\mathcal{F}v : v \in L^1(\mathbb{R}^n)\}$, then there exists a unique function in $L^1(\mathbb{R}^n)$, written $\mathcal{F}^{-1}u$, such that $u = \mathcal{F}(\mathcal{F}^{-1}u)$. In particular, $\mathcal{F}^{-1}u$ is the inverse Fourier transform of u if $u \in \mathcal{S}(\mathbb{R}^n)$ (the space of rapidly decreasing functions on $\mathbb{R}^n$). We define $u(A) \in B(X)$ by

$$u(A)x = \int_{\mathbb{R}^n} \left(\mathcal{F}^{-1}u\right)(\xi) e^{-i(\xi,A)} x d\xi, \quad x \in X, \tag{3.1}$$

where $(\xi, A) = \sum_{j=1}^{n} \xi_j A_j$.

We will need the following lemma, in which the statements (a) and (b) are well-known, (c) and (d) can be found in [14] and [6], respectively.

Lemma 3.1. (a) *$\mathcal{F}L^1(\mathbb{R}^n)$ is a Banach algebra under pointwise multiplication and addition with norm $\|u\|_{\mathcal{F}L^1} := \|\mathcal{F}^{-1}u\|_{L^1}$.*

(b) *$u \mapsto u(A)$ is an algebra homomorphism from $\mathcal{F}L^1(\mathbb{R}^n)$ into $\mathbf{B}(X)$, and there exists a constant $M > 0$ such that $\|u(A)\| \le M\|u\|_{\mathcal{F}L^1}$.*

(c) *$E := \{\phi(A)x : \phi \in \mathcal{S}(\mathbb{R}^n),\ x \in X\} \subset \cap_{\mu \in \mathbb{N}_0^n} D(A^\mu)$, $\overline{E} = X$, $\overline{P(A)|_E} = \overline{P(A)}$ and $\phi(A)P(A) \subset P(A)\phi(A) = (P\phi)(A)$ for $\phi \in \mathcal{S}(\mathbb{R}^n)$.*

(d) *Let $u \in C^j(\mathbb{R}^n)$ $(j > n/2)$. Suppose that there exist constants L, M_0, $a > 0$, and $b \in [-1, 2a/n - 1)$ such that*

$$\left|D^k u(\xi)\right| \le \begin{cases} M_0^{|k|} |\xi|^{b|k|-a}, & \text{for } |\xi| \ge L,\ |k| \le j, \\ M_0^{|k|}, & \text{for } |\xi| < L,\ |k| \le j, \end{cases} \tag{3.2}$$

where $k \in \mathbb{N}_0^n$, then $u \in \mathcal{F}L^1(\mathbb{R}^n)$ and $\|u\|_{\mathcal{F}L^1} \le M M_0^{n/2}$ for some constant $M > 0$.

Recall that the Mittag-Leffler function (see [15, 16]) is defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i} \int_{\mathcal{T}} \frac{\mu^{\alpha-\beta} e^{\mu}}{\mu^\alpha - z}\, d\mu, \quad \alpha, \beta > 0,\ z \in \mathbb{C}, \tag{3.3}$$

where the path $\mathcal{T}$ is a loop which starts and ends at $-\infty$ and encircles the disc $|t| \le |z|^{1/\alpha}$ in the positive sense. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \quad \operatorname{Re}\lambda > \omega^{1/\alpha},\ \omega > 0 \tag{3.4}$$

and with their asymptotic expansion as $z \to \infty$. If $0 < \alpha < 2$, $\beta > 0$, then

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp\left(z^{1/\alpha}\right) + \varepsilon_{\alpha,\beta}(z), \quad \left|\arg z\right| \le \frac{1}{2}\alpha\pi, \tag{3.5}$$

$$E_{\alpha,\beta}(z) = \varepsilon_{\alpha,\beta}(z), \quad \left|\arg(-z)\right| < \left(1 - \frac{1}{2}\alpha\right)\pi, \tag{3.6}$$

where

$$\varepsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O\left(|z|^{-N}\right) \tag{3.7}$$

as $z \to \infty$, and the O-term is uniform in $\arg z$ if $|\arg(-z)| \leq (1-\alpha/2-\epsilon)\pi$. Note that for $\beta > 0$,

$$\left|E_{\alpha,\beta}(z)\right| \leq E_{\alpha,\beta}(|z|), \qquad z \in \mathbb{C}. \tag{3.8}$$

The following two lemmas are about derivatives of the Mittag-Leffler functions.

Lemma 3.2.

$$E'_{\alpha,\beta}(z) = \frac{1}{\alpha} E_{\alpha,\alpha+\beta-1}(z) - \frac{\beta-1}{\alpha} E_{\alpha,\alpha+\beta}(z). \tag{3.9}$$

Proof. By the definition of $E_{\alpha,\beta}(z)$,

$$\begin{aligned}
E'_{\alpha,\beta}(z) &= \left(\sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}\right)' \\
&= \sum_{n=1}^{\infty} \frac{n z^{n-1}}{\Gamma(\alpha n + \beta)} \\
&= \sum_{n=1}^{\infty} \frac{n z^{n-1}}{\Gamma(\alpha n + \beta - 1 + 1)} \\
&= \sum_{n=1}^{\infty} \frac{z^{n-1} \cdot (\alpha n + \beta - 1)}{\alpha(\alpha n + \beta - 1)\Gamma(\alpha n + \beta - 1)} - \sum_{n=1}^{\infty} \frac{\beta-1}{\alpha} \cdot \frac{z^{n-1}}{\Gamma(\alpha n + \beta)} \\
&= \frac{1}{\alpha} E_{\alpha,\alpha+\beta-1}(z) - \frac{\beta-1}{\alpha} E_{\alpha,\alpha+\beta}(z),
\end{aligned} \tag{3.10}$$

as we wanted to show. □

For short, $E_\alpha(z) := E_{\alpha,1}(z)$.

Lemma 3.3. *Suppose that $1 < \alpha < 2$. For every $n \in \mathbb{N}$ and $\epsilon > 0$ there exist constants $M > 0$ and $L > 0$ such that for $k = 0, \ldots, n$,*

$$\left|E_\alpha^{(k)}(z)\right| \leq \frac{M}{|z|}, \quad \text{if } |z| \geq L, \ \left|\arg(-z)\right| \leq \left(1 - \frac{\alpha}{2} - \epsilon\right)\pi. \tag{3.11}$$

Proof. First note that $E'_\alpha(z) = (1/\alpha) E_{\alpha,\alpha}(z)$, and by induction on k one can prove that

$$E_\alpha^{(k)}(z) = \sum_{j=1}^{k} a_j E_{\alpha,\alpha k-(k-j)}(z), \tag{3.12}$$

where a_j only depend on α and k. Since $\alpha > 1$ we have that $\alpha k - (k - j) > 0$ whence, by the asymptotic formula for Mittag-Leffler functions (3.6), we obtain (3.11). □

Now let us recall the definition of coercive polynomials. For fixed $r > 0$, a polynomial $P(\xi)$ is called *r-coercive* if $|P(\xi)|^{-1} = O(|\xi|^{-r})$ as $|\xi| \to \infty$. In the sequel, M is a generic constant independent of t which may vary from line to line.

Lemma 3.4. *Suppose that $P(\xi)$ is an r-coercive polynomial of order m and $\{P(\xi) : \xi \in \mathbb{R}^n\} \subset \mathbb{C} \setminus \Sigma_{\alpha'\pi/2}$, where $1 < \alpha' < 2$. Let $k_0 = [n/2] + 1$. Then for $1 < \alpha < \alpha'$, $\gamma > 0$, $a \in \Sigma_{\alpha'\pi/2}$, there exist constants $M, L \geq 0$ such that*

$$\left|D^\mu\left[E_\alpha(t^\alpha P)(a-P)^{-\gamma}\right]\right| \leq M\left(1 + t^{\alpha|\mu|}\right)|\xi|^{(m-1)|\mu|-r\gamma}, \quad |\xi| \geq L, \ |\mu| \leq k_0, \ t \geq 0. \tag{3.13}$$

Proof. Suppose that for $|\xi| \geq L$, (3.11) holds up to order k_0 and

$$|P(\xi)| \geq M|\xi|^r, \qquad |a - P(\xi)| \geq M|\xi|^r. \tag{3.14}$$

By induction, one can show that

$$D^\mu E_\alpha(t^\alpha P) = \sum_{j=1}^{|\mu|} t^{\alpha j} E_\alpha^{(j)}(t^\alpha P) Q_j, \tag{3.15}$$

where $\deg Q_j \leq mj - |\mu|$. Thus if $|\xi| \geq L$ and $|t^\alpha P| \geq L$,

$$\begin{aligned} |D^\mu E_\alpha(t^\alpha P)| &\leq M\left(1 + t^{\alpha(|\mu|-1)}\right)|\xi|^{m|\mu|-|\mu|-r} \\ &\leq M\left(1 + t^{\alpha|\mu|}\right)|\xi|^{(m-1)|\mu|-r}, \end{aligned} \tag{3.16}$$

and if $|\xi| \geq L$ with $|t^\alpha P| \leq L$, by (3.8) and (3.12) we know that

$$|D^\mu E_\alpha(t^\alpha P)| \leq M\left(t^\alpha + t^{\alpha|\mu|}\right)|\xi|^{(m-1)|\mu|}. \tag{3.17}$$

Altogether, we have

$$|D^\mu E_\alpha(t^\alpha P)| \leq M\left(1 + t^{\alpha|\mu|}\right)|\xi|^{(m-1)|\mu|}, \quad |\xi| \geq L. \tag{3.18}$$

And by

$$\left|D^\mu (a-P)^{-\gamma}\right| \leq M|\xi|^{(m-r-1)|\mu|-r\gamma}, \quad |\xi| \geq L \tag{3.19}$$

and Leibniz's formula we have

$$\left|D^{\mu}\left(E_{\alpha}(t^{\alpha}P)(a-P)^{-\gamma}\right)\right| \leq M\left(1+t^{\alpha|\mu|}\right)|\xi|^{(m-1)|\mu|-r\gamma}, \quad |\xi| \geq L. \tag{3.20}$$

□

Lemma 3.5. *This proves (3.13). Suppose that the assumptions of Lemma 3.4 are satisfied. Let $\gamma > nm/2r$. Then $E_{\alpha}(t^{\alpha}P)(a-P)^{-\gamma} \in \mathcal{F}L^1(\mathbb{R}^n)$ and*

$$\left\|E_{\alpha}(t^{\alpha}P)(a-P)^{-\gamma}\right\|_{\mathcal{F}L^1} \leq M\left(1+t^{\alpha n/2}\right), \quad t \geq 0. \tag{3.21}$$

The same result holds with $E_{\alpha}(t^{\alpha}P)$ replaced by $E_{\alpha,\alpha}(t^{\alpha}P)$.

Proof. By Lemma 3.1(d), it remains to prove that for $|\xi| \leq L$,

$$\left|D^{\mu}\left(E_{\alpha}(t^{\alpha}P)(a-P)^{-\gamma}\right)\right| \leq M\left(1+t^{\alpha|\mu|}\right), \quad |\mu| \leq k_0,\ t \geq 0. \tag{3.22}$$

To show this we can use (3.15) and then give the estimates according to the values $t^{\alpha}P$. For $|\xi| \leq L$ with $|t^{\alpha}P| \geq L$ the estimate (3.8) can be applied, and for $|\xi| \leq L$ with $|t^{\alpha}P| \leq L$ note that all the functions $E_{\alpha}^{(j)}(t^{\alpha}P)$ are uniformly bounded.

For the second part of the lemma, note that $E_{\alpha,\alpha}(z) = \alpha E_{\alpha}'(z)$. □

4. Existence of α-Times Regularized Resolvents for Operator Polynomials

In this section, we will construct the fractional regularized resolvent families for coercive differential operators on Banach spaces.

Theorem 4.1. *Suppose that P is an r-coercive polynomial of order m, and $\{P(\xi) : \xi \in \mathbb{R}^n\} \subset \mathbb{C} \setminus \Sigma_{\alpha'\pi/2}$, where $1 < \alpha' < 2$. Then for $1 < \alpha < \alpha'$, $a \in \Sigma_{\alpha'\pi/2}$, $\gamma > nm/2r$, $C = (a-P)^{-\gamma}(A)$, there exists an analytic α-times C-regularized resolvent family $S_{\alpha}(t)$ for $\overline{P(A)}$, and $S_{\alpha}(t) = (E_{\alpha}(t^{\alpha}P)(a-P)^{-\gamma})(A)$ with*

$$\|S_{\alpha}(t)\| \leq M\left(1+t^{\alpha n/2}\right), \quad t \geq 0. \tag{4.1}$$

Proof. Let $u_t = E_{\alpha}(t^{\alpha}P)(a-P)^{-\gamma}$, $t \geq 0$. By Lemma 3.5, $u_t \in \mathcal{F}L^1$ and $\|u_t\|_{\mathcal{F}L^1} \leq M(1+t^{\alpha n/2})$. Define $S_{\alpha}(t) = u_t(A)$. Then by Lemma 3.1(b), $S_{\alpha}(t) \in \mathbf{B}(X)$, $\|S_{\alpha}(t)\| \leq M(1+t^{\alpha n/2})$, and in particular $C = S_\alpha(0) \in \mathbf{B}(X)$. To check the strong continuity of $S_\alpha(t)$, take $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then for $t, t+h \geq 0$, by Lemma 3.5

$$\begin{aligned}
&\left\| S_\alpha(t+h)\phi(A) - S_\alpha(t)\phi(A) \right\| \\
&\quad \leq M \left\| \left(E_\alpha((t+h)^\alpha P) - E_\alpha(t^\alpha P)\right)(a-P)^{-\gamma}\phi \right\|_{\mathcal{F}L^1} \\
&\quad \leq M \left\| \int_t^{t+h} s^{\alpha-1} E_{\alpha,\alpha}(s^\alpha P)(a-P)^{-\gamma} P\phi\, ds \right\|_{\mathcal{F}L^1} \\
&\quad \leq M \int_t^{t+h} s^{\alpha-1}\left(1 + s^{\alpha n/2}\right) ds \cdot \|P\phi\|_{\mathcal{F}L^1} \longrightarrow 0, \quad \text{as } h \longrightarrow 0.
\end{aligned} \tag{4.2}$$

Since the set E of Lemma 3.1 is dense in X, we have done. Next we will show that

$$\lambda^{\alpha-1}\left(\lambda^\alpha - \overline{P(A)}\right)^{-1} C = \int_0^\infty e^{-\lambda t} S_\alpha(t)\, dt, \quad \lambda > 0. \tag{4.3}$$

In fact, for $\phi \in \mathcal{S}(\mathbb{R}^n)$, by Lemma 3.1(b) and (c) we have

$$P(A)S_\alpha(t)\phi(A) = \left(Pu_t\phi\right)(A) = S_\alpha(t)P(A)\phi(A). \tag{4.4}$$

Since $\mathcal{F}L^1(\mathbb{R}^n)$ is a Banach algebra, it follows that $u_t, u_t(\lambda - P)\phi \in \mathcal{F}L^1$. Thus by Lemmas 3.1, 3.5, (3.4), and Fubini's theorem one obtains that for $x \in X$, $\lambda > 0$,

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} S_\alpha(t)(\lambda^\alpha - P(A))\phi(A)x\, dt &= \int_0^\infty e^{-\lambda t}\left(u_t(\lambda^\alpha - P)\phi\right)(A)x\, dt \\
&= \left(\int_0^\infty e^{-\lambda t} u_t\, dt(\lambda^\alpha - P)\phi\right)(A)x \\
&= \lambda^{\alpha-1} C\phi(A)x.
\end{aligned} \tag{4.5}$$

This implies that

$$\int_0^\infty e^{-\lambda t} S_\alpha(t)\left(\lambda^\alpha - \overline{P(A)}\right)x\, dt = \lambda^{\alpha-1}Cx, \quad x \in D\left(\overline{P(A)}\right), \tag{4.6}$$

once again by the density of the set E of Lemma 3.1. A similar argument works to get

$$\left(\lambda^\alpha - \overline{P(A)}\right)\int_0^\infty e^{-\lambda t} S_\alpha(t)x\, dt = \lambda^{\alpha-1}Cx, \quad x \in X. \tag{4.7}$$

Therefore, we have proved (4.3). And it is routine to show that $C^{-1}\overline{P(A)}C = \overline{P(A)}$, thus by Theorem 2.3 we know that $S_\alpha(t)$ is the α-times C-regularized resolvent family for $\overline{P(A)}$.

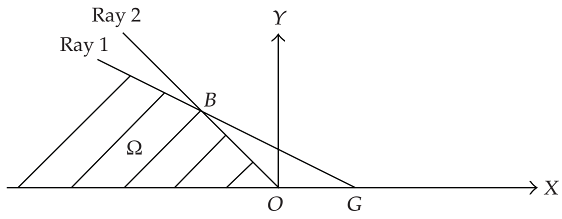

Figure 1

Moreover, since $\alpha < \alpha'$ is arbitrary, by the subordination principle (Theorem 2.4) we know that $S_\alpha(t)$ is analytic. □

We can extend this result to a more general case.

Theorem 4.2. *Let $P(\xi)$ be an r-coercive polynomial of order m such that $\{P(\xi) : \xi \in \mathbb{R}^n\} \subset \mathbb{C} \setminus (\omega + \Sigma_{\alpha'\pi/2})$ for some $\omega \geq 0$, and let $\alpha' > 1$. Then for $1 < \alpha < \alpha'$, $a \in \omega + \Sigma_{\alpha'\pi/2}$, $\gamma > nm/2r$, and $C = (a - P)^{-\gamma}(A)$, there exists an analytic α-times C-regularized resolvent family $S_\alpha(t)$ for $\overline{P(A)}$ with*

$$\|S_\alpha(t)\| \leq M\left(1 + t^{\alpha n/2}\right) \exp\left(\omega^{1/\alpha} t\right), \quad t \geq 0. \tag{4.8}$$

Proof. We only consider the area above the x-axis, the lower area can be treated similarly.

Let Ray $1 := \{\omega + \rho e^{i\alpha'\pi/2} : 0 \leq \rho < \infty\}$, and let Ray $2 := \{\rho e^{i\alpha\pi/2} : 0 \leq \rho < \infty\}$, where $1 < \alpha < \alpha' < 2$. Let G be the point $(\omega, 0)$, and set B to denote the intersection point of the two above rays. Let Ω denote the region to the left side of Ray 1 and 2 (see Figure 1).

If $P(\xi)$ falls into Ω, the asymptotic formula (3.6) can be applied to get estimates similarly as in the proof of Theorem 4.1. It remains to consider the values $P(\xi)$ within the triangle ΔGOB. To estimate $D^\mu E_\alpha(t^\alpha P(\xi))$ for such values $P(\xi)$, we use (3.12), (3.15), and (3.5) to obtain

$$|D^\mu E_\alpha(t^\alpha P(\xi))| \leq M\left(1 + t^{\alpha|\mu|}\right) \exp\left(\omega^{1/\alpha} t\right) \tag{4.9}$$

if $\operatorname{Re}((\rho e^{i\theta})^{1/\alpha}) < \omega^{1/\alpha}$, where $\rho e^{i\theta}$ denotes an arbitrary point on the line segment from G to B.

Since

$$\frac{\rho}{\sin(\alpha'\pi/2)} = \frac{\omega}{\sin(\alpha'\pi/2 - \theta)}, \tag{4.10}$$

we have

$$\operatorname{Re}\left((\rho e^{i\theta})^{1/\alpha}\right) = \omega^{1/\alpha} \left(\frac{\sin(\alpha'\pi/2)}{\sin(\alpha'\pi/2 - \theta)}\right)^{1/\alpha} \cos(\theta/\alpha). \tag{4.11}$$

Thus, to show that $\mathrm{Re}((\rho e^{i\theta})^{1/\alpha}) < \omega^{1/\alpha}$ $(0 < \theta \le \alpha\pi/2)$ one needs to check that

$$\cos^{\alpha}(\theta/\alpha) < \cos\theta + \sin\theta \cdot \tan\left(\frac{\alpha' - 1}{2}\pi\right), \quad 0 < \theta \le \frac{\alpha\pi}{2}; \tag{4.12}$$

and this is true if

$$\cos(\theta/\alpha) \le \cos\theta + \sin\theta \cdot \tan\left(\frac{\alpha - 1}{2}\pi\right), \quad 0 < \theta \le \frac{\alpha\pi}{2}, \tag{4.13}$$

since $1 < \alpha < \alpha' < 2$.

We first consider the case when $\pi/2 \le \theta \le \alpha\pi/2$. Let $g(\theta) = \cos\theta + \sin\theta \cdot \tan(((\alpha - 1)/2)\pi) - \cos(\theta/\alpha)$, then $g'(\theta) = -\sin\theta + (1/\alpha)\sin(\theta/\alpha) + \cos\theta \cdot \tan(((\alpha-1)/2)\pi) \le 0$ since $\sin\theta > (1/\alpha)\sin(\theta/\alpha)$ and $\cos\theta \le 0$ for $\pi/2 \le \theta \le \alpha\pi/2$. So $g(\theta)$ decreases with respect to θ, which means that $g(\theta) \ge 0$ since $g(\alpha\pi/2) = 0$.

For $0 < \theta < \pi/2$, we will show that

$$\cos(\theta/\alpha) \le \cos\theta + \sin\theta \cdot \frac{\alpha - 1}{2}\pi, \tag{4.14}$$

which implies (4.13). Now for fixed $\theta \in (0, \pi/2)$, denote by $h(\alpha) = \cos\theta + \sin\theta \cdot ((\alpha-1)/2)\pi - \cos(\theta/\alpha)$. Since $\alpha > 1$, we have $h'(\alpha) = (\pi/2)\sin\theta - (\theta/\alpha^2)\sin(\theta/\alpha) > 0$; it thus follows that $h(\alpha) \ge h(1) = 0$. Therefore we have proved (4.14).

Now by (3.19) and (4.9) one obtains, for $|\xi| \ge L$,

$$\left|D^{\mu}(E_{\alpha}(t^{\alpha}P)(a - P)^{-\gamma})\right| \le M\left(1 + t^{\alpha|\mu|}\right)|\xi|^{(m-1)|\mu| - r\gamma}\exp\left(\omega^{1/\alpha}t\right), \tag{4.15}$$

and for $|\xi| \le L$,

$$\left|D^{\mu}(E_{\alpha}(t^{\alpha}P)(a - P)^{-\gamma})\right| \le M\left(1 + t^{\alpha|\mu|}\right)\exp\left(\omega^{1/\alpha}\ t\right). \tag{4.16}$$

An argument similar to that one of the proof of Theorem 4.1 gives our claim. □

In the following theorem, we do not assume that P is coercive, but the choice of C is different.

Theorem 4.3. *Suppose that $P(\xi)$ is a polynomial of order m, and $\{P(\xi) : \xi \in \mathbb{R}^n\} \subset \mathbb{C} \setminus (\omega + \Sigma_{\alpha'\pi/2})$, where $1 < \alpha' < 2$. Then for $1 < \alpha < \alpha' < 2$, $\beta > n/2$, $C = (1 + |A|^2)^{-m\beta/2}$ (which is defined by (3.1) with $u(x) = (1 + |x|^2)^{-m\beta/2}$), there exists an analytic α-times C-regularized resolvent family $S_{\alpha}(t)$ for $\overline{P(A)}$ such that*

$$\|S_{\alpha}(t)\| \le M\left(1 + t^{\alpha n/2}\right)\exp\left(\omega^{1/\alpha}t\right), \quad t \ge 0. \tag{4.17}$$

Proof. From (4.9) and

$$\left|D^{\mu}\left(1+|\xi|^2\right)^{-\beta/2}\right| \leq M|\xi|^{-|\mu|-\beta}, \quad |\xi| \geq L,\ \mu \in \mathbb{N}_0^n, \tag{4.18}$$

we have for $|\xi| \geq L$,

$$\left|D^{\mu}\left(E_{\alpha}(t^{\alpha}P)\left(1+\xi|^2\right)^{-\beta/2}\right)\right| \leq M\left(1+t^{\alpha|\mu|}\right)|\xi|^{(m-1)|\mu|-\beta}\exp\left(\omega^{1/\alpha}t\right), \tag{4.19}$$

and for $|\xi| \leq L$,

$$\left|D^{\mu}\left(E_{\alpha}(t^{\alpha}P)(1+|\xi|^2)^{-\beta/2}\right)\right| \leq M\left(1+t^{\alpha|\mu|}\right)\exp\left(\omega^{1/\alpha}t\right). \tag{4.20}$$

It thus follows from Lemma 3.1 that when $\beta > nm/2$, $E_{\alpha}(t^{\alpha}P)(1+|\xi|^2)^{-\beta/2} \in \mathcal{F}L^1(\mathbb{R}^n)$. Similarly as in the proof of Theorem 4.1 we can show that there is an analytic α-times C-regularized resolvent family for $\overline{P(A)}$. □

From now on X will be $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ or $C_0(\mathbb{R}^n) := \{f \in C(\mathbb{R}^n) : \lim_{|x|\to\infty} f(x) = 0\}$. The partial differential operator $P(D)$ defined by

$$P(D)f = \mathcal{F}^{-1}(P\mathcal{F}f) \tag{4.21}$$

with

$$D(P(D)) = \left\{f \in X : \mathcal{F}^{-1}(P\mathcal{F}f) \in X\right\} \tag{4.22}$$

is closed and densely defined on X. Since $iD_j = \partial/\partial x_j$ $(1 \leq j \leq n)$ is the generator of the bounded C_0-group $\{T_j(t)\}_{t\in\mathbb{R}}$ given by

$$T_j(t)f(x_1,\ldots,x_n) = f(x_1,\ldots,x_{j-1},x_j+t,x_{j+1},\ldots,x_n) \quad t \in \mathbb{R} \tag{4.23}$$

on X, we can apply the above results to $P(D)$ on X. It is remarkable that when $X = L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ these results can be improved. In fact, if $A = D = (D_1,\ldots,D_n)$, then the functions u_t's in the proofs of the above theorems give rise to Fourier multipliers on $L^p(\mathbb{R}^n)$ having norm of polynomial growth t^{n_p} at infinity, where $n_p = n|1/2 - 1/p|$. For details we refer to [3, 8]). We summarize these conclusions in the following two theorems.

Theorem 4.4. *Suppose that the assumptions of Theorem 4.2 are satisfied.*

(a) *For $X = L^1(\mathbb{R}^n)$ or $C_0(\mathbb{R}^n)$, $C = (a-P)^{-\gamma}(D)$, where $\gamma > nm/2r$, there exists an analytic α-times C-regularized resolvent family $S_{\alpha}(t)$ for $P(D)$ and*

$$\|S_{\alpha}(t)\| \leq M\left(1+t^{\alpha n/2}\right)\exp\left(\omega^{1/\alpha}t\right), \quad t \geq 0. \tag{4.24}$$

(b) *For $X = L^p(\mathbb{R}^n)$, $C = (a-P)^{-\gamma}(D)$, where $\gamma > n_p m/r$, $n_p = n|1/2-1/p|$, there exists an analytic α-times C-regularized resolvent family $S_\alpha(t)$ for $P(D)$ and*

$$\|S_\alpha(t)\| \leq M(1+t^{\alpha n_p})\exp\left(\omega^{1/\alpha}t\right), \quad t \geq 0. \tag{4.25}$$

Theorem 4.5. *Suppose that the assumptions of Theorem 4.3 are satisfied.*

(a) *For $X = L^1(\mathbb{R}^n)$ or $C_0(\mathbb{R}^n)$, $C = (1-\Delta)^{-m\beta/2}$, where $\beta > n/2$, there exists an analytic α-times C-regularized resolvent family $S_\alpha(t)$ for $P(D)$ and*

$$\|S_\alpha(t)\| \leq M\left(1+t^{\alpha n/2}\right)\exp\left(\omega^{1/\alpha}t\right), \quad t \geq 0. \tag{4.26}$$

(b) *For $X = L^p(\mathbb{R}^n)$, $C = (1-\Delta)^{-m\beta/2}$, where $\beta > n_p$, there exists an analytic α-times C-regularized resolvent family $S_\alpha(t)$ for $P(D)$ and*

$$\|S_\alpha(t)\| \leq M(1+t^{\alpha n_p})\exp\left(\omega^{1/\alpha}t\right), \quad t \geq 0. \tag{4.27}$$

We end this paper with some examples to demonstrate the applications of our results.

Example 4.6. (a) The polynomial corresponding to the Laplacian Δ on $L^p(\mathbb{R}^n)$ $(n > 1, p \neq 2)$ is $P(\xi) = -|\xi|^2$. By Theorem 4.4, for every $1 < \alpha < 2$ there exists an analytic α-times $(1-\Delta)^{-\gamma}$-regularized resolvent family for the operator Δ, where $\gamma > n_p$.

(b) Consider $P(D)$ on $L^p(\mathbb{R}^2)$ $(1 < p < \infty)$ with

$$P(\xi) = -\left(1+\xi_1^2\right)\left(1+(\xi_2-\xi_1^l)^2\right) \quad (l \in \mathbb{N}). \tag{4.28}$$

Then $P(\xi) \leq -1 (\xi \in \mathbb{R}^2)$. We claim that P is $(2/l)$-coercive. Indeed, if $|\xi_2| \geq 2|\xi_1^l|$, then

$$|P(\xi)| \geq \left(1+\xi_1^2\right)\left(1+\frac{1}{4}\xi_2^2\right) \geq \frac{1}{4}|\xi|^2. \tag{4.29}$$

If $|\xi_2| < 2|\xi_1^l|$, then

$$|P(\xi)| \geq 1+|\xi_1|^2 \geq c|\xi|^{2/l} \quad \text{for } |\xi| \geq 1, \tag{4.30}$$

for some proper constant c, as desired. By Theorems 4.4 and 4.5, for every $1 < \alpha < 2$ there exists an analytic α-times C-regularized resolvent family for $P(D)$, where $C = (1-P)^{-\gamma}(D)$ with $\gamma > 2(l^2+l)|1/2-1/p|$ or $C = (1-\Delta)^{-(l+1)\beta}$ with $\beta > 2|1/2-1/p|$. We remark that if $l \geq 5$ and $|1/2-1/p| \geq 1/4+1/l$, then $0 \in \sigma(P(D))$ (see [17]). Since $0 \notin P(\mathbb{R}^2)$, it follows from [18, Theorem 1] that $\rho(P(D)) = \emptyset$. Consequently, in this case there is no α-times resolvent family for $P(D)$ for any α.

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