Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2007, Article ID 61659, 16 pages
doi:10.1155/2007/61659

Research Article
A Note on the Parabolic Differential and Difference Equations

Allaberen Ashyralyev, Yasar Sozen, and Pavel E. Sobolevskii

Received 13 August 2006; Revised 12 December 2006; Accepted 6 March 2007

Recommended by Martin J. Bohner

The differential equation $u'(t)+\mathbf{A}u(t)=f(t)$ $(-\infty<t<\infty)$ in a general Banach space $\mathbf{E}$ with the strongly positive operator $\mathbf{A}$ is ill-posed in the Banach space $C(\mathbf{E})=C(\mathbb{R},\mathbf{E})$ with norm $\|\varphi\|_{C(\mathbf{E})}=\sup_{-\infty<t<\infty}\|\varphi(t)\|_{\mathbf{E}}$. In the present paper, the well-posedness of this equation in the Hölder space $C^{\alpha}(\mathbf{E})=C^{\alpha}(\mathbb{R},\mathbf{E})$ with norm $\|\varphi\|_{C^{\alpha}(\mathbf{E})}=\sup_{-\infty<t<\infty}\|\varphi(t)\|_{\mathbf{E}}+\sup_{-\infty<t<t+s<\infty}(\|\varphi(t+s)-\varphi(t)\|_{\mathbf{E}}/s^{\alpha})$, $0<\alpha<1$, is established. The almost coercivity inequality for solutions of the Rothe difference scheme in $C(\mathbb{R}_{\tau},\mathbf{E})$ spaces is proved. The well-posedness of this difference scheme in $C^{\alpha}(\mathbb{R}_{\tau},\mathbf{E})$ spaces is obtained.

1. Introduction

The role played by coercivity inequalities (maximal regularity, well-posedness) in the study of boundary value problems for parabolic and elliptic differential equations is well known (see, e.g., [1–3]).

Coercivity inequalities approach permits to investigate the general boundary value problems for both elliptic and parabolic differential equations.

The coercivity inequalities also hold for various difference analogues of such problems. These inequalities evidently enable us to prove not only the existence of solutions but also the well-posedness of such problems. Main role of the coercivity inequalities for difference problems lies in that they present a special type of stability, which allows the existence of exact, that is, two-sided estimates of rate of convergence approximate solutions with respect to the corresponding coercivity norms.

It is quite possible that there are cases where the difference problem is well-posed, although the differential problem is not.

Well-posedness of local and nonlocal boundary value problems for abstract parabolic differential and difference equations in Banach spaces have been studied extensively by many researchers (see [3–21] and the references therein).

In present paper, the well-posedness of the parabolic equation is investigated. The paper is organized as follows. In Section 2, the parabolic differential equation in the Banach space $\mathbf{E}$ is considered. The well-posedness of this equation in the Hölder space is presented. In Section 3, the first order of accuracy Rothe difference scheme for parabolic differential equation is studied. The almost coercivity inequality for solutions of this difference scheme is established. Section 4 presents the well-posedness of this difference scheme in difference analogues of Hölder spaces.

2. Well-posedness of the parabolic differential equation

In the arbitrary Banach space $\mathbf{E}$, the parabolic differential equation

$$\frac{du(t)}{dt}+\mathbf{A}u(t)=f(t), \quad -\infty<t<\infty, \tag{2.1}$$

is considered. Here $u(t)$ and $f(t)$ are unknown and given abstract functions, defined on $\mathbb{R}=(-\infty,\infty)$ with values in $\mathbf{E}$; $\mathbf{A}$ is a linear unbounded closed operator acting in $\mathbf{E}$ with dense domain $D(\mathbf{A})\subset\mathbf{E}$.

A function $u(t)$ is called a solution of the problem (2.1) if the following conditions are satisfied:

(i) $u(t)$ is continuously differentiable bounded on $\mathbb{R}$;

(ii) The element $u(t)$ belongs to $D(\mathbf{A})$ for all $t\in\mathbb{R}$ and the function $Au(t)$ is continuously bounded on $\mathbb{R}$;

(iii) $u(t)$ satisfies (2.1).

A solution of problem (2.1) defined in this manner will from now on be referred to as a solution of problem (2.1) in the space $C(\mathbf{E})=C(\mathbb{R},\mathbf{E})$ of all continuously bounded functions $\varphi(t)$ defined on $\mathbb{R}$ with values in $\mathbf{E}$ equipped with the norm

$$\|\varphi\|_{C(\mathbf{E})}=\sup_{-\infty<t<\infty}\|\varphi(t)\|_{\mathbf{E}}. \tag{2.2}$$

We say that the problem (2.1) is well-posed in $C(\mathbf{E})$ if the following conditions are satisfied.

(1) Problem (2.1) is uniquely solvable for any $f(t)\in C(\mathbf{E})$. This means that an additive and homogeneous operator $u(t)\equiv u(t;f(t))$ acting from $C(\mathbf{E})$ to $C(\mathbf{E})$ is defined and gives the solution of problem (2.1) in $C(\mathbf{E})$. Moreover, the operators $(d/dt)[u(t;f(t))]$ and $\mathbf{A}u(t;f(t))$ acting in $C(\mathbf{E})$ have these properties also (see, e.g., [10]).

(2) $u(t;f(t))$, regarded as an operator from $C(\mathbf{E})$ to $C(\mathbf{E})$, is continuous. It means that inequality

$$\|u(t;f(t))\|_{C(\mathbf{E})}\le M\|f\|_{C(\mathbf{E})} \tag{2.3}$$

holds for some $1\le M<\infty$, which does not depend on $f(t)\in C(\mathbf{E})$.

In this paper, we will indicate with M positive constants which can be different from time to time and we are not interested to precise. We will write $M(\alpha,\beta,\ldots)$ to stress the fact that the constant depends only on $\alpha,\beta,\ldots$.

From the well-posedness of problem (2.1) in $C(\mathbf{E})$ it follows that the operator $u(t;f(t))$ is continuous in $C(\mathbf{E})$, and the operator $\mathbf{A}u(t;f(t))$ is defined on the entire space $C(\mathbf{E})$. The operator $\mathbf{A}$, which acts in the Banach space $\mathbf{E}$ with domain $D(\mathbf{A})$, generates via the formula $\mathcal{A}u = \mathbf{A}u(t)$ an operator $\mathcal{A}$, which acts in the Banach space $C(\mathbf{E})$ and is defined on the functions $u(t) \in C(\mathbf{E})$ with the property that $\mathbf{A}u(t) \in C(\mathbf{E})$. From the fact that the operator $\mathbf{A}^{-1}$ exists and is bounded, it follows that the operator $\mathcal{A}^{-1}$ exists and is bounded, and hence $\mathcal{A}$ is closed in $C(\mathbf{E})$. As a result, the operator $\mathbf{A}u(t;f(t)) = \mathcal{A}(\cdot,f)$ is closed in $C(\mathbf{E})$. By Banach's theorem, this operator is continuous, that is, for any $f(t) \in C(\mathbf{E})$ one has the inequality

$$\|\mathbf{A}u(t;f(t))\|_{\mathbf{E}} \le M\|f\|_{\mathbf{E}}, \tag{2.4}$$

where M does not depend $f(t)$.

This leads us to coercivity inequality

$$\|u'\|_{C(\mathbf{E})} + \|\mathbf{A}u(t)\|_{C(\mathbf{E})} \le M_C\|f\|_{C(\mathbf{E})} \tag{2.5}$$

for solution of well-posed in $C(\mathbf{E})$ problem (2.1) with some $1 \le M_C < \infty$, which does not depend on $f(t) \in C(E)$.

It is assumed that the operator $-\mathbf{A}$ generates a semigroup $\exp\{-t\mathbf{A}\}$ $(t \ge 0)$ with exponentially decreasing norm when $t \to +\infty$, that is, the following estimates hold:

$$\|\mathbf{e}^{-t\mathbf{A}}\|_{\mathbf{E}\to\mathbf{E}} \le M\mathbf{e}^{-\delta t}. \tag{2.6}$$

Now let us consider the function $v(t)$ defined by

$$\begin{gathered} (2\mathbf{A})^{-1}\mathbf{e}^{t\mathbf{A}}v \quad \text{if } t < 0, \\ (2\mathbf{A})^{-1}\mathbf{e}^{-t\mathbf{A}}v + t\mathbf{e}^{-t\mathbf{A}}v \quad \text{if } t \ge 0. \end{gathered} \tag{2.7}$$

If $v \in D(\mathbf{A})$, then $v(t)$ is the solution $C(\mathbf{E})$ of (2.1) with $f(t) = \mathbf{e}^{-|t|\mathbf{A}}v$.

For $t > 0$, using (2.3), (2.4), and (2.6), we get the estimate

$$\begin{aligned} \|t\mathbf{A}\mathbf{e}^{-t\mathbf{A}}v\|_{\mathbf{E}} &\le \|(2^{-1}\mathbf{e}^{-t\mathbf{A}} + t\mathbf{A}\mathbf{e}^{-t\mathbf{A}})v\|_{\mathbf{E}} + \|2^{-1}\mathbf{e}^{-t\mathbf{A}}v\|_{\mathbf{E}} \\ &\le \sup_{0\le t<\infty}\|(2^{-1}\mathbf{e}^{-t\mathbf{A}} + t\mathbf{A}\mathbf{e}^{-t\mathbf{A}})v\|_{\mathbf{E}} + \frac{M}{2}\|v\|_{\mathbf{E}} \\ &\le \sup_{-\infty\le t<\infty}\|\mathbf{A}v(t)\|_{\mathbf{E}} + \frac{M}{2}\|v\|_{\mathbf{E}} \\ &\le M\|\mathbf{e}^{-|t|\mathbf{A}}v\|_{\mathbf{E}} + \frac{M}{2}\|v\|_{\mathbf{E}} \le M_1\|v\|_{\mathbf{E}}. \end{aligned} \tag{2.8}$$

Since $D(\mathbf{A})$ is dense in $\mathbf{E}$, this implies that $\mathbf{A}\mathbf{e}^{-t\mathbf{A}}$ is bounded and obeys the estimate

$$\|\mathbf{A}\mathbf{e}^{-t\mathbf{A}}\|_{\mathbf{E}\to\mathbf{E}} \le Mt^{-1}. \tag{2.9}$$

This means the analyticity of the semigroup $\mathbf{e}^{-t\mathbf{A}}$ for $t>0$ [10]. Finally, the foregoing argument shows that the analyticity of the semigroup $\mathbf{e}^{-t\mathbf{A}}$ is a necessary condition for the well-posedness of problem (2.1) in $C(\mathbf{E})$ [10].

Let $u(t)$ be a solution of the problem (2.1). Then, for any $-\infty<s\leq t$, we have the identity

$$\frac{d}{ds}\left(\mathbf{e}^{-(t-s)\mathbf{A}}\mathbf{A}^{-1}u(s)\right)=\mathbf{e}^{-(t-s)\mathbf{A}}\mathbf{A}^{-1}f(s). \tag{2.10}$$

Integrating with respect to s over the interval $[x,t]$, we obtain

$$\mathbf{A}^{-1}u(t)-\mathbf{e}^{-(t-x)\mathbf{A}}\mathbf{A}^{-1}u(x)=\int_x^t \mathbf{e}^{-(t-s)\mathbf{A}}\mathbf{A}^{-1}f(s)ds. \tag{2.11}$$

Since $\mathbf{A}$ is closed, we have

$$u(t)=\mathbf{e}^{-(t-x)\mathbf{A}}u(x)+\int_x^t \mathbf{e}^{-(t-s)\mathbf{A}}f(s)ds. \tag{2.12}$$

By the fact that the analytic semigroup $\mathbf{e}^{-p\mathbf{A}}$ has norm decaying property as $p\to\infty$,

$$u(t)=\int_{-\infty}^t \mathbf{e}^{-(t-s)\mathbf{A}}f(s)ds. \tag{2.13}$$

It is easy to see that formula (2.13) defines solution of problem (2.1) in $C(\mathbf{E})$, if, for example, $\mathbf{A}f(t)\in C(\mathbf{E})$ or $f'(t)\in C(E)$. It turns out that formula (2.13) defines solution of problem (2.1) in $C(\mathbf{E})$ under essentially less restriction on smoothness of function $f(t)$. Finally, from (2.6), (2.9), the following estimate follow:

$$\left\|\mathbf{A}^{\beta}\left[\mathbf{e}^{-t\mathbf{A}}-\mathbf{e}^{-(t+\tau)\mathbf{A}}\right]\right\|_{\mathbf{E}\to\mathbf{E}}\leq M\frac{\tau^{\alpha}}{t^{\alpha+\beta}} \tag{2.14}$$

for any $0<t<t+\tau$, $0\leq\beta\leq 1$, and $0\leq\alpha\leq 1$.

The well-posedness of problem (2.1) can be established on the assumption (2.6), (2.9) if one considers this problem in the Hölder space $C^{\alpha}(\mathbf{E})=C^{\alpha}(\mathbb{R},\mathbf{E})$, $\alpha\in(0,1)$, of all $\mathbf{E}$-valued abstract functions $\varphi(t)$ defined on $\mathbb{R}$ with the norm

$$\|\varphi\|_{C^{\alpha}(\mathbf{E})}=\sup_{-\infty<t<\infty}\|\varphi(t)\|_{\mathbf{E}}+\sup_{-\infty<t<t+\tau<\infty}\frac{\|\varphi(t+\tau)-\varphi(t)\|_{\mathbf{E}}}{\tau^{\alpha}}. \tag{2.15}$$

A function $u(t)$ is said to be a solution of problem (2.1) in $C^{\alpha}(\mathbf{E})$ if it is a solution of this problem in $C(\mathbf{E})$ and the functions $u'(t),\mathbf{A}u(t)\in C^{\alpha}(\mathbf{E})$. The well-posedness in $C^{\alpha}(\mathbf{E})$ of problem (2.1) means that coercivity inequality

$$\|u'\|_{C^{\alpha}(\mathbf{E})}+\|\mathbf{A}u\|_{C^{\alpha}(\mathbf{E})}\leq M(\alpha)\|f\|_{C^{\alpha}(\mathbf{E})} \tag{2.16}$$

holds for its solution $u(t)$ in $C^{\alpha}(\mathbf{E})$ with some $1\leq M(\alpha)<\infty$, which is independent of $f(t)\in C^{\alpha}(\mathbf{E})$.

Theorem 2.1. *The problem (2.1) is well-posed in* $C^{\alpha}(\mathbf{E})$ *and the following coercivity inequality:*

$$\|u'\|_{C^{\alpha}(\mathbf{E})}+\|\mathbf{A}u\|_{C^{\alpha}(\mathbf{E})}\leq\frac{M}{\alpha(1-\alpha)}\|f\|_{C^{\alpha}(\mathbf{E})}\tag{2.17}$$

holds for some $1\leq M<\infty$.

Proof. From formula (2.13), it follows that

$$\mathbf{A}u(t)=f(t)+\int_{-\infty}^{t}\mathbf{A}\mathbf{e}^{-\mathbf{A}(t-s)}(f(s)-f(t))ds.\tag{2.18}$$

Let us estimate $\|\mathbf{A}u\|_{C(\mathbf{E})}$. Using formula (2.18), we have

$$||\mathbf{A}u(t)||_{\mathbf{E}}\leq||f(t)||_{\mathbf{E}}+\int_{-\infty}^{t}||\mathbf{A}\mathbf{e}^{-\mathbf{A}(t-s)}||_{\mathbf{E}\to\mathbf{E}}||f(s)-f(t)||_{\mathbf{E}}ds.\tag{2.19}$$

Using estimates (2.6), (2.9), we get

$$||\mathbf{A}\mathbf{e}^{-\mathbf{A}(t-s)}||_{\mathbf{E}\to\mathbf{E}}\leq||\mathbf{A}\mathbf{e}^{-\mathbf{A}((t-s)/2)}||_{\mathbf{E}\to\mathbf{E}}||\mathbf{e}^{-\mathbf{A}(t-s)/2}||_{\mathbf{E}\to\mathbf{E}}\leq M\mathbf{e}^{-(\delta/2)(t-s)}\frac{M}{(t-s)/2}.\tag{2.20}$$

Hence,

$$||\mathbf{A}u(t)||_{\mathbf{E}}\leq\|f\|_{C^{\alpha}(\mathbf{E})}\left\{1+M_1\int_{-\infty}^{t}\frac{\mathbf{e}^{-(\delta/2)(t-s)}}{(t-s)^{1-\alpha}}ds\right\}\tag{2.21}$$

for all $t\in\mathbb{R}$.

From the substitution $u=t-s$, it becomes

$$\int_{-\infty}^{t}\frac{\mathbf{e}^{-(\delta/2)(t-s)}}{(t-s)^{1-\alpha}}ds=\int_{0}^{\infty}\frac{\mathbf{e}^{-(\delta/2)x}}{x^{1-\alpha}}dx\leq\int_{0}^{1}\frac{dx}{x^{1-\alpha}}+\int_{1}^{\infty}\mathbf{e}^{-(\delta/2)x}dx.\tag{2.22}$$

Hence,

$$||\mathbf{A}u(t)||_{E}\leq\|f\|_{C^{\alpha}(\mathbf{E})}\left\{1+M_1\left(\frac{1}{\alpha}+\frac{\mathbf{e}^{-\delta/2}}{\delta/2}\right)\right\}=\frac{M(\delta)}{\alpha}\|f\|_{C^{\alpha}(\mathbf{E})}\tag{2.23}$$

for all $t\in\mathbb{R}$. So, from that it follows

$$\|\mathbf{A}u\|_{C(\mathbf{E})}\leq\frac{M(\delta)}{\alpha}\|f\|_{C^{\alpha}(\mathbf{E})}.\tag{2.24}$$

Now, let us establish similar bound for the α-Hölder norm $H^\alpha(\mathbf{A}u)$ of $\mathbf{A}u$, where $H^\alpha(\varphi)$ denotes $\sup_{-\infty<t<t+\tau<\infty}(\|\varphi(t+\tau)-\varphi(t)\|_{\mathbf{E}}/\tau^\alpha)$. For $-\infty<t<t+\tau<\infty$, we can write

$$
\begin{aligned}
\mathbf{A}u(t+\tau)-\mathbf{A}u(t) &= f(t+\tau)-f(t)+\int_{t-\tau}^{t+\tau}\mathbf{A}e^{-A(t+\tau-s)}(f(s)-f(t+\tau))ds \\
&\quad -\int_{t-\tau}^{t}\mathbf{A}e^{-\mathbf{A}(t-s)}(f(s)-f(t))ds \\
&\quad -\int_{-\infty}^{t-\tau}\mathbf{A}[e^{-\mathbf{A}(t+\tau-s)}-e^{-\mathbf{A}(t-s)}](f(s)-f(t))ds \\
&\quad +\int_{-\infty}^{t-\tau}\mathbf{A}e^{-\mathbf{A}(t+\tau-s)}(f(t)-f(t+\tau))ds \\
&= J_1+J_2+J_3+J_4+J_5.
\end{aligned}
\tag{2.25}
$$

Clearly,

$$\|f(t+\tau)-f(t)\|_{\mathbf{E}}\le\tau^\alpha\|f\|_{C^\alpha(\mathbf{E})} \tag{2.26}$$

for all $t\in\mathbb{R}$. Then

$$\|J_1\|_{\mathbf{E}}\le\tau^\alpha\|f\|_{C^\alpha(\mathbf{E})}. \tag{2.27}$$

Using estimates (2.6), (2.9), we get

$$\|J_2\|_{\mathbf{E}}\le\int_{t-\tau}^{t+\tau}M\|\mathbf{A}e^{-\mathbf{A}(t+\tau-s)}\|_{\mathbf{E}\to\mathbf{E}}\|f(s)-f(t+\tau)\|_{\mathbf{E}}ds\le M\|f\|_{C^\alpha(\mathbf{E})}\int_{t-\tau}^{t+\tau}\frac{1}{(t+\tau-s)^{1-\alpha}}ds. \tag{2.28}$$

The use of the substitution $x=t+\tau-s$ gives

$$\int_{t-\tau}^{t+\tau}\frac{1}{(t+\tau-s)^{1-\alpha}}ds=\int_0^{2\tau}\frac{1}{x^{1-\alpha}}dx=\frac{(2\tau)^\alpha}{\alpha}, \tag{2.29}$$

from that it follows

$$\|J_2\|_{\mathbf{E}}\le M\frac{(2\tau)^\alpha}{\alpha}\|f\|_{C^\alpha(\mathbf{E})} \tag{2.30}$$

for all $t\in\mathbb{R}$. In a similar manner one establishes the estimate

$$\|J_3\|_{\mathbf{E}}\le\frac{M\tau^\alpha}{\alpha}\|f\|_{C^\alpha(\mathbf{E})}. \tag{2.31}$$

Using estimate (2.14) for $\beta=1$ and $\alpha=1$, we get

$$\|J_4\|_{\mathbf{E}}\le\int_{-\infty}^{t-\tau}M\frac{e^{-\delta(t-s)}\tau}{(t-s)^2}\|f(s)-f(t)\|_{\mathbf{E}}ds\le M\|f\|_{C^\alpha(\mathbf{E})}\tau\int_{-\infty}^{t-\tau}\frac{ds}{(t-s)^{2-\alpha}}=\frac{M\tau^\alpha}{1-\alpha}\|f\|_{C^\alpha(\mathbf{E})} \tag{2.32}$$

for all $t \in \mathbb{R}$. Then

$$\|J_4\|_{\mathbf{E}} \leq \frac{M\tau^{\alpha}}{1-\alpha}\|f\|_{C^{\alpha}(\mathbf{E})}. \tag{2.33}$$

Finally, using the formula

$$\int_{-\infty}^{t-\tau} \mathbf{A}\mathbf{e}^{-\mathbf{A}(t+\tau-s)}ds = \mathbf{e}^{-2\tau A}, \tag{2.34}$$

and estimates (2.6), (2.9), we get

$$\|J_5\|_{\mathbf{E}} \leq \|\mathbf{e}^{-2\tau\mathbf{A}}\|_{\mathbf{E}\to\mathbf{E}}\|f(t)-f(t+\tau)\|_{\mathbf{E}} \leq M\tau^{\alpha}\|f\|_{C^{\alpha}(\mathbf{E})} \tag{2.35}$$

for all $t \in \mathbb{R}$. Then

$$\|J_5\|_{\mathbf{E}} \leq M\tau^{\alpha}\|f\|_{C^{\alpha}(\mathbf{E})}. \tag{2.36}$$

Combining all these, and using estimate (2.24), we get

$$\|\mathbf{A}u\|_{C^{\alpha}(\mathbf{E})} \leq \frac{M}{\alpha(1-\alpha)}\|f\|_{C^{\alpha}(\mathbf{E})}. \tag{2.37}$$

By the triangle inequality, this last estimate and (2.1) yield

$$\|u'\|_{C^{\alpha}(\mathbf{E})} \leq \frac{M}{\alpha(1-\alpha)}\|f\|_{C^{\alpha}(\mathbf{E})}. \tag{2.38}$$

Theorem 2.1 is proved. □

Note that the proof of Theorem 2.1 can also be considered a new proof of a particular case of a well-known result [21]. More precisely, if we assume that

(i) $\sqrt{-1}\mathbb{R} \subset \rho(\mathbf{A})$;

(ii) there is $M \in \mathbb{R}^+ = \{z \in \mathbb{R};\ z > 0\}$, such that

$$\forall\omega \in \mathbb{R}, \quad \|(\sqrt{-1}\omega + A)^{-1}\|_{\mathbf{E}\to\mathbf{E}} \leq (1+|\omega|)^{-1}, \tag{2.39}$$

then as a consequence of [21, Theorem 8.2], Theorem 2.1 can be obtained.

3. Almost coercivity inequality

The difference analogue of the differential equation (2.1)

$$\frac{u_k - u_{k-1}}{\tau} + \mathbf{A}u_k = f_k, \quad k \in \mathbb{Z}, \tag{3.1}$$

will be considered. Here $u_k \in D(\mathbf{A})$ and $f_k \in \mathbf{E}$ are unknown and given elements, τ is a positive small number.

The Banach space $\mathscr{C}(\mathbb{R}_{\tau}, \mathbf{E})$ of all bounded grid functions $v^{\tau} = \{v_k\}_{k=-\infty}^{\infty}$ defined on $\mathbb{R}_{\tau} = \{t_k = k\tau;\ k \in \mathbb{Z}\}$ with the norm

$$\|v^{\tau}\|_{\mathscr{C}(\mathbb{R}_{\tau},\mathbf{E})} = \sup_{-\infty<k<\infty}\|v_k\|_{\mathbf{E}} \tag{3.2}$$

is introduced and the operator $\mathfrak{D}_\tau$, acting from the space $\mathcal{C}(\mathbb{R}_\tau,\mathbf{E})$ into the space $\mathcal{C}(\mathbb{R}_\tau,\mathbf{E})$, by the rule

$$v^\tau = \mathfrak{D}_\tau u^\tau, \qquad v_k = \frac{u_k - u_{k-1}}{\tau}, \quad k \in \mathbb{Z}, \tag{3.3}$$

is defined. Then the difference equation (3.1) will be considered as operator equation

$$\mathfrak{D}_\tau u^\tau + \mathbf{A}u^\tau = f^\tau \tag{3.4}$$

in the Banach space $\mathcal{C}(\mathbb{R}_\tau,\mathbf{E})$. Here $\mathbf{A}u^\tau = \{\mathbf{A}u_k\}_{k=-\infty}^{\infty}$ and $f^\tau = \{f_k\}_{k=-\infty}^{\infty}$.

From the property (2.6), (2.9), it follows that there exists the bounded operator $(I+\tau\mathbf{A})^{-1}$, that is, the resolvent $\mathfrak{R}(\tau\mathbf{A})$, defined on whole space $\mathbf{E}$. Therefore, for every f^τ, there exists a unique solution $u^\tau = u^\tau(f^\tau)$ of the problem (3.4) and the following formula holds:

$$u_k = \sum_{i=-\infty}^{k} \mathfrak{R}^{k-i+1}(\tau\mathbf{A}) f_i \quad \tau, k \in \mathbb{Z}. \tag{3.5}$$

Let the assumption (2.6), (2.9) be satisfied. Since the semigroup $\mathbf{e}^{-t\mathbf{A}}$ obeys the exponential decay estimates (2.6), (2.9), we have that

$$\|\mathfrak{R}^k(\tau\mathbf{A})\|_{\mathbf{E}\to\mathbf{E}} \le M(1+\tau\delta)^{-k}, \quad k \ge 1, \tag{3.6}$$

$$\|k\tau\mathbf{A}\mathfrak{R}^k(\tau\mathbf{A})\|_{\mathbf{E}\to\mathbf{E}} \le M, \quad k \ge 1. \tag{3.7}$$

Actually, from the formula connecting the resolvent of the generator of a semigroup with the semigroup (see [20]) it follows that

$$(I+\tau\mathbf{A})^{-k} = \frac{1}{(k-1)!}\int_0^\infty t^{k-1}\mathbf{e}^{-t}\mathbf{e}^{-\tau t\mathbf{A}}dt. \tag{3.8}$$

Using this formula and (2.6), we get

$$\|(I+\tau\mathbf{A})^{-k}\|_{\mathbf{E}\to\mathbf{E}} \le \frac{M}{(k-1)!}\int_0^\infty t^{k-1}\mathbf{e}^{-t(1+\delta\tau)}dt = M(1+\delta\tau)^{-k}. \tag{3.9}$$

Estimate (3.6) is proved. For $k \ge 2$ using (2.6), (2.9), (3.8) and the fact that the operator $\mathbf{A}$ is closed, this yields the estimate

$$\|\mathbf{A}(I+\tau\mathbf{A})^{-k}\|_{\mathbf{E}\to\mathbf{E}} \le \frac{M}{\tau(k-1)!}\int_0^\infty t^{k-2}\mathbf{e}^{-t}dt = \frac{M}{\tau(k-1)(1+\delta\tau)^{k-1}} \le \frac{4\delta M}{\tau k(1+\delta\tau)^k}, \tag{3.10}$$

where the last inequality results from $0 \le \tau \le 1$. Therefore, (3.7) is proved for $k \ge 2$. For $k = 1$, the estimate is obvious.

From (3.6), (3.7), the following estimates follow:

$$||\mathbf{A}^{\beta}[\mathfrak{R}^{k}(\tau\mathbf{A})-\mathfrak{R}^{k+m}(\tau\mathbf{A})]||_{\mathbf{E}\to\mathbf{E}}\leq M\frac{(m\tau)^{\alpha}}{(k\tau)^{\alpha+\beta}} \tag{3.11}$$

for any $1\leq k<k+m$, $0\leq\alpha\leq 1$, and $0\leq\beta\leq 1$.

The problem (3.4) is said to be stable in $\mathscr{C}(\mathbb{R}_{\tau},\mathbf{E})$ if we have the stability inequality

$$||u^{\tau}||_{\mathscr{C}(\mathbb{R}_{\tau},\mathbf{E})}\leq M||f^{\tau}||_{\mathscr{C}(\mathbb{R}_{\tau},\mathbf{E})}, \tag{3.12}$$

where M is independent not only of f^{τ} but also of τ.

Theorem 3.1. *The problem (3.4) is stable in $\mathscr{C}(\mathbb{R}_{\tau},\mathbf{E})$ norm.*

The proof of Theorem 3.1 is based on formula (3.5) and estimates (3.6), (3.7).

The problem (3.4) is said to be coercively stable (well-posed) in $\mathscr{C}(\mathbb{R}_{\tau},\mathbf{E})$ if we have the coercive stability

$$||\mathbf{A}u^{\tau}||_{\mathscr{C}(\mathbb{R}_{\tau},\mathbf{E})}\leq M||f^{\tau}||_{\mathscr{C}(\mathbb{R}_{\tau},\mathbf{E})}, \tag{3.13}$$

where M is independent not only of f^{τ} but also of τ.

Since the problem (2.1) in the space $C(\mathbb{R},\mathbf{E})$ is not well-posed for the general positive operator $\mathbf{A}$ and space $\mathbf{E}$, then the well-posedness of the difference problem (3.4) in $\mathscr{C}(\mathbb{R}_{\tau},\mathbf{E})$ norm does not take place uniformly with respect to $\tau>0$. This means that the coercivity norm

$$||u^{\tau}||_{\mathscr{K}_{\tau}(\mathbf{E})}=||\mathbf{A}u^{\tau}||_{\mathscr{C}(\mathbb{R}_{\tau},\mathbf{E})}+||\mathfrak{D}_{\tau}u^{\tau}||_{\mathscr{C}(\mathbb{R}_{\tau},\mathbf{E})} \tag{3.14}$$

tends to ∞ as $\tau\to 0^{+}$. The investigation of the difference problem (3.4) permits to establish the order of growth of this norm to ∞.

Theorem 3.2. *For the solution of the difference problem (3.4), we have the almost coercivity inequality*

$$||u^{\tau}||_{\mathscr{K}_{\tau}(\mathbf{E})}\leq M\min\left\{\ln\frac{1}{\tau},\ 1+|\ln\|\mathbf{A}\|_{\mathbf{E}\to\mathbf{E}}|\right\}||f^{\tau}||_{\mathscr{C}(\mathbb{R}_{\tau},\mathbf{E})}. \tag{3.15}$$

Proof. Using formula (3.5) and the substitution $m=k-i+1$, we get

$$\begin{aligned}\mathbf{A}u_{k}&=\sum_{i=-\infty}^{k}\mathbf{A}\mathfrak{R}^{k-i+1}(\tau\mathbf{A})f_{i}\tau=\sum_{m=1}^{\infty}\mathbf{A}\mathfrak{R}^{m}(\tau\mathbf{A})f_{k-m+1}\tau\\&=\sum_{m=1}^{[1/\tau]}\mathbf{A}\mathfrak{R}^{m}(\tau\mathbf{A})f_{k-m+1}\tau+\sum_{m=[1/\tau]+1}^{\infty}\mathbf{A}\mathfrak{R}^{m}(\tau\mathbf{A})f_{k-m+1}\tau=J_{1}+J_{2},\end{aligned} \tag{3.16}$$

where $[\cdot]$ stands for the integer part.

Let us estimate J_2. Using estimates (3.6), (3.7), we obtain

$$\begin{aligned}||J_2||_{\mathbf{E}} &\leq \sum_{m=[1/\tau]+1}^{\infty} ||\mathbf{A}\mathfrak{R}^{[m/2]}\mathfrak{R}^{m-[m/2]}||_{\mathbf{E}\to\mathbf{E}}||f_{k-m+1}||_{\mathbf{E}} \\ &\leq M||f^{\tau}||_{\mathscr{C}(\mathbb{R}_\tau,\mathbf{E})} \sum_{m=[1/\tau]+1}^{\infty} \frac{1}{(1+\tau\delta)^{m/2}m} \\ &\leq M||f^{\tau}||_{\mathscr{C}(\mathbb{R}_\tau,\mathbf{E})} \frac{1}{[1/\tau]+1} \sum_{m=[1/\tau]+1}^{\infty} \frac{1}{(1+\tau\delta)^{m/2}} \\ &\leq M(\delta)||f^{\tau}||_{\mathscr{C}(\mathbb{R}_\tau,\mathbf{E})}.\end{aligned} \tag{3.17}$$

Let us estimate J_1. It is clear that

$$\sum_{m=1}^{[1/\tau]} \tau||\mathbf{A}\mathfrak{R}^{m}(\tau\mathbf{A})f_{k-m+1}||_{E} \leq \sum_{m=1}^{[1/\tau]} \tau||\mathbf{A}\mathfrak{R}^{m}(\tau\mathbf{A})||_{E\to E}||f^{\tau}||_{\mathscr{C}(\mathbb{R}_\tau,\mathbf{E})}. \tag{3.18}$$

By [10, Theorem 1.2, page 87],

$$\sum_{m=1}^{[1/\tau]} \tau||\mathbf{A}\mathfrak{R}^{m}(\tau\mathbf{A})||_{\mathbf{E}\to\mathbf{E}} \leq M\min\left\{\ln\left[\frac{1}{\tau}\right],\ 1+|\ln\|\mathbf{A}\|_{\mathbf{E}\to\mathbf{E}}|\right\}. \tag{3.19}$$

Thus,

$$||J_1||_{\mathbf{E}} \leq M\min\left\{\ln\left[\frac{1}{\tau}\right],\ 1+|\ln\|\mathbf{A}\|_{\mathbf{E}\to\mathbf{E}}|\right\}||f^{\tau}||_{\mathscr{C}(\mathbb{R}_\tau,\mathbf{E})}. \tag{3.20}$$

Combining the estimates for $\|J_1\|_{\mathbf{E}}$ and $\|J_2\|_{\mathbf{E}}$, we obtain

$$||\mathbf{A}u_k||_{\mathbf{E}} \leq M\min\left\{\ln\left[\frac{1}{\tau}\right],\ 1+|\ln\|\mathbf{A}\|_{\mathbf{E}\to\mathbf{E}}|\right\}||f^{\tau}||_{\mathscr{C}(\mathbb{R}_\tau,\mathbf{E})} \tag{3.21}$$

for all k. It follows from that

$$||\mathbf{A}u^{\tau}||_{\mathscr{C}(\mathbb{R}_\tau,\mathbf{E})} \leq M\min\left\{\ln\frac{1}{\tau},\ 1+|\ln\|\mathbf{A}\|_{\mathbf{E}\to\mathbf{E}}|\right\}||f^{\tau}||_{\mathscr{C}(\mathbb{R}_\tau,\mathbf{E})}. \tag{3.22}$$

By the triangle inequality, this last estimate and (3.4) yield

$$||\mathfrak{D}_\tau u^{\tau}||_{\mathscr{C}(\mathbb{R}_\tau,\mathbf{E})} \leq M_1\min\left\{\ln\frac{1}{\tau},\ 1+|\ln\|\mathbf{A}\|_{\mathbf{E}\to\mathbf{E}}|\right\}||f^{\tau}||_{\mathscr{C}(\mathbb{R}_\tau,\mathbf{E})}. \tag{3.23}$$

Theorem 3.2 is proved. □

Finally, in the next section the theorem on the well-posedness of difference scheme (3.1) in the difference analogy of $C^{\alpha}(\mathbb{R},\mathbf{E})$, $0<\alpha<1$, spaces are established.

4. Well-posedness of difference scheme

Now, the difference equation (3.1) is considered as operator equation (3.4) in the Banach space $\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})$ $(0<\alpha<1)$ of grid functions $\varphi^\tau=\{\varphi_k\}_{k=-\infty}^{\infty}$ with norm

$$\|\varphi^\tau\|_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})}=\|\varphi^\tau\|_{\mathscr{C}(\mathbb{R}_\tau,\mathbf{E})}+\sup_{-\infty<i<i+r<\infty}\frac{\|\varphi_{i+r}-\varphi_i\|_{\mathbf{E}}}{(r\tau)^\alpha}. \tag{4.1}$$

The well-posedness of (3.4) in the space $\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})$ means that for solutions u^τ of (3.4) in $\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})$ coercive inequality

$$\|\mathfrak{D}_\tau u^\tau\|_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})}+\|\mathbf{A}u^\tau\|_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})}\le M_C(\alpha)\|f^\tau\|_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})} \tag{4.2}$$

holds for some $1\le M_C(\alpha)<\infty$, which is independent of f^τ and positive small number τ.

Theorem 4.1. *The difference equation (3.4) is well-possed in Banach space $\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})$ $(0<\alpha<1)$ and for its solutions coercivity inequality*

$$\|\mathfrak{D}_\tau u^\tau\|_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})}+\|\mathbf{A}u^\tau\|_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})}\le \frac{M}{\alpha(1-\alpha)}\|f^\tau\|_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})} \tag{4.3}$$

holds for some $1\le M<\infty$, which does not depend on $f^\tau\in\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})$, $\alpha\in(0,1)$, and positive small number τ.

Proof. Let us estimate $\|\mathbf{A}u^\tau\|_{\mathscr{C}(\mathbb{R}_\tau,\mathbf{E})}$. Using formula (3.5), estimates (3.6), (3.7), and the identity

$$\tau\mathbf{A}\mathfrak{R}^{k-i+1}=\mathfrak{R}^{k-i}-\mathfrak{R}^{k-i+1}, \tag{4.4}$$

we obtain

$$\mathbf{A}u_k=\sum_{i=-\infty}^{k-1}\tau\mathbf{A}\mathfrak{R}^{k-i+1}(\tau\mathbf{A})(f_i-f_k)+f_k. \tag{4.5}$$

The estimates (3.6), (3.7), (3.11), and the substitution $j=k-i+1$ will imply

$$\begin{aligned}\|\mathbf{A}u_k\|_{\mathbf{E}}&\le\sum_{i=-\infty}^{k-1}\|\tau\mathbf{A}\mathfrak{R}^{k-i+1}(\tau\mathbf{A})\|_{\mathbf{E}\to\mathbf{E}}\|f_i-f_k\|_{\mathbf{E}}+\|f_k\|_{\mathbf{E}}\\&=\sum_{j=2}^{\infty}\|\tau\mathbf{A}\mathfrak{R}^{j}(\tau\mathbf{A})\|_{\mathbf{E}\to\mathbf{E}}\|f_{k-j+1}-f_k\|_{\mathbf{E}}+\|f_k\|_{\mathbf{E}}\\&\le\|f^\tau\|_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})}\left(\sum_{j=2}^{\infty}\frac{M\tau^\alpha}{(1+\tau\delta)^{j/2}j^{1-\alpha}}+1\right)\end{aligned} \tag{4.6}$$

for all k. Splitting the sum into two parts as $j < N = [1/\tau]$ and $j \geq N$, we can write

$$\begin{aligned}||\mathbf{A}u_k||_{\mathbf{E}} &\leq ||f^\tau||_{\mathcal{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})}\left(\sum_{j=2}^{N-1}\frac{M\tau^\alpha}{(1+\tau\delta)^{j/2}j^{1-\alpha}}+\sum_{j=N}^{\infty}\frac{M\tau^\alpha}{(1+\tau\delta)^{j/2}j^{1-\alpha}}+1\right)\\ &= ||f^\tau||_{\mathcal{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})}(S_1+S_2+1).\end{aligned} \tag{4.7}$$

From $N = [1/\tau]$, it follows that S_2 is bounded by

$$\sum_{j=N}^{\infty}\frac{M\tau}{(1+\tau\delta)^{j/2}} \leq M(\delta). \tag{4.8}$$

Next, let us estimate S_1. Using the estimate $N\tau \leq 1$ and

$$\sum_{j=1}^{N-1}\frac{M\tau}{(\tau j)^{1-\alpha}} \leq \int_0^1\frac{M ds}{s^{1-\alpha}}, \tag{4.9}$$

we obtain

$$S_1 \leq \frac{M}{\alpha}. \tag{4.10}$$

Therefore, from (4.8) and (4.10), it follows that for all k,

$$||\mathbf{A}u_k||_{\mathbf{E}} \leq \frac{M(\delta)}{\alpha}||f^\tau||_{\mathcal{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})}. \tag{4.11}$$

Hence, we obtain

$$||\mathbf{A}u^\tau||_{\mathcal{C}(\mathbb{R}_\tau,\mathbf{E})} \leq \frac{M(\delta)}{\alpha}||f^\tau||_{\mathcal{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})}. \tag{4.12}$$

Next, the estimate for the α-Hölder norm $H^\alpha(\mathbf{A}u^\tau)$ of $\mathbf{A}u^\tau$ will be established, where

$$H^\alpha(\varphi^\tau) = \sup_{-\infty<k<k+r<\infty}\frac{||\varphi_{k+r}-\varphi_k||_{\mathbf{E}}}{(r\tau)^\alpha}. \tag{4.13}$$

From (4.5), it follows that

$$\begin{aligned}\mathbf{A}u_{k+r}-\mathbf{A}u_k &= \sum_{i=k-r}^{k+r-1}\tau\mathbf{A}\mathfrak{R}^{k+r-i+1}(\tau\mathbf{A})(f_i-f_{k+r})-\sum_{i=k-r}^{k-1}\tau\mathbf{A}\mathfrak{R}^{k-i+1}(\tau\mathbf{A})(f_i-f_k)\\ &\quad+\sum_{i=-\infty}^{k-r-1}\tau\mathbf{A}\left[\mathfrak{R}^{k+r-i+1}(\tau\mathbf{A})-\mathfrak{R}^{k-i+1}(\tau\mathbf{A})\right](f_i-f_k)\\ &\quad+\sum_{i=-\infty}^{k-r-1}\tau\mathbf{A}\mathfrak{R}^{k+r-i+1}(\tau\mathbf{A})(f_k-f_{k+r})+(f_{k+r}-f_k)\\ &= J_1+J_2+J_3+J_4+J_5.\end{aligned} \tag{4.14}$$

Let us estimate $\{J_m\}_{m=1}^{5}$ for $m = 1,\dots,5$ separately. Let us estimate $\|J_1\|_{\mathbf{E}}$. Using estimates (3.6), (3.7), and the substitution $j = k+r-i$, we get

$$\begin{aligned} ||J_1||_{\mathbf{E}} &\le \sum_{i=k-r}^{k+r-1} ||\tau\mathbf{A}\mathfrak{R}^{k+r-i+1}(\tau\mathbf{A})||_{\mathbf{E}\to\mathbf{E}}||f_i - f_{k+r}||_{\mathbf{E}} \\ &\le M||f^\tau||_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})} \sum_{i=k-r}^{k+r-1} \frac{\tau^\alpha}{(k+r-i)^{1-\alpha}} = M||f^\tau||_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})} \sum_{j=1}^{2r} \frac{\tau^\alpha}{j^{1-\alpha}}. \end{aligned} \tag{4.15}$$

It results in

$$||J_1||_{\mathbf{E}} \le (r\tau)^\alpha \frac{M2^\alpha}{\alpha}||f^\tau||_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})}. \tag{4.16}$$

In a similar manner, we can show that

$$||J_2||_{\mathbf{E}} \le (r\tau)^\alpha \frac{M}{\alpha}||f^\tau||_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})}. \tag{4.17}$$

Now, let us estimate $\|J_3\|_{\mathbf{E}}$. Using the estimate (3.11) for $\alpha = 1$ and $\beta = 1$, we obtain

$$\begin{aligned} ||J_3||_{\mathbf{E}} &\le \sum_{i=-\infty}^{k-r-1} ||\tau\mathbf{A}[\mathfrak{R}^{k+r-i+1}(\tau\mathbf{A}) - \mathfrak{R}^{k-i+1}(\tau\mathbf{A})]||_{\mathbf{E}\to\mathbf{E}}||f_i - f_k||_{\mathbf{E}} \\ &\le M||f^\tau||_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})} \sum_{i=-\infty}^{k-r-1} \frac{r\tau^\alpha}{(k-i+1)^{2-\alpha}}. \end{aligned} \tag{4.18}$$

The substitution $j = k-i+1$ gives

$$||J_3||_{\mathbf{E}} \le M||f^\tau||_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})} \sum_{j=r+2}^{\infty} \frac{r\tau^\alpha}{j^{2-\alpha}} \le (r\tau)^\alpha M_1||f^\tau||_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})} \frac{1}{(1-\alpha)} \tag{4.19}$$

for all k. Thus, we have

$$||J_3||_{\mathbf{E}} \le (r\tau)^\alpha \frac{M_1}{1-\alpha}||f^\tau||_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})}. \tag{4.20}$$

Next, let us establish an estimate for $\|J_4\|_{\mathbf{E}}$. Using estimates (3.6), (3.7), and the identity (4.4), we get

$$J_4 = \mathfrak{R}^{2r+1}(\tau\mathbf{A})(f_k - f_{k+r}). \tag{4.21}$$

From estimates (3.6), (3.7), it follows

$$||J_4||_{\mathbf{E}} \le ||\mathfrak{R}^{2r+1}(\tau\mathbf{A})||_{\mathbf{E}\to\mathbf{E}}||f_k - f_{k+r}||_{\mathbf{E}} \le M(r\tau)^\alpha||f^\tau||_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})} \tag{4.22}$$

for all k. Then

$$||J_4||_{\mathbf{E}} \le M(r\tau)^\alpha||f^\tau||_{\mathscr{C}^\alpha(\mathbb{R}_\tau,\mathbf{E})}. \tag{4.23}$$

Combining all the estimates for $\{J_m\}_{m=1}^{5}$, we get

$$H^{\alpha}(\mathbf{A}u^{\tau}) \leq \frac{M_2}{(1-\alpha)\alpha}\|f^{\tau}\|_{\mathscr{C}^{\alpha}(\mathbb{R}_{\tau},\mathbf{E})}. \tag{4.24}$$

Using estimates (4.12) and (4.24), we get

$$\|\mathbf{A}u^{\tau}\|_{\mathscr{C}^{\alpha}(\mathbb{R}_{\tau},\mathbf{E})} \leq \frac{M_2}{(1-\alpha)\alpha}\|f^{\tau}\|_{\mathscr{C}^{\alpha}(\mathbb{R}_{\tau},\mathbf{E})}. \tag{4.25}$$

Then estimate

$$\|\mathfrak{D}_{\tau}u^{\tau}\|_{\mathscr{C}^{\alpha}(\mathbb{R}_{\tau},\mathbf{E})} \leq \frac{M_2}{(1-\alpha)\alpha}\|f^{\tau}\|_{\mathscr{C}^{\alpha}(\mathbb{R}_{\tau},\mathbf{E})} \tag{4.26}$$

follows from the triangle inequality, estimate (4.25), and (3.4).

This finishes the proof of Theorem 4.1. □

Note that for any $0<\alpha<1$ the norms in the spaces $\mathbf{E}_{\alpha}(\mathscr{C}(\mathbb{R}_{\tau}),\mathfrak{D}_{\tau}+I_{\tau})$ and $\mathscr{C}^{\alpha}(\mathbb{R}_{\tau})$ are equivalent uniformly in τ (see [9, 22]). Here, $\mathbf{E}_{\alpha}=\mathbf{E}_{\alpha}(\mathbf{E},\mathbf{B})$ $(0<\alpha<1)$ denotes the Banach space of all $v\in\mathbf{E}$ for which the following norm is finite:

$$\|v\|_{\mathbf{E}_{\alpha}} = \sup_{\lambda>0}\lambda^{\alpha}\|\mathbf{B}(\lambda+\mathbf{B})^{-1}v\|_{\mathbf{E}}. \tag{4.27}$$

Then the application of Grisvard's theory also permits to establish the well-posedness of difference problem (3.4) in $\mathscr{C}^{\alpha}(\mathbb{R}_{\tau},\mathbf{E})$ and to obtain the following coercivity inequality:

$$\|\mathfrak{D}_{\tau}u^{\tau}\|_{\mathscr{C}^{\alpha}(\mathbb{R}_{\tau},\mathbf{E})} + \|\mathbf{A}u^{\tau}\|_{\mathscr{C}^{\alpha}(\mathbb{R}_{\tau},\mathbf{E})} \leq \frac{M}{\alpha^{2}(1-\alpha)}\|f^{\tau}\|_{\mathscr{C}^{\alpha}(\mathbb{R}_{\tau},\mathbf{E})} \tag{4.28}$$

for some $1\leq M<\infty$, which does not depend on $f^{\tau}\in\mathscr{C}^{\alpha}(\mathbb{R}_{\tau},\mathbf{E})$, $\alpha\in(0,1)$ and positive small number τ.

If the coercive stability estimates of Theorem 4.1 and the passing to the limit for $\tau\to 0^{+}$ are considered, one can recover Theorem 2.1 on the well-posedness of the problem (2.1) in $C^{\alpha}(\mathbf{E})$, $0<\alpha<1$, spaces.

Of course, well-posedness and almost coercivity inequality could be also established for the more general Padé difference schemes of the high order of accuracy (see [18, 19]).

Acknowledgments

The authors are grateful to all referees for their critical reading and insightful suggestions to improve the manuscript. The part of the article was done during the third author's visit to Universidade Federal do Ceará, Brasil. This research was supported by the CNPq Grant no. 310092/2003-5.

References

[1] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, "Nauka", Moscow, Russia, 1967.

[2] M. I. Vishik, A. D. Myshkis, and O. A. Oleinik, "Partial differential equations," in *Mathematics in USSR in the Last 40 Years, 1917–1957, Vol. 1*, pp. 563–599, Fizmatgiz, Moscow, Russia, 1959.

[3] P. E. Sobolevskii, "Well-posedness of difference elliptic equation," *Discrete Dynamics in Nature and Society*, vol. 1, no. 3, pp. 219–231, 1997.

[4] P. E. Sobolevskii, "Coerciveness inequalities for abstract parabolic equations," *Doklady Akademii Nauk SSSR*, vol. 157, no. 1, pp. 52–55, 1964 (Russian).

[5] P. E. Sobolevskii, "Some properties of the solutions of differential equations in fractional spaces," *Trudy Naucno-Issledovatel'skogo Instituta Matematiki VGU*, vol. 14, pp. 68–74, 1975 (Russian), [RZh. Mat. 1975:7 B825].

[6] G. Da Prato and P. Grisvard, "Sommes d'opérateurs linéaires et équations différentielles opérationnelles," *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, vol. 54, no. 3, pp. 305–387, 1975.

[7] G. Da Prato and P. Grisvard, "Équations d'évolution abstraites non linéaires de type parabolique," *Comptes Rendus de l'Académie des Sciences Série A-B*, vol. 283, no. 9, pp. A709–A711, 1976.

[8] A. Ashyralyev, A. Hanalyev, and P. E. Sobolevskii, "Coercive solvability of the nonlocal boundary value problem for parabolic differential equations," *Abstract and Applied Analysis*, vol. 6, no. 1, pp. 53–61, 2001.

[9] A. Ashyralyev and P. E. Sobolevskii, *Positive Operators and the Fractional Spaces. The Methodical Instructions for the Students of Engineering Grups*, Offset Laboratory VSU, Voronezh, Russia, 1989.

[10] A. Ashyralyev and P. E. Sobolevskii, *Well-Posedness of Parabolic Difference Equations*, vol. 69 of *Operator Theory: Advances and Applications*, Birkhäuser, Basel, Switzerland, 1994.

[11] A. Ashyralyev, I. Karatay, and P. E. Sobolevskii, "On well-posedness of the nonlocal boundary value problem for parabolic difference equations," *Discrete Dynamics in Nature and Society*, vol. 2004, no. 2, pp. 273–286, 2004.

[12] A. E. Polička and P. E. Sobolevskii, "Correct solvability of parabolic difference equations in Bochner spaces," *Trudy Moskovskogo Matematicheskogo Obshchestva*, vol. 36, pp. 29–57, 294, 1978 (Russian).

[13] A. Ashyralyev, "Nonlocal boundary-value problems for abstract parabolic equations: well-posedness in Bochner spaces," *Journal of Evolution Equations*, vol. 6, no. 1, pp. 1–28, 2006.

[14] P. E. Sobolevskii, "The coercive solvability of difference equations," *Doklady Akademii Nauk SSSR*, vol. 201, no. 5, pp. 1063–1066, 1971 (Russian).

[15] A. Ashyralyev and P. E. Sobolevskii, "The theory of interpolation of linear operators and the stability of difference schemes," *Doklady Akademii Nauk SSSR*, vol. 275, no. 6, pp. 1289–1291, 1984 (Russian).

[16] A. Ashyralyev, S. Piskarev, and L. Weis, "On well-posedness of difference schemes for abstract parabolic equations in $L^p([0,1];E)$ spaces," *Numerical Functional Analysis & Optimization*, vol. 23, no. 7-8, pp. 669–693, 2002.

[17] D. Guidetti, B. Karasözen, and S. Piskarev, "Approximation of abstract differential equations," *Journal of Mathematical Sciences (New York)*, vol. 122, no. 2, pp. 3013–3054, 2004.

[18] A. Ashyralyev and P. E. Sobolevskii, "Well-posed solvability of the Cauchy problem for difference equations of parabolic type," *Nonlinear Analysis*, vol. 24, no. 2, pp. 257–264, 1995.

[19] A. Ashiraliev and P. E. Sobolevskii, "Differential schemes of the high-order accuracy for parabolic equations with variable-coefficients," *Dopovidi Akademii Nauk Ukrainskoi RSR, Seriya À-Fiziko-Matematichni ta Technichni Nauki*, vol. 6, pp. 3–7, 1988 (Russian).

[20] A. Ashyralyev and P. E. Sobolevskii, *New Difference Schemes for Partial Differential Equations*, vol. 148 of *Operator Theory: Advances and Applications*, Birkhäuser, Basel, Switzerland, 2004.

[21] H. Amann, "Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications," *Mathematische Nachrichten*, vol. 186, pp. 5–56, 1997.

[22] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, vol. 18 of *North-Holland Mathematical Library*, North-Holland, Amsterdam, The Netherlands, 1978.

Allaberen Ashyralyev: Department of Mathematics, Fatih University, 34500 Istanbul, Turkey
Email address: aashyr@fatih.edu.tr

Yasar Sozen: Department of Mathematics, Fatih University, 34500 Istanbul, Turkey
Email address: ysozen@fatih.edu.tr

Pavel E. Sobolevskii: Institute of Mathematics, Hebrew University, 91904 Jerusalem, Israel
Current address: Universidade Federal do Ceará, 60020 Ceará, Brazil
Email address: pavels@math.huji.ac.il