Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2009, Article ID 417473, 14 pages
doi:10.1155/2009/417473

Research Article

Solution and Stability of a Mixed Type Cubic and Quartic Functional Equation in Quasi-Banach Spaces

M. Eshaghi Gordji,[1] S. Zolfaghari,[1] J. M. Rassias,[2] and M. B. Savadkouhi[1]

[1] *Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran*
[2] *Section of Mathematics and Informatics, Pedagogical Department, National and Capodistrian University of Athens, 4, Agamemnonos Street, Aghia Paraskevi, 15342 Athens, Greece*

Correspondence should be addressed to M. Eshaghi Gordji, madjid.eshaghi@gmail.com

Received 24 January 2009; Accepted 6 August 2009

Recommended by Elena Litsyn

We obtain the general solution and the generalized Ulam-Hyers stability of the mixed type cubic and quartic functional equation $f(x+2y)+f(x-2y) = 4(f(x+y)+f(x-y))-24f(y)-6f(x)+3f(2y)$ in quasi-Banach spaces.

1. Introduction

We recall some basic facts concerning quasiBanach space. A quasinorm is a real-valued function on X satisfying the following.

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasinormed space if $\|\cdot\|$ is a quasinorm on X. A quasiBanach space is a complete quasinormed space. A quasinorm $\|\cdot\|$ is called a p-norm $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p \tag{1.1}$$

for all $x, y \in X$. In this case, a quasiBanach space is called a p-Banach space. Given a p-norm, the formula $d(x,y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem [1] (see also [2]), each quasinorm is equivalent to some p-norm. Since it is much easier to work with p-norms, henceforth we restrict our attention mainly to p-norms. The stability problem of functional equations originated from a question of Ulam [3] in 1940, concerning the stability of group homomorphisms. Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, Under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [4] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \le \delta \tag{1.2}$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \le \delta \tag{1.3}$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. Rassias [5] succeeded in extending the result of Hyers' Theorem by weakening the condition for the Cauchy difference controlled by $(\|x\|^p + \|y\|^p)$, $p \in [0,1)$ to be unbounded. This condition has been assumed further till now, through the complete Hyers direct method, in order to prove linearity for generalized Hyers-Ulam stability problem forms. A number of mathematicians were attracted to the pertinent stability results of Rassias [6], and stimulated to investigate the stability problems of functional equations. The stability phenomenon that was introduced and proved by Rassias is called Hyers-Ulam-Rassias stability. And then the stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 5, 7–18]).

The following cubic functional equation, which is the oldest cubic functional equation, was introduced by the third author of this paper, Rassias [6] (in 2001):

$$f(x+2y) + 3f(x) = 3f(x+y) + f(x-y) + 6f(y). \tag{1.4}$$

Jun and Kim [19] introduced the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.5}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.5). The function $f(x) = x^3$ satisfies the functional equation (1.5), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function f between real vector spaces X and Y is a solution of (1.5) if and only if there exists a unique function $C : X \times X \times X \to Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables (see also [20]).

The quartic functional equation (1.6) was introduced by Rassias [21] (in 2000) and then (in 2005) was employed by Park and Bae [22] and others, such that:

$$f(x+2y) + f(x-2y) = 4[f(x+y) + f(x-y)] + 24f(y) - 6f(x). \tag{1.6}$$

In fact they proved that a function f between real vector spaces X and Y is a solution of (1.6) if and only if there exists a unique symmetric multiadditive function $Q : X \times X \times X \times X \to Y$ such that $f(x) = Q(x, x, x, x)$ for all x (see also [21–29]). It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.6), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function. In this paper we deal with the following functional equation:

$$f(x+2y) + f(x-2y) = 4(f(x+y) + f(x-y)) - 24f(y) - 6f(x) + 3f(2y) \tag{1.7}$$

in quasiBanach spaces. It is easy to see that the function $f(x) = ax^3 + bx^4$ is a solution of the functional equation (1.7). In the present paper we investigate the general solution of functional equation (1.7) when f is a mapping between vector spaces, and we establish the generalized Hyers-Ulam-Rassias stability of the functional equation (1.7) whenever f is a mapping between two quasiBanach spaces. We only mention here the papers [30, 31] concerning the stability of the mixed type functional equations.

2. General Solution

Throughout this section, X and Y will be real vector spaces. Before proceeding to the proof of Theorem 2.3 which is the main result in this section, we shall need the following two lemmas.

Lemma 2.1. *If an even function $f : X \to Y$ satisfies* (1.7), *then f is quartic.*

Proof. Putting $x = y = 0$ in (1.7), we get $f(0) = 0$. Setting $x = 0$ in (1.7), by evenness of f we obtain

$$f(2y) = 16f(y) \tag{2.1}$$

for all $y \in X$. Hence (1.7) can be written as

$$f(x+2y) + f(x-2y) = 4(f(x+y) + f(x-y)) + 24f(y) - 6f(x). \tag{2.2}$$

This means that f is quartic function,which completes the proof of the lemma. □

Lemma 2.2. *If an odd function $f : X \to Y$ satisfies* (1.7), *then f is a cubic function.*

Proof. Setting $x = y = 0$ in (1.7) gives $f(0) = 0$. Putting $x = 0$ in (1.7), then by oddness of f, we have

$$f(2y) = 8f(y). \tag{2.3}$$

Hence (1.7) can be written as

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x). \tag{2.4}$$

Replacing x by $x+y$ in (2.4), we obtain

$$f(x+3y) + f(x-y) = 4f(x+2y) - 6f(x+y) + 4f(x). \tag{2.5}$$

Substituting $-y$ for y in (2.5) gives

$$f(x-3y) + f(x+y) = 4f(x-2y) - 6f(x-y) + 4f(x). \tag{2.6}$$

If we subtract (2.5) from (2.6), we obtain

$$f(x+3y) - f(x-3y) = 4f(x+2y) - 4f(x-2y) - 5f(x+y) + 5f(x-y). \tag{2.7}$$

Let us interchange x and y in (2.7). Then we see that

$$f(3x+y) + f(3x-y) = 4f(2x+y) + 4f(2x-y) - 5f(x+y) - 5f(x-y). \tag{2.8}$$

With the substitution $y := x+y$ in (2.4), we have

$$f(3x+2y) - f(x+2y) = 4f(2x+y) - 4f(y) - 6f(x). \tag{2.9}$$

From the substitution $y := -y$ in (2.9) it follows that

$$f(3x-2y) - f(x-2y) = 4f(2x-y) + 4f(y) - 6f(x). \tag{2.10}$$

If we add (2.9) to (2.10), we have

$$f(3x+2y) + f(3x-2y) = 4f(2x+y) + 4f(2x-y) + f(x+2y) + f(x-2y) - 12f(x). \tag{2.11}$$

Replacing x by $2x$ in (2.7) and using (2.3), we obtain

$$f(2x+3y) - f(2x-3y) = 32f(x+y) - 32f(x-y) - 5f(2x+y) + 5f(2x-y). \tag{2.12}$$

Interchanging x with y in (2.12) gives the equation

$$f(3x+2y)+f(3x-2y)=32f(x+y)+32f(x-y)-5f(x+2y)-5f(x-2y). \tag{2.13}$$

If we compare (2.11) and (2.13) and employ (2.4), we conclude that

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{2.14}$$

This means that f is cubic function. This completes the proof of Lemma. □

Theorem 2.3. *A function $f: X \to Y$ satisfies (1.7) for all $x, y \in X$ if and only if there exists a unique function $C: X\times X\times X \to Y$ and a unique symmetric multiadditive function $Q: X\times X\times X\times X \to Y$ such that $f(x)=C(x,x,x)+Q(x,x,x,x)$ for all $x \in X$, and that C is symmetric for each fixed one variable and is additive for fixed two variables.*

Proof. Let f satisfy (1.7). We decompose f into the even part and odd part by setting

$$f_e(x)=\frac{1}{2}(f(x)+f(-x)), \qquad f_o(x)=\frac{1}{2}(f(x)-f(-x)) \tag{2.15}$$

for all $x \in X$. By (1.7), we have

$$\begin{aligned} f_e(x+2y)+f_e(x-2y)&=\frac{1}{2}[f(x+2y)+f(-x-2y)+f(x-2y)+f(-x+2y)] \\ &=4(f_e(x+y)+f_e(x-y))-24f_e(y)-6f_e(x)+3f_e(2y) \end{aligned} \tag{2.16}$$

for all $x, y \in X$. This means that f_e satisfies in (1.7). Similarly we can show that f_o satisfies (1.7). By Lemmas 2.1 and 2.2, f_e and f_o are quartic and cubic, respectively. Thus there exists a unique function $C: X\times X\times X \to Y$ and a unique symmetric multiadditive function $Q: X\times X\times X\times X \to Y$ such that $f_e(x)=Q(x,x,x,x)$ and that $f_o(x)=C(x,x,x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables. Thus $f(x)=C(x,x,x)+Q(x,x,x,x)$ for all $x \in X$. The proof of the converse is trivial. □

3. Stability

Throughout this section, X and Y will be a uniquely two-divisible abelian group and a quasiBanach spaces respectively, and p will be a fixed real number in $[0,1]$. We need the following lemma in the main theorems. Now before taking up the main subject, given $f: X \to Y$, we define the difference operator $D_f: X\times X \to Y$ by

$$D_f(x,y)=f(x+2y)+f(x-2y)-4[f(x+y)+f(x-y)]-3f(2y)+24f(y)+6f(x) \tag{3.1}$$

for all $x, y \in X$. We consider the following functional inequality:

$$\|D_f(x,y)\| \le \phi(x,y) \tag{3.2}$$

for an upper bound $\phi : X \times X \to [0,\infty)$.

Lemma 3.1. *Let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers. Then*

$$\left(\sum_{i=1}^{n} x_i\right)^p \le \sum_{i=1}^{n} x_i{}^p. \tag{3.3}$$

Theorem 3.2. *Let $l \in \{1,-1\}$ be fixed and let $\varphi : X \times X \to \mathbb{R}^+$ be a function such that*

$$\lim_{n\to\infty} 16^{\ln} \varphi\left(\frac{x}{2^{\ln}}, \frac{y}{2^{\ln}}\right) = 0 \tag{3.4}$$

for all $x, y \in X$ and

$$\sum_{i=1}^{\infty} 16^{ilp} \varphi^p\left(0, \frac{y}{2^{li}}\right) < \infty \tag{3.5}$$

for all $y \in X$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x,y)\|_Y \le \varphi(x,y), \tag{3.6}$$

for all $x, y \in X$. Then the limit

$$Q(x) := \lim_{n\to\infty} 16^{\ln} f\left(\frac{x}{2^{\ln}}\right) \tag{3.7}$$

exists for all $x \in X$ and $Q : X \to Y$ is a unique quartic function satisfying

$$\|f(x) - Q(x)\|_Y \le \frac{K}{16} \left[\widetilde{\varphi}_e(x)\right]^{1/p}, \tag{3.8}$$

where

$$\widetilde{\varphi}_e(x) := \sum_{i=|l+1|/2}^{\infty} 16^{ilp} \varphi^p\left(0, \frac{x}{2^{li}}\right) \tag{3.9}$$

for all $x \in X$.

Proof. Let $l = 1$. By putting $x = 0$ in (3.6), we get

$$\|f(2y) - 16f(y)\|_Y \leq \varphi(0, y) \tag{3.10}$$

for all $y \in X$. Replacing y by x in (3.10) yields

$$\|f(2x) - 16f(x)\|_Y \leq \varphi(0, x) \tag{3.11}$$

for all $x \in X$. Let $\psi(x) = \varphi(0, x)$ for all $x \in X$, then by (3.11), we get

$$\|f(2x) - 16f(x)\|_Y \leq \psi(x) \tag{3.12}$$

for all $x \in X$. Interchanging x with $x/2^{n+1}$ in (3.12), and multiplying by 16^n it follows that

$$\left\|16^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 16^n f\left(\frac{x}{2^n}\right)\right\|_Y \leq K 16^n \psi\left(\frac{x}{2^{n+1}}\right) \tag{3.13}$$

for all $x \in X$ and all nonnegative integers n. Since Y is p-Banach space, then by (3.13) we have

$$\begin{aligned}\left\|16^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 16^m f\left(\frac{x}{2^m}\right)\right\|_Y^p &\leq \sum_{i=m}^{n}\left\|16^{i+1} f\left(\frac{x}{2^{i+1}}\right) - 16^i f\left(\frac{x}{2^i}\right)\right\|_Y^p \\ &\leq K^p \sum_{i=m}^{n} 16^{ip} \psi^p\left(\frac{x}{2^{i+1}}\right)\end{aligned} \tag{3.14}$$

for all nonnegative integers n and m with $n \geq m$ and all $x \in X$. Since $\psi^p(x) = \varphi^p(0, x)$ for all $x \in X$. Therefore by (3.5) we have

$$\sum_{i=1}^{\infty} 16^{ip} \psi^p\left(\frac{x}{2^i}\right) < \infty \tag{3.15}$$

for all $x \in X$. Therefore we conclude from (3.14) and (3.15) that the sequence $\{16^n f(x/2^n)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it follows that the sequence $\{16^n f(x/2^n)\}$ converges for all $x \in X$. We define the mapping $Q : X \to Y$ by (3.7) for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (3.14), we get

$$\|f(x) - Q(x)\|_Y^p \leq K^p \sum_{i=0}^{\infty} 16^{ip} \psi^p\left(\frac{x}{2^{i+1}}\right) = \frac{K^p}{16^p} \sum_{i=1}^{\infty} 16^{ip} \psi^p\left(\frac{x}{2^i}\right) \tag{3.16}$$

for all $x \in X$. Therefore (3.8) follows from (3.9) and (3.16). Now we show that Q is quartic. It follows from (3.4), (3.6) and (3.7)

$$\|D_Q(x, y)\|_Y = \lim_{n\to\infty} 16^n \left\|D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\|_Y \leq \lim_{n\to\infty} 16^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{3.17}$$

for all $x, y \in X$. Therefore the mapping $Q : X \to Y$ satisfies (1.7). Since $Q(0) = 0$, then by Lemma 2.1 we get that the mapping $Q : X \to Y$ is quartic. To prove the uniqueness of Q, let $T : X \to Y$ be another quartic mapping satisfies (3.8). Since

$$\lim_{n\to\infty} 16^{np} \sum_{i=1}^{\infty} 16^{ip} \varphi^p\left(\frac{x}{2^{i+n}}, \frac{y}{2^{i+n}}\right) = \lim_{n\to\infty} \sum_{i=n+1}^{\infty} 16^{ip} \varphi^p\left(\frac{x}{2^i}, \frac{y}{2^i}\right) = 0 \tag{3.18}$$

for all $y \in X$ and all $x \in \{0\}$, then

$$\lim_{n\to\infty} 16^{np} \widetilde{\varphi}_e\left(\frac{x}{2^n}\right) = 0 \tag{3.19}$$

for all $x \in X$. It follows from (3.8), (3.19)

$$\|Q(x) - T(x)\|_Y^p = \lim_{n\to\infty} 16^{np} \left\| f\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\|_Y^p \le \frac{K^p}{16^p} \lim_{n\to\infty} 16^{np} \widetilde{\varphi}_e\left(\frac{x}{2^n}\right) = 0 \tag{3.20}$$

for all $x \in X$. Hence $Q = T$. For $l = -1$, we obtain

$$\left\| \frac{f(2^n x)}{16^n} - f(x) \right\|_Y^p \le \frac{K^p}{16^p} \sum_{i=0}^{\infty} \frac{\varphi^p(0, 2^i x)}{16^{ip}}, \tag{3.21}$$

from which one can prove the result by a similar technique. □

Corollary 3.3. *Let θ, r, s, u, v be nonnegative real numbers such that $s \ne 4 \ne u + v$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\left\| D_f(x, y) \right\|_Y \le \theta\left(\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s \right) \tag{3.22}$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\|_Y \le K\theta \left\{ \frac{1}{|16^p - 2^{sp}|} \right\}^{1/p} \|x\|_X^s \tag{3.23}$$

for all $x \in X$.

Proof. It follows from Theorem 3.2that $\varphi(x, y) := \theta(\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s)$ for all $x, y \in X$. □

Theorem 3.4. *Let $l \in \{1, -1\}$ be fixed and let $\varphi : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 8^{\ln} \varphi\left(\frac{x}{2^{\ln}}, \frac{y}{2^{\ln}}\right) = 0 \tag{3.24}$$

for all $x, y \in X$ *and*

$$\sum_{i=1}^{\infty} 8^{ilp} \varphi^{p}\left(0, \frac{y}{2^{il}}\right) < \infty \tag{3.25}$$

for all $y \in X$. *Suppose that an odd function* $f : X \to Y$ *satisfies the inequality*

$$\left\| D_f(x, y) \right\|_Y \leq \varphi(x, y), \tag{3.26}$$

for all $x, y \in X$. *Then the limit*

$$C(x) := \lim_{n \to \infty} 8^{\ln} f\left(\frac{x}{2^{\ln}}\right) \tag{3.27}$$

exists for all $x \in X$ *and* $C : X \to Y$ *is a unique cubic function satisfying*

$$\left\| f(x) - C(x) \right\|_Y \leq \frac{K}{24} \left[\widetilde{\phi}_o(x)\right]^{1/p} \tag{3.28}$$

for all $x \in X$, *where*

$$\widetilde{\phi}_o(x) := \sum_{i=|l+1|/2}^{\infty} 8^{ilp} \varphi^{p}\left(0, \frac{x}{2^{il}}\right). \tag{3.29}$$

Proof. Let $l = 1$. Setting $x = 0$ in (3.26), we get

$$\left\| 3f(2y) - 24f(y) \right\|_Y \leq \varphi(0, y) \tag{3.30}$$

for all $y \in X$. If we replace y in (3.30) by x and divide both sides of (3.30) by 3, we get

$$\left\| f(2x) - 8f(x) \right\|_Y \leq \frac{1}{3} \varphi(0, x) \tag{3.31}$$

for all $x \in X$. Let $\phi(x) = (1/3)\varphi(0, x)$ for all $x \in X$, then by (3.31), we get

$$\left\| f(2x) - 8f(x) \right\|_Y \leq \phi(x) \tag{3.32}$$

for all $x \in X$. Multiply (3.32) by 8^n and replace x by $x/2^{n+1}$, we obtain that

$$\left\| 8^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 8^n f\left(\frac{x}{2^n}\right) \right\|_Y \leq K 8^n \phi\left(\frac{x}{2^{n+1}}\right) \tag{3.33}$$

for all $x \in X$ and all nonnegative integers n. Since Y is a p-Banach space, (3.33) follows that

$$\begin{aligned}\left\|8^{n+1} f\left(\frac{x}{2^{n+1}}\right)-8^{m} f\left(\frac{x}{2^{m}}\right)\right\|_{Y}^{p} &\leq \sum_{i=m}^{n}\left\|8^{i+1} f\left(\frac{x}{2^{i+1}}\right)-8^{i} f\left(\frac{x}{2^{i}}\right)\right\|_{Y}^{p} \\ &\leq K^{p} \sum_{i=m}^{n} 8^{i p} \phi^{p}\left(\frac{x}{2^{i+1}}\right)\end{aligned} \tag{3.34}$$

for all nonnegative integers n and m with $n \geq m$ and all $x \in X$. Since $\phi^{p}(x)=(1/3^{p})\varphi^{p}(0, x)$ for all $x \in X$. Therefore it follows from (3.25) that

$$\sum_{i=1}^{\infty} 8^{i p} \phi^{p}\left(\frac{x}{2^{i}}\right)<\infty \tag{3.35}$$

for all $x \in X$, therefore we conclude from (3.34) and (3.35) that the sequence $\{8^{n} f(x/2^{n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{8^{n} f(x/2^{n})\}$ converges for all $x \in X$. So one can define the mapping $C: X \rightarrow Y$ by (3.27) for all $x \in X$. Letting $m=0$ and passing the limit $n \rightarrow \infty$ in (3.34), we get

$$\|f(x)-C(x)\|_{Y}^{p} \leq K^{p} \sum_{i=0}^{\infty} 8^{i p} \phi^{p}\left(\frac{x}{2^{i+1}}\right)=\frac{K^{p}}{8^{p}} \sum_{i=1}^{\infty} 8^{i p} \phi^{p}\left(\frac{x}{2^{i}}\right) \tag{3.36}$$

for all $x \in X$. Therefore (3.28) follows from (3.29) and (3.36). Now we show that C is cubic. It follows from (3.24), (3.26) and (3.27)

$$\left\|D_{C}(x, y)\right\|_{Y}=\lim_{n \rightarrow \infty} 8^{n}\left\|D_{f}\left(\frac{x}{2^{n}}, \frac{y}{2^{n}}\right)\right\|_{Y} \leq \lim_{n \rightarrow \infty} 8^{n} \varphi\left(\frac{x}{2^{n}}, \frac{y}{2^{n}}\right)=0 \tag{3.37}$$

for all $x, y \in X$. Therefore the mapping $C: X \rightarrow Y$ satisfies (1.7). Since f is an odd function, then (3.27) implies that the mapping odd. Therefore by Lemma 2.2 we get that the mapping $C: X \rightarrow Y$ is cubic. The rest of proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let θ be a nonnegative real number and r, s be real numbers such that $s \neq 3 \neq u+v$. Suppose that an odd function $f: X \rightarrow Y$ satisfies the inequality*

$$\left\|D_{f}(x, y)\right\|_{Y} \leq \theta\left(\|x\|_{X}^{u}\|y\|_{X}^{v}+\|x\|_{X}^{r}+\|y\|_{X}^{s}\right) \tag{3.38}$$

for all $x, y \in X$. Then there exists a unique cubic function $C: X \rightarrow Y$ satisfying

$$\|f(x)-C(x)\|_{Y} \leq \frac{K \theta}{3}\left\{\frac{1}{\left|8^{p}-2^{s p}\right|}\right\}^{1 / p}\|x\|_{X}^{s} \tag{3.39}$$

for all $x \in X$.

Proof. It follows from (3.38) and Theorem 3.4that $\varphi(x, y):=\theta(\|x\|_{X}^{u}\|y\|_{X}^{v}+\|x\|_{X}^{r}+\|y\|_{X}^{s})$ for all $x, y \in X$. □

Theorem 3.6. *Let $l \in \{1,-1\}$ be fixed and let $\varphi : X \times X \to [0,\infty)$ be a function which satisfies*

$$\lim_{n\to\infty}\left\{\left(\frac{|l|+l}{2}\right)16^{\ln}\varphi\left(\frac{x}{2^{\ln}},\frac{y}{2^{\ln}}\right)+\left(\frac{|l|-l}{2}\right)8^{\ln}\varphi\left(\frac{x}{2^{\ln}},\frac{y}{2^{\ln}}\right)\right\}=0 \tag{3.40}$$

for all $x, y \in X$ and

$$\sum_{i=|l+1|/2}^{\infty}\left\{\left(\frac{|l|+l}{2}\right)16^{ilp}\varphi^p\left(0,\frac{y}{2^{li}}\right)+\left(\frac{|l|-l}{2}\right)8^{ilp}\varphi^p\left(0,\frac{y}{2^{li}}\right)\right\}<\infty \tag{3.41}$$

for all $y \in X$. Suppose that a function $f : X \to Y$ with $f(0)=0$ satisfies the inequality

$$\|D_f(x,y)\|_Y \le \varphi(x,y) \tag{3.42}$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$ and a unique cubic function $C : X \to Y$ satisfying (1.7) *and*

$$\|f(x)-Q(x)-C(x)\|_Y \le \frac{K^3}{32}\left[\widetilde{\varphi}_e(x)+\widetilde{\varphi}_e(-x)\right]^{1/p}+\frac{K^3}{48}\left[\widetilde{\phi}_o(x)+\widetilde{\phi}_o(-x)\right]^{1/p} \tag{3.43}$$

for all $x \in X$, where $\widetilde{\varphi}_e(x)$ and $\widetilde{\phi}_o(x)$ that have been defined in (3.9) *and* (3.29), *respectively.*

Proof. Let $f_e(x) = (1/2)(f(x)+f(-x))$ for all $x \in X$. Then $f_e(0)=0$, $f_e(-x)=f_e(x)$, and $\|D_{f_e}(x,y)\| \le (K/2)[\varphi(x,y)+\varphi(-x,-y)]$ for all $x, y \in X$. Let

$$\Phi(x,y)=\frac{K}{2}\left[\varphi(x,y)+\varphi(-x,-y)\right] \tag{3.44}$$

for all $x, y \in X$. So

$$\lim_{n\to\infty}16^{\ln}\Phi\left(\frac{x}{2^{\ln}},\frac{y}{2^{\ln}}\right)=0 \tag{3.45}$$

for all $x, y \in X$. Since

$$\Phi^p(x,y)\le\frac{K^p}{2^p}\left[\varphi^p(x,y)+\varphi^p(-x,-y)\right] \tag{3.46}$$

for all $x, y \in X$, then

$$\sum_{i=1}^{\infty}16^{ilp}\Phi^p\left(\frac{x}{2^{il}},\frac{y}{2^{il}}\right)<\infty \tag{3.47}$$

for all $y \in X$ and all $x \in \{0\}$. Hence, in view of Theorem 3.2, there exists a unique quartic function $Q : X \to Y$ satisfying

$$\left\| f_e(x) - Q(x) \right\|_Y \leq \frac{K}{16}\left[\widetilde{\Psi}_e(x)\right]^{1/p} \tag{3.48}$$

for all $x \in X$, where

$$\widetilde{\Psi}_e(x) := \sum_{i=1}^{\infty} 16^{ilp} \Phi^p\left(0, \frac{x}{2^{il}}\right). \tag{3.49}$$

We have

$$\widetilde{\Psi}_e(x) \leq \frac{K^p}{2^p}\left[\widetilde{\varphi}_e(x) + \widetilde{\varphi}_e(-x)\right] \tag{3.50}$$

for all $x \in X$. Therefore it follows from (3.48) that,

$$\left\| f_e(x) - Q(x) \right\|_Y \leq \frac{K^2}{32}\left[\widetilde{\varphi}_e(x) + \widetilde{\varphi}_e(-x)\right]^{1/p} \tag{3.51}$$

for all $x \in X$. Let $f_o(x) = (1/2)(f(x) - f(-x))$ for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$, and $\|D_{f_o}(x, y)\| \leq \Phi(x, y)$ for all $x, y \in X$. From Theorem 3.4, it follows that there exists a unique cubic function $C : X \to Y$ satisfying

$$\left\| f_o(x) - C(x) \right\|_Y \leq \frac{K}{24}\left[\widetilde{\Phi}_o(x)\right]^{1/p} \tag{3.52}$$

for all $x \in X$, where

$$\widetilde{\Phi}_o(x) := \sum_{i=1}^{\infty} 8^{ip} \Phi^p\left(0, \frac{x}{2^{i}}\right). \tag{3.53}$$

Since

$$\widetilde{\Phi}_o(x) \leq \frac{K^p}{2^p}\left[\widetilde{\phi}_o(x) + \widetilde{\phi}_o(-x)\right] \tag{3.54}$$

for all $x \in X$, it follows from (3.52) that,

$$\left\| f_o(x) - C(x) \right\|_Y \leq \frac{K^2}{48}\left[\widetilde{\phi}_o(x) + \widetilde{\phi}_o(-x)\right]^{1/p} \tag{3.55}$$

for all $x \in X$. Hence (3.43) follows from (3.51) and (3.55). □

Corollary 3.7. *Let θ, r, s be nonnegative real numbers such that $u+v, s \in (4,\infty)\cup(-\infty,3)$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\left\|D_f(x,y)\right\|_Y \leq \theta\left(\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s\right) \tag{3.56}$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$ and a unique cubic function $C : X \to Y$ satisfying (1.7) *and*

$$\left\|f(x) - Q(x) - C(x)\right\|_Y \leq \frac{K^3\theta}{3}\left\{3\left[\frac{1}{|16^p - 2^{sp}|}\right]^{1/p} + \left[\frac{1}{|8^p - 2^{sp}|}\right]^{1/p}\right\}\|x\|_X^s \tag{3.57}$$

for all $x \in X$.

Proof. It follows from Theorem 3.6that

$$\varphi(x,y) := \theta\left(\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s\right) \tag{3.58}$$

for all $x, y \in X$. □

Acknowledgment

The second and fourth authors would like to thank the office of gifted students at Semnan University for its financial support.

References

[1] S. Rolewicz, *Metric Linear Spaces*, Polish Scientific, Warsaw, Poland, 2nd edition, 1984.

[2] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis. Vol. 1*, vol. 48 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI, USA, 2000.

[3] S. M. Ulam, *Problems in Modern Mathematics*, chapter 6, John Wiley & Sons, New York, NY, USA, 1940.

[4] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[5] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[6] J. M. Rassias, "Solution of the Ulam stability problem for cubic mappings," *Glasnik Matematički*, vol. 36(56), no. 1, pp. 63–72, 2001.

[7] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[8] P. W. Cholewa, "Remarks on the stability of functional equations," *Aequationes Mathematicae*, vol. 27, no. 1-2, pp. 76–86, 1984.

[9] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

[10] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[11] A. Grabiec, "The generalized Hyers-Ulam stability of a class of functional equations," *Publicationes Mathematicae Debrecen*, vol. 48, no. 3-4, pp. 217–235, 1996.

[12] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, vol. 34 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, Boston, Mass, USA, 1998.

[13] J. M. Rassias, "On a new approximation of approximately linear mappings by linear mappings," *Discussiones Mathematicae*, vol. 7, pp. 193–196, 1985.
[14] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Bulletin des Sciences Mathématiques*, vol. 108, no. 4, pp. 445–446, 1984.
[15] J. M. Rassias, "Solution of a problem of Ulam," *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.
[16] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.
[17] Th. M. Rassias, Ed., *Functional Equations and Inequalities*, vol. 518 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.
[18] Th. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.
[19] K.-W. Jun and H.-M. Kim, "The generalized Hyers-Ulam-Rassias stability of a cubic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 274, no. 2, pp. 267–278, 2002.
[20] P. Găvruţa and L. Cădariu, "General stability of the cubic functional equation," *Buletinul Ştiinţific al Universităţii Politehnica din Timişoara. Seria Matematică-Fizică*, vol. 47(61), no. 1, pp. 59–70, 2002.
[21] J. M. Rassias, "Solution of the Ulam stability problem for quartic mappings," *The Journal of the Indian Mathematical Society*, vol. 67, no. 1–4, pp. 169–178, 2000.
[22] W.-G. Park and J.-H. Bae, "On a bi-quadratic functional equation and its stability," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 62, no. 4, pp. 643–654, 2005.
[23] J. K. Chung and P. K. Sahoo, "On the general solution of a quartic functional equation," *Bulletin of the Korean Mathematical Society*, vol. 40, no. 4, pp. 565–576, 2003.
[24] L. Cădariu, "Fixed points in generalized metric space and the stability of a quartic functional equation," *Buletinul Ştiinţific al Universităţii Politehnica din Timişoara. Seria Matematică-Fizică*, vol. 50(64), no. 2, pp. 25–34, 2005.
[25] S. H. Lee, S. M. Im, and I. S. Hwang, "Quartic functional equations," *Journal of Mathematical Analysis and Applications*, vol. 307, no. 2, pp. 387–394, 2005.
[26] A. Najati, "On the stability of a quartic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 1, pp. 569–574, 2008.
[27] C.-G. Park, "On the stability of the orthogonally quartic functional equation," *Bulletin of the Iranian Mathematical Society*, vol. 31, no. 1, pp. 63–70, 2005.
[28] K. Ravi and M. Arunkumar, "Hyers-Ulam-Rassias stability of a quartic functional equation," *International Journal of Pure and Applied Mathematics*, vol. 34, no. 2, pp. 247–260, 2007.
[29] E. Thandapani, K. Ravi, and M. Arunkumar, "On the solution of the generalized quartic functional equation," *Far East Journal of Applied Mathematics*, vol. 24, no. 3, pp. 297–312, 2006.
[30] A. Gilányi, "On Hyers-Ulam stability of monomial functional equations," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 68, pp. 321–328, 1998.
[31] A. Gilányi, "Hyers-Ulam stability of monomial functional equations on a general domain," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 96, no. 19, pp. 10588–10590, 1999.