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Research Article

Existence and Uniqueness of Periodic Solutions of Mixed Monotone Functional Differential Equations

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This paper deals with the existence and uniqueness of periodic solutions for the first-order functional differential equation $y'(t) = -a(t)y(t) + f_1(t, y(t-\tau(t))) + f_2(t, y(t-\tau(t)))$ with periodic coefficients and delays. We choose the mixed monotone operator theory to approach our problem because such methods, besides providing the usual existence results, may also sometimes provide uniqueness as well as additional numerical schemes for the computation of solutions.

1. Introduction

In this paper, we are concerned with the existence and uniqueness of periodic solutions for the first-order functional differential equation (cf., e.g., [1–5])

$$y'(t) = -a(t)y(t) + f_1\big(t, y(t-\tau(t))\big) + f_2\big(t, y(t-\tau(t))\big), \tag{1.1}$$

$$x'(t) = a(t)x(t) - f_1(t, x(t-\tau(t))) - f_2(t, x(t-\tau(t))), \tag{1.2}$$

where we will assume that $a = a(t)$ and $\tau = \tau(t)$ are continuous T-periodic functions, that $T > 0$, that $f_1, f_2 \in C(R^2, R)$ and T-periodic with respect to the first variable, and that $a(t) > 0$ for $t \in R$.

Functional differential equations with periodic delays such as those stated above appear in a number of ecological, economical, control and physiological, and other models. One important question is whether these equations can support periodic solutions, and whether they are unique. The existence question has been studied extensively by many authors (see, e.g., [1–5]). The uniqueness problem seems to be more difficult, and less studies are known.

We will tackle the existence and uniqueness question by fixed point theorems for mixed monotone operators. We choose this approach because such fixed point methods, besides providing the usual existence and uniqueness results, sometimes may also provide additional numerical schemes for the computation of solutions.

We first recall some useful terminologies (see [6, 7]). Let E be a real Banach space with zero element θ. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions: (i) $x \in P$ and $\lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$, if and only if $y - x \in P$. A cone P is called normal if there is $M > 0$ such that $x, y \in E$ and $\theta \leq x \leq y \Rightarrow \|x\| \leq M\|y\|$. P is said to be solid if the interior P^0 of P is nonempty.

Assume that $u_0, v_0 \in E$ and $u_0 \leq v_0$. The set $\{x \in E : u_0 \leq x \leq v_0\}$ is denoted by $[u_0, v_0]$. Assume that $h > \theta$. Let $P_h = \{x \in E : \exists \lambda, \mu > 0 \text{ such that } \lambda h \leq x \leq \mu h\}$. Obviously if P is a solid cone and $h \in P^0$, then $P_h = P^0$.

Definition 1.1. Let E be an ordered Banach space, and let $D \subset E$. An operator is called *mixed monotone* on $D \times D$ if $A : D \times D \to E$ and $A(x_1, y_1) \leq A(x_2, y_2)$ for any $x_1, x_2, y_1, y_2 \in D$ that satisfy $x_1 \leq x_2$ and $y_2 \leq y_1$.Also, $x^* \in D$ is called a fixed point of A if $A(x^*, x^*) = x^*$.

A function $f : I \subset R \to R$ is said to be convex in I if $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$ for any $t \in [0,1]$ and any $x, y \in I$. We say that the function f is a concave function if $-f$ is a convex function.

Definition 1.2. Assume $f : I \subset R \to R$ and $0 \leq \alpha < 1$.Then, f is said to be an α-concave or $-\alpha$-convex function if $f(tx) \geq t^{\alpha} f(x)$ or, respectively, $f(tx) \leq t^{-\alpha} f(x)$ for $x \in I$ and $t \in (0,1)$.

Definition 1.3. Let $D \subset E$, and let $A : D \times D \to E$. The operator A is called (ϕ-concave)-($-\psi$-convex) if there exist functions $\phi : (0,1] \times D \to (0,\infty)$ and $\psi : (0,1] \times D \to (0,\infty)$ such that

(H_0) $t < \phi(t,x)\psi(t,x) \leq 1$ for $x \in D$ and $t \in (0,1)$,

(H_1) $A(tx, y) \geq \phi(t,x)A(x,y)$ for any $t \in (0,1)$ and $(x,y) \in D \times D$,

(H_2) $A(x, ty) \leq A(x,y)/\psi(t,y)$ for any $t \in (0,1)$ and $(x,y) \in D \times D$.

Assume that $I \subset R$ and $x_0 \in I$. Recall that a function $f : I \to R$ is said to be left lower semicontinuous at x_0 if $\liminf_{n\to\infty} f(x_n) \geq f(x_0)$ for any monotonically increasing sequence $\{x_n\} \subset I$ that converges to x_0.

The proof of the following theorem can be found in [7].

Theorem 1.4. *Let P be a normal cone of E. Let $u_0, v_0 \in E$ such that $u_0 \leq v_0$, and let $A : [u_0, v_0] \times [u_0, v_0] \to E$ be a mixed monotone operator. If A is a (ϕ-concave)-($-\psi$-convex) operator and satisfies the following three conditions:*

(A1) *there exists $r_0 > 0$ such that $u_0 \geq r_0 v_0$;*

(A2) *$u_0 \leq A(u_0, v_0)$ and $A(v_0, u_0) \leq v_0$;*

(A3) *there exists $\omega_0 \in [u_0, v_0]$ such that $\min_{x\in[u_0,v_0]}\phi(t,x)\psi(t,x) = \phi(t,\omega_0)\psi(t,\omega_0)$ for each $t \in (0,1)$, and $\phi(t,\omega_0)\psi(t,\omega_0)$ is left lower semicontinuous at any $t \in (0,1)$,*

then A has a unique fixed point $x^* \in [u_0, v_0]$, that is, $x^* = A(x^*, x^*)$, and for any $x_0, y_0 \in [u_0, v_0]$, if we set $x_n = A(x_{n-1}, y_{n-1})$ and $y_n = A(y_{n-1}, x_{n-1})$ for $n \in N$, then $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = x^*$.

Remark 1.5. Condition (A3) in Theorem 1.4 can be replaced by (A3') $\phi(t,x)\psi(t,x)$ is monotone in x and left lower semicontinuous at any $t \in (0,1)$.

2. Main Results

A real T-periodic continuous function $y : R \to R$ is said to be a T-periodic solution of (1.1) if substitution of it into (1.1) yields an identity for all $t \in R$.

It is well known (see, e.g., [1, 2]) that (1.1) has a T-periodic solution $y(t)$ if, and only if, $y(t)$ is a T-periodic solution of the equation

$$y(t) = \int_t^{t+T} G(t,s) f_1(s, y(s-\tau(s)))ds + \int_t^{t+T} G(t,s) f_2(s, y(s-\tau(s)))ds, \tag{2.1}$$

where

$$G(t,s) = \frac{\exp\left(\int_t^s a(u)du\right)}{\exp\left(\int_0^T a(u)du\right) - 1}, \tag{2.2}$$

and (1.2) has a T-periodic solution $x(t)$ if, and only if, $x(t)$ is a T-periodic solution of the equation

$$x(t) = \int_{t-T}^{t} H(t,s) f_1(s, x(s-\tau(s)))ds + \int_{t-T}^{t} H(t,s) f_2(s, x(s-\tau(s)))ds, \tag{2.3}$$

where

$$H(t,s) = \frac{\exp\left(\int_s^t a(u)du\right)}{\exp\left(\int_0^T a(u)du\right) - 1}. \tag{2.4}$$

Furthermore, the Cauchy function $G(t,s)$ satisfies

$$0 < m \equiv \lim_{0\le t,s\le T} G(t,s) \le G(t,s) \le \max_{0\le t,s\le T} G(t,s) \equiv M < \infty. \tag{2.5}$$

Now let $C_T(R)$ be the Banach space of all real T-periodic continuous functions $y : R \to R$ endowed with the usual linear structure as well as the norm

$$\|y\| = \sup_{t\in[0,T]} |y(t)|. \tag{2.6}$$

Then $P = \{\phi \in C_T(R) : \phi(x) \ge 0, x \in R\}$ is a normal cone of $C_T(R)$.

Definition 2.1. The functions $v_0, \omega_0 \in C_T^1(R)$ are said to form a pair of lower and upper quasisolutions of (1.1) if $v_0(t) \le \omega_0(t)$ and

$$v_0'(t) \le -a(t)v_0(t) + f_1(t, v_0(t-\tau(t))) + f_2(t, \omega_0(t-\tau(t))), \tag{2.7}$$

as well as

$$\omega_0'(t) \ge -a(t)\omega_0(t) + f_1(t, \omega_0(t-\tau(t))) + f_2(t, v_0(t-\tau(t))). \tag{2.8}$$

We remark that the term quasi is used in the above definition to remind us that they are different from the traditional concept of lower and upper solutions (cf. (2.7) with $v_0'(t) \le -a(t)v_0(t) + f_1(t, v_0(t-\tau(t))) + f_2(t, v_0(t-\tau(t)))$).

Let $A : P \times P \to C_T(R)$ be defined by

$$A(u,v)(t) = \int_t^{t+T} G(t,s) f_1(s, u(s-\tau(s)))ds + \int_t^{t+T} G(t,s) f_2(s, v(s-\tau(s)))ds. \tag{2.9}$$

We need two basic assumptions in the main results:

(B_1) for any $s \in R$, $f_1(s,x)$ is an increasing function of x, and $f_2(s,x)$ is a decreasing function of x;

(B_2) there exist $u_0, v_0 \in P$ such that u_0 and v_0 form a respective pair of lower and upper quasisolutions for (1.1).

Theorem 2.2. *Suppose that conditions (B_1) and (B_2) hold, and*

(C1) *for any $s \in R$, $f_1(s,\cdot)$ is an α-concave function, $f_2(s,\cdot)$ is a convex function;*

(C2) *there exist $\varepsilon \ge 1/(2-\alpha)$ such that $A(u_0, v_0) \geqslant \varepsilon A(v_0, \theta)$.*

Then (1.1) *has a unique solution $x^* \in [u_0, v_0]$, and for any $x_0, y_0 \in [u_0, v_0]$, if we set $x_n = A(x_{n-1}, y_{n-1})$ and $y_n = A(y_{n-1}, x_{n-1})$, then $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = x^*$.*

Proof. The mapping $A : P \times P \to C_T(R)$ is a mixed monotone operator in view of (B1). Let

$$u_1(z) = \int_z^{z+T} G(z,s) f_1(s, u_0(s-\tau(s)))ds + \int_z^{z+T} G(z,s) f_2(s, v_0(s-\tau(s)))ds. \tag{2.10}$$

Then

$$\begin{aligned}
u_1'(z) &= -a(z)u_1(z) + G(z, z+T) f_1(z+T, u_0(z+T-\tau(z+T))) \\
&\quad - G(z,z) f_1(z, u_0(z-\tau(z))) \\
&\quad + G(z, z+T) f_2(z+T, v_0(z+T-\tau(z+T))) - G(z,z) f_2(z, v_0(z-\tau(z))) \\
&= -a(z)u_1(z) + G(z, z+T) f_1(z, u_0(z-\tau(z))) - G(z,z) f_1(z, u_0(z-\tau(z))) \\
&\quad + G(z, z+T) f_2(z, v_0(z-\tau(z))) - G(z,z) f_2(z, v_0(z-\tau(z))) \\
&= -a(z)u_1(z) + f_1(z, u_0(z-\tau(z))) + f_2(z, v_0(z-\tau(z))).
\end{aligned} \tag{2.11}$$

Set $m(z) = u_1(z) - u_0(z)$. Then

$$m'(z) = u_1'(z) - u_0'(z) \geqslant -a(z)m(z). \tag{2.12}$$

Next, we will prove that $m(z) \geqslant 0$. Suppose to the contrary that there exists $z_0 \in R$ such that

$$m(z_0) = \min_{z \in R} m(z) < 0. \tag{2.13}$$

Then $m'(z_0) \geq -a(z_0)m(z_0) > 0$, which is a contradiction since $m(z_0) = \min_{z\in R} m(z)$. Thus $u_0 \leq A(u_0, v_0)$. Similarly, we can prove $A(v_0, u_0) \leq v_0$. Then we have

$$\begin{gathered} u_1 \leq A(u_1, v_1), \qquad A(v_1, u_1) \leq v_1, \\ u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_2 \leq v_1 \leq v_0. \end{gathered} \tag{2.14}$$

From condition (C2), we know that $u_1 \geq \varepsilon v_1$. Since $u_1 \leq v_1$, we must have $0 < \varepsilon \leq 1$.

We will prove that $A : [u_1, v_1] \times [u_1, v_1] \to C_T(R)$ is a (ϕ-concave)-($-\psi$-convex) operator, where

$$\phi(t, u) = t^{\alpha}, \quad \psi(t, v) = \frac{\varepsilon}{1 - (1-\varepsilon)t}, \quad t \in (0,1),\ u, v \in [u_0, v_0]. \tag{2.15}$$

In fact, for any $u, v \in [u_0, v_0]$, $t \in (0, 1)$, and $z \in G$, we have

$$\begin{aligned} A(u, tv)(z) &= A(u, tv + (1-t)\theta)(z) \\ &= \int_z^{z+T} G(z,s) f_1(s, u(s-\tau(s)))ds \\ &\quad + \int_z^{z+T} G(z,s) f_2(s, (tv + (1-t)\theta)(s-\tau(s)))ds \\ &\leq \int_z^{z+T} G(z,s) f_1(s, u(s-\tau(s)))ds + t\int_z^{z+T} G(z,s) f_2(s, v(s-\tau(s)))ds \\ &\quad + (1-t)\int_z^{z+T} G(z,s) f_2(s, \theta(s-\tau(s)))ds \\ &= tA(u,v)(z) + (1-t)A(u,\theta)(z) \\ &\leq tA(u,v)(z) + (1-t)A(v_0,\theta)(z) \leq tA(u,v)(z) + \frac{1-t}{\varepsilon}A(u_0,v_0)(z) \\ &\leq tA(u,v)(z) + \frac{1-t}{\varepsilon}A(u,v)(z) \\ &= \frac{1}{\psi(t,v)}A(u,v)(z), \end{aligned} \tag{2.16}$$

thus

$$\begin{aligned}
A(u,tv) &\leq \frac{1}{\psi(t,v)}A(u,v), \\
A(tu,v)(z) &= \int_z^{z+T} G(z,s)f_1(s,tu(s-\tau(s)))ds + \int_z^{z+T} G(z,s)f_2(s,v(s-\tau(s)))ds \\
&\geq t^\alpha \int_z^{z+T} G(z,s)f_1(s,u(s-\tau(s)))ds + \int_z^{z+T} G(z,s)f_2(s,v(s-\tau(s)))ds \\
&\geq t^\alpha A(u,v)(z) \\
&= \phi(t,u)A(u,v)(z),
\end{aligned} \tag{2.17}$$

so that

$$A(tu,v) \geq \phi(t,u)A(u,v). \tag{2.18}$$

Further we can prove

$$t < \phi(t,u)\psi(t,u) \leq 1 \tag{2.19}$$

for any $t \in (0,1)$ and $u \in [u_0, v_0]$. Indeed, since

$$\phi(t,u)\psi(t,u) = \frac{\varepsilon t^\alpha}{1-t+\varepsilon t}, \quad t \in (0,1), \; u \in [u_0, v_0], \tag{2.20}$$

hence, we only need to prove

$$t < \frac{\varepsilon t^\alpha}{1-t+\varepsilon t} \leq 1, \quad t \in (0,1). \tag{2.21}$$

From $0 < \varepsilon \leq 1$, we know that $\varepsilon t^\alpha - \varepsilon t + t \leq t^\alpha \leq 1$ for any $0 < t < 1$, therefore

$$\frac{\varepsilon t^\alpha}{1-t+\varepsilon t} \leq 1, \quad t \in (0,1). \tag{2.22}$$

On the other hand, the function

$$g(t) = \varepsilon t^{\alpha-1} + (1-\varepsilon)t - 1, \quad t \in [0,1] \tag{2.23}$$

satisfies $g(1) = 0$ and $g'(t) = \varepsilon(\alpha-1)t^{\alpha-2}+1-\varepsilon$. From $\varepsilon \geq 1/(2-\alpha)$, we have $\varepsilon(1-\alpha)/(1-\varepsilon) \geq 1$. Then $t^{2-\alpha} < \varepsilon(1-\alpha)/(1-\varepsilon)$ for $0 < t < 1$. Thus $\varepsilon(\alpha-1)t^{\alpha-2}+1-\varepsilon < 0$, that is, $g'(t) < 0$. Therefore, $g(t) > 0$ for any $0 < t < 1$. Finally,

$$t < \frac{\varepsilon t^\alpha}{1-t+\varepsilon t}, \quad t \in (0,1). \tag{2.24}$$

Therefore, $A : [u_1, v_1] \times [u_1, v_1] \to C_T(R)$ is a (ϕ-concave)-($-\psi$-convex) operator. From (2.20), $\phi(t,u)\psi(t,u)$ is monotone in u and is left lower semicontinuous at t. By Theorem 1.4, we know that A has a unique fixed point $x^* \in [u_1, v_1] \subset [u_0, v_0]$. Hence (1.1) has a unique solution $x^* \in [u_0, v_0]$, and for any $x_0, y_0 \in [u_0, v_0]$, if we set $x_n = A(x_{n-1}, y_{n-1})$ and $y_n = A(y_{n-1}, x_{n-1})$, then $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = x^*$. The proof is complete. □

Theorem 2.3. *Suppose that conditions (B_1) and (B_2) hold, and*

(D1) *there exist $r_0 > 0$ such that $u_0 \geq r_0 v_0$;*

(D2) *for any $s \in R$, $f_1(s,\cdot)$ is an α-concave function and $f_2(s,ty) \leq [(1+\eta)t]^{-1} f_2(s,y)$ for any $y \in P$ and $t \in [0,1]$, where $\eta = \eta(t,y)$ satisfies the following conditions:*

(DH_1) *$\eta(t,y)$ is monotone in y and left lower semicontinuous in t;*

(DH_2) *for any $(t,y) \in (0,1) \times [u_0, v_0]$,*

$$\frac{1}{t^\alpha} - 1 < \eta(t,y) \leq \frac{1}{t} - 1 < \frac{1}{t^{1+\alpha}} - 1. \tag{2.25}$$

Then (1.1) *has a unique solution $x^* \in [u_0, v_0]$, and for any $x_0, y_0 \in [u_0, v_0]$, if we set $x_n = A(x_{n-1}, y_{n-1})$ and $y_n = A(y_{n-1}, x_{n-1})$ for $n \in N$, then $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n = x^*$.*

Proof. We assert that $A : [u_0, v_0] \times [u_0, v_0] \to C_T(R)$ is a (ϕ-concave)-($-\psi$-convex) mixed monotone operator, where

$$\phi(t,u) = t^\alpha, \quad \psi(t,v) = \left[1 + \eta(t,v)\right] t \quad \text{for } t \in (0,1),\ u, v \in [u_0, v_0]. \tag{2.26}$$

In fact,

$$\begin{aligned} A(tu, v) &\geq t^\alpha A(u,v) = \phi(t,u) A(u,v), \\ A(u, tv) &\leq \frac{1}{t\left[1 + \eta(t,v)\right]} A(u,v) = \frac{1}{\psi(t,v)} A(u,v) \end{aligned} \tag{2.27}$$

for any $u, v \in [u_0, v_0]$ and $t \in (0,1)$. From (2.25), we know that $t < \phi(t,u)\psi(t,u) \leq 1$. Thus $A : [u_0, v_0] \times [u_0, v_0] \to C_T(R)$ is a (ϕ-concave)-($-\psi$-convex) mixed monotone operator. We may now complete our proof by Theorem 1.4. □

Theorem 2.4. *Suppose that conditions (B_1) and (B_2) hold, and*

(E1) *for any $s \in R$, $f_1(s,\cdot)$ is a concave function; $f_2(s,ty) \leq [(1+\eta)t]^{-1} f_2(s,y)$ for any $y \in P$ and $t \in [0,1]$, and $\eta = \eta(t,y)$ satisfies the following conditions:*

(EH_1) *there exists $\varepsilon \in (0,1]$ such that $A(\theta, v_0) \geq \varepsilon A(v_0, u_0)$;*

(EH_2) *for any $(t,y) \in (0,1) \times [u_0, v_0]$,*

$$\frac{1}{t + \varepsilon(1-t)} - 1 < \eta(t,y) \leq \frac{1}{t} - 1 \leq \frac{1}{t^2 + \varepsilon t(1-t)} - 1. \tag{2.28}$$

Then (1.1) *has unique solution* $x^* \in [u_0, v_0]$, *and for any* $x_0, y_0 \in [u_0, v_0]$, *if we set* $x_n = A(x_{n-1}, y_{n-1})$ *and* $y_n = A(y_{n-1}, x_{n-1})$ *for* $n \in N$, *then* $\lim_{n\to\infty} x_n = x^*$ *and* $\lim_{n\to\infty} y_n = x^*$.

Proof. Set $u_n = A(u_{n-1}, v_{n-1})$ and $v_n = A(v_{n-1}, u_{n-1})$ for $n \in N$. Then we know that

$$\begin{gathered} u_1 \le A(u_1, v_1), \qquad A(v_1, u_1) \le v_1\ , \\ u_0 \le u_1 \le u_2 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_2 \le v_1 \le v_0. \end{gathered} \tag{2.29}$$

From (EH_2) we have $u_1 \ge \varepsilon v_1$. Next we will prove that $A : [u_1, v_1] \times [u_1, v_1] \to C_T(R)$ is a (ϕ-concave)-($-\psi$-convex) operator, where

$$\phi(t, u) = t + \varepsilon(1-t), \quad \psi(t, v) = \left[1 + \eta(t, v)\right]t \quad \text{for } t \in (0,1),\ u, v \in [u_0, v_0]. \tag{2.30}$$

In fact, for any $u, v \in [u_0, v_0]$ and $t \in (0,1)$,

$$\begin{aligned} A(tu, v) &= A(tu + (1-t)\theta, v) \ge tA(u, v) + (1-t)A(\theta, v) \\ &\ge tA(u, v) + (1-t)A(\theta, v_0) \ge tA(u, v) + \varepsilon(1-t)A(v_0, u_0) \\ &\ge tA(u, v) + \varepsilon(1-t)A(u, v) = \phi(t, u)A(u, v), \\ A(u, tv) &\le \frac{1}{\left[1 + \eta(t, v)\right]t} A(u, v) = \frac{1}{\psi(t, v)} A(u, v). \end{aligned} \tag{2.31}$$

From (2.28), we know that $t < \phi(t, u)\psi(t, u) \le 1$. Thus $A : [u_1, v_1] \times [u_1, v_1] \to C_T(R)$ is a (ϕ-concave)-($-\psi$-convex) mixed monotone operator. We may now complete our proof by Theorem 1.4. □

Theorem 2.5. *Suppose that conditions* (B_1) *and* (B_2) *hold, and*

(F1) *there exists* $r_0 > 0$ *such that* $u_0 \ge r_0 v_0$;

(F2) $f_1(s, x) > 0$ *and* $f_2(s, x) > 0$ *for any* $s, x \in R$, *and there exist* $e > 0$, $f_1(s, tx) \ge (1 + \eta)tf_1(s, x)$ *for any* $x \in P_e$ *and* $t \in (0,1)$, *where* $P_e = \{x \in E : \exists\lambda, \mu > 0 \text{ such that } \lambda e \le x \le \mu e\}$, $f_2(s, tx) \le [(1+\zeta)t]^{-1} f_2(s, x)$ *for any* $x \in P$ *and* $t \in [0,1]$; $\eta = \eta(t, x)$, $\zeta = \zeta(t, x)$ *satisfies the following conditions:*

(FH_1) $(1 + \eta(t, x))(1 + \zeta(t, x))$ *is monotone in* x *and left lower semicontinuous in* t;

(FH_2) *for any* $(t, x) \in (0,1) \times [u_0, v_0]$,

$$\begin{gathered} 1 + \eta(t, x) \le \frac{1}{t}, \qquad 1 + \zeta(t, x) \le \frac{1}{t}, \\ \frac{1}{t} - 1 < \eta(t, x) + \zeta(t, x) + \eta(t, x)\zeta(t, x) \le \frac{1}{t^2} - 1. \end{gathered} \tag{2.32}$$

Then (1.1) *has a unique solution* $x^* \in [u_0, v_0]$, *and for any* $x_0, y_0 \in [u_0, v_0]$, *if we set* $x_n = A(x_{n-1}, y_{n-1})$ *and* $y_n = A(y_{n-1}, x_{n-1})$ *for* $n \in N$, *then* $\lim_{n\to\infty} x_n = x^*$ *and* $\lim_{n\to\infty} y_n = x^*$.

Proof. We may easily prove that $A : [u_0, v_0] \times [u_0, v_0] \to C_T(R)$ is a (ϕ-concave)-($-\psi$-convex) mixed monotone operator, where

$$\phi(t,u) = \left[1 + \eta(t,u)\right]t, \quad \psi(t,v) = \left[1 + \zeta(t,v)\right]t \quad \text{for } t \in (0,1),\ u,v \in [u_0, v_0]. \tag{2.33}$$

And from (FH_2) we know that

$$t < \phi(t,u)\psi(t,u) \le 1 \tag{2.34}$$

for any $t \in (0,1)$ and $u \in [u_0, v_0]$. Now the proof can be completed by means of Theorem 1.4. □

Theorem 2.6. *Suppose that conditions* (B_1) *and* (B_2) *hold, and*

(G1) *if* $u_0 \le v_0$, *there exists* r_0 *such that* $u_0 \ge r_0 v_0$;

(G2) $f_1(s,x) > 0$ *and* $f_2(s,x) > 0$ *for any* $s, x \in R$; *there exist* $e > 0$ *and* $\eta = \eta(t,x)$ *such that* $f_1(s,tx) \ge (1+\eta)t f_1(s,x)$ *for any* $x \in P_e$ *and* $t \in (0,1)$, *where* $P_e = \{x \in E : \exists \lambda, \mu > 0$ *such that* $\lambda e \le x \le \mu e\}$; *for any* $s \in R$, $f_2(s,\cdot)$ *is a* $(-\alpha)$-*convex function, and* $\eta = \eta(t,x)$ *satisfies the following conditions:*

(GH_1) $\eta(t,x)$ *is monotone in* x *and left lower semicontinuous in* t;

(GH_2) *for any* $(t,x) \in (0,1) \times [u_0, v_0]$,

$$\begin{gathered} 1 + \eta(t,x) \le \frac{1}{t}, \\ \frac{1}{t^{\alpha}} - 1 < \eta(t,x) \le \frac{1}{t} - 1 < \frac{1}{t^{1+\alpha}} - 1. \end{gathered} \tag{2.35}$$

Then (1.1) *has a unique solution* $x^* \in [u_0, v_0]$, *and for any* $x_0, y_0 \in [u_0, v_0]$, *if we set* $x_n = A(x_{n-1}, y_{n-1})$ *and* $y_n = A(y_{n-1}, x_{n-1})$ *for* $n \in N$, *then* $\lim_{n\to\infty} x_n = x^*$ *and* $\lim_{n\to\infty} y_n = x^*$.

Proof. It is easily seen that $A : [u_0, v_0] \times [u_0, v_0] \to C_T(R)$ is a (ϕ-concave)-($-\psi$-convex) mixed monotone operator, where

$$\phi(t,u) = \left[1 + \eta(t,u)\right]t, \quad \psi(t,v) = t^{\alpha} \quad \text{for } t \in (0,1),\ u,v \in [u_0, v_0]. \tag{2.36}$$

From (GH_2), we know that $t < \phi(t,u)\psi(t,u) \le 1$. Then $A : [u_0, v_0] \times [u_0, v_0] \to C_T(R)$ is a (ϕ-concave)-($-\psi$-convex) mixed monotone operator. The proof may now be completed by means of Theorem 1.4. □

Theorem 2.7. *Suppose that conditions* (B_1) *and* (B_2) *hold, and*

(J1) $f_1(s,x) > 0$ *and* $f_2(s,x) > 0$ *for any* $s, x \in R$; $f_1(s,tx) \ge (1+\eta)t f_1(s,x)$ *for any* $x \in P_e$ *and* $t \in (0,1)$, *where* $P_e = \{x \in E : \exists \lambda, \mu > 0$ *such that* $\lambda e \le x \le \mu e\}$; *for any* $s \in R$, $f_2(s,\cdot)$ *is a convex function;* $\eta = \eta(t,x)$ *satisfies the following conditions:*

(JH_1) *$\eta(t,x)$ is monotone in x and left lower semicontinuous in t;*

(JH_2) *there exists $\varepsilon \in (1/2, 1)$ such that $A(u_0, v_0) \geq \varepsilon A(v_0, \theta)$ and*

$$\frac{(1-t)(1-\varepsilon)}{\varepsilon} < \eta(t,x) \leq \frac{1}{t} - 1 < \frac{1-t}{\varepsilon t} \tag{2.37}$$

for any $(t,x) \in (0,1) \times [u_0, v_0]$.

Then (1.1) *has unique solution $x^* \in [u_0, v_0]$, and for any $x_0, y_0 \in [u_0, v_0]$, if we set $x_n = A(x_{n-1}, y_{n-1})$ and $y_n = A(y_{n-1}, x_{n-1})$ for $n \in N$, then $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = x^*$.*

Proof. Set $u_n = A(u_{n-1}, v_{n-1})$ and $v_n = A(v_{n-1}, u_{n-1})$ for $n \in N$. Then we have $u_1 \leq A(u_1, v_1), A(v_1, u_1) \leq v_1$, and

$$u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_2 \leq v_1 \leq v_0. \tag{2.38}$$

From (JH_2) we can see that $u_1 \geq \varepsilon v_1$.

Next we will prove that $A : [u_1, v_1] \times [u_1, v_1] \to C_T(R)$ is a (ϕ-concave)-($-\psi$-convex) operator. We need only to verify that $A : [u_0, v_0] \times [u_0, v_0] \to C_T(R)$ is a (ϕ-concave)-($-\psi$-convex) operator, where

$$\phi(t,u) = [1 + \eta(t,u)]t, \quad \psi(t,v) = \frac{\varepsilon}{1-(1-\varepsilon t)}t \quad \text{for } t \in (0,1),\ u, v \in [u_0, v_0]. \tag{2.39}$$

In fact, for any $u, v \in [u_0, v_0]$ and $t \in (0,1)$, we have

$$\begin{aligned}
A(tu, v) &\geq [1 + \eta(t,u)]tA(u,v) = \phi(t,u)A(u,v), \\
A(u, tv) &= A(u, tv + (1-t)\theta) \leq tA(u,v) + (1-t)A(u,\theta) \\
&\leq tA(u,v) + (1-t)A(v_0,\theta) \leq tA(u,v) + \frac{1-t}{\varepsilon}A(u_0,v_0) \\
&\leq tA(u,v) + \frac{1-t}{\varepsilon}A(u,v) = \frac{1}{\psi(t,v)}A(u,v).
\end{aligned} \tag{2.40}$$

From (JH_2), we have $t < \phi(t,u)\psi(t,u) \leq 1$. Then $A : [u_1, v_1] \times [u_1, v_1] \to C_T(R)$ is a (ϕ-concave)-($-\psi$-convex) mixed monotone operator. The rest of the proof follows from Theorem 1.4. □

Theorem 2.8. *Suppose that conditions (B_1) and (B_2) hold, and*

(K1) *for any $s \in R$, $f_1(s,\cdot)$ is an α_1-concave function, $f_2(s,\cdot)$ is a $(-\alpha_2)$-convex function; where $0 \leq \alpha_1 + \alpha_2 < 1$;*

(K2) *there exist $r_0 > 0$ such that $u_0 \geq r_0 v_0$.*

Then (1.1) *has unique solution $x^* \in [u_0, v_0]$, and for any $x_0, y_0 \in [u_0, v_0]$, if we set $x_n = A(x_{n-1}, y_{n-1})$ and $y_n = A(y_{n-1}, x_{n-1})$ for $n \in N$, then $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = x^*$.*

Indeed, it is easily seen that $A : [u_0, v_0] \times [u_0, v_0] \to C_T(R)$ is a (ϕ-concave)-($-\psi$-convex) mixed monotone operator, where

$$\phi(t,u) = t^{\alpha_1}, \quad \psi(t,v) = t^{\alpha_2} \quad \text{for } t \in (0,1),\ u,v \in [u_0,v_0]. \tag{2.41}$$

The rest of the proof now follows from Theorem 1.4.

If P is a solid cone, we have the following result.

Theorem 2.9. *Suppose that P is a solid cone of E, that condition (B_1) holds, and that*

(L1) *for any $s \in R$, $f_1(s,\cdot)$ is a α_1-concave function, $f_2(s,\cdot)$ is a $(-\alpha_2)$-convex function, where $0 \le \alpha_1 + \alpha_2 < 1$;*

(L2) *there exist $u_0, v_0 \in P^0$ such that $u_0(t)$ and $v_0(t)$ form a pair of lower and upper quasisolutions for* (1.1).

Then (1.1) *has unique solution $x^* \in [u_0, v_0]$, and for any $x_0, y_0 \in [u_0, v_0]$, if we set $x_n = A(x_{n-1}, y_{n-1})$ and $y_n = A(y_{n-1}, x_{n-1})$, then $x_n \to x^*$, $y_n \to x^*(n \to \infty)$.*

Indeed, from $u_0, v_0 \in P^0$, we know that there exists $r_0 > 0$ such that $u_0 \ge r_0 v_0$. The rest of the proof is similar to that of Theorem 2.7.

3. An Example

As an example, consider the equation

$$y'(t) = -a(t)y(t) + \left[p(t)y^{1/3}(t-\tau(t)) + q(t)y^{-1/2}(t-\tau(t))\right], \tag{3.1}$$

where $p(t)$ and $q(t)$ are nonnegative continuous T-periodic functions; $a(t)$ and $\tau(t)$ are continuous T-periodic functions and satisfy

$$p_{\max} + 10^{3/2} q_{\max} \le a(t) \le 10^2 p_{\min} + 10^3 q_{\min}, \tag{3.2}$$

where $p_{\max} = \max_{t\in[0,T]} p(t)$, $p_{\min} = \min_{t\in[0,T]} p(t)$, $q_{\max} = \max_{t\in[0,T]} q(t)$, $q_{\min} = \min_{t\in[0,T]} q(t)$, and $p_{\max} + \sqrt{1000} q_{\max} \le 100 p_{\min} + 1000 q_{\min}$. Then (3.1) will have a unique solution $y = y^*(t)$ that satisfies $10^{-3} \le y^*(t) \le 1$. Furthermore, if we set $v_0(t) = 10^{-3}$, $\omega_0(t) = 1$,

$$\begin{aligned} v_n(t) &= \int_t^{t+T} G(t,s)\left[p(s)v_{n-1}^{1/3}(s-\tau(s)) + q(s)\omega_{n-1}^{-1/2}(s-\tau(s))\right]ds \quad n \in N, \\ \omega_n(t) &= \int_t^{t+T} G(t,s)\left[p(s)\omega_{n-1}^{1/3}(s-\tau(s)) + q(s)v_{n-1}^{-1/2}(s-\tau(s))\right]ds \quad n \in N, \end{aligned} \tag{3.3}$$

then $\{v_n\}$ and $\{\omega_n\}$ converge uniformly to y^*.

Indeed, let $C_T(R)$ be the Banach space of all real T-periodic continuous functions defined on R and endowed with the usual linear structure as well as the norm

$$\|y\| = \sup_{t\in[0,1]} |y(t)|. \tag{3.4}$$

The set $P = \{\phi \in C_T(R) : \phi(x) \geq 0,\ x \in R\}$ is a normal cone of $C_T(R)$. Equation (3.1) has a T-periodic solution $y(t)$, if and only if, $y(t)$ is a T-periodic solution of the equation

$$y(t) = \int_t^{t+T} G(t,s)\left[p(s)y^{1/3}(s-\tau(s)) + q(s)y^{-1/2}(s-\tau(s))\right]ds, \tag{3.5}$$

where

$$G(t,s) = \frac{\exp(\int_t^s a(u)du)}{\exp\left(\int_0^T a(u)du\right) - 1}. \tag{3.6}$$

Set

$$A(x,y) = \int_t^{t+T} G(t,s)\left[p(s)x^{1/3}(s-\tau(s)) + q(s)y^{-1/2}(s-\tau(s))\right]ds, \tag{3.7}$$

$v_0(t) = 10^{-3},\ \omega_0(t) = 1,\ \alpha_1 = 1/3$, and $\alpha_2 = 1/2$. Then $v_0(t)$ and $\omega_0(t)$ form a pair of lower and upper quasisolutions for (3.1). By Theorem 2.8, we know that (3.1) has a unique solution $y^* \in [10^{-3}, 1]$, and if we set $v_n = A(v_{n-1}, \omega_{n-1}),\ \omega_n = A(\omega_{n-1}, v_{n-1})$ for $n \in N$, then $\lim_{n\to\infty} v_n = y^*$ and $\lim_{n\to\infty}\omega_n = y^*$.

Other examples can be constructed to illustrate the other results in the previous section.

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