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Research Article
The Convolution on Time Scales

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The main theme in this paper is an initial value problem containing a dynamic version of the transport equation. Via this problem, the delay (or shift) of a function defined on a time scale is introduced, and the delay in turn is used to introduce the convolution of two functions defined on the time scale. In this paper, we give some elementary properties of the delay and of the convolution and we also prove the convolution theorem. Our investigation contains a study of the initial value problem under consideration as well as some results about power series on time scales. As an extensive example, we consider the q-difference equations case.

1. Introduction

As is known, the methods connected to the employment of integral transformations are very useful in mathematical analysis. Those methods are successfully applied to solve differential and integral equations, to study special functions, and to compute integrals. The main advantage of the method of integral transformations is the possibility to prepare tables of direct and inverse transformations of various functions frequently encountered in applications (the role of those tables is similar to that of derivative and integral tables in calculus).

One of the more widely used integral transformations is the Laplace transform. If f is a given function and if the integral

$$F(z) = \int_0^{\infty} f(t)e^{-zt}dt \tag{1.1}$$

exists, then the function F of a complex variable is called the Laplace transform of the original function f. The operation which yields F from a given f is also called the Laplace transform (or Laplace transformation). The original function f is called the inverse transform or inverse of F.

A discrete analogue of the Laplace transform is the so-called Z-transform, which can be used to solve linear difference equations as well as certain summation equations being discrete analogues of integral equations. If we have a sequence $\{y_k\}_{k=0}^{\infty}$, then its Z-transform is a function Y of a complex variable defined by

$$Y(z) = \sum_{k=0}^{\infty} \frac{y_k}{z^k}. \tag{1.2}$$

The Laplace transform on time scales (note that time scales analysis unifies and extends continuous and discrete analysis, see [1, 2]) is introduced by Hilger in [3], but in a form that tries to unify the (continuous) Laplace transform and the (discrete) Z-transform. For arbitrary time scales, the Laplace transform is introduced and investigated by Bohner and Peterson in [4] (see also [1, Section 3.10]).

An important general property of the classical Laplace transform is the so-called convolution theorem. It often happens that we are given two transforms $F(z)$ and $G(z)$ whose inverses f and g we know, and we would like to calculate the inverse h of the product $H(z) = F(z)G(z)$ from those known inverses f and g. The inverse h is written $f * g$ and is called the convolution of f and g. The classical convolution theorem states that H is the Laplace transform of the convolution h of f and g defined by

$$h(t) = (f * g)(t) = \int_0^t f(t-s)g(s)ds. \tag{1.3}$$

The main difficulty which arises when we try to introduce the convolution for functions defined on an arbitrary time scale $\mathbb{T}$ is that, if t and s are in the time scale $\mathbb{T}$, then it does not necessarily follow that the difference $t-s$ is also an element of $\mathbb{T}$ so that $f(t-s)$, the shift (or delay) of the function f, is not necessarily defined if f is defined only on the time scale $\mathbb{T}$. In [4], this difficulty is overcome in case of pairs of functions, in which one of the functions is an "elementary function." In the present paper, we offer a way to define the "shift" and hence the convolution of two "arbitrary" functions defined on the time scale $\mathbb{T}$. The idea of doing so is the observation that the usual shift $f(t-s) = u(t,s)$ of the function f defined on the real line $\mathbb{R}$ is the unique solution of the problem (for the first-order partial differential equation)

$$\frac{\partial u(t,s)}{\partial t} = -\frac{\partial u(t,s)}{\partial s}, \quad t,s \in \mathbb{R}, \qquad u(t,0) = f(t), \quad t \in \mathbb{R}. \tag{1.4}$$

This suggests to introduce the "shift" $\widehat{f}(t,s)$ of the given function f defined on a time scale $\mathbb{T}$ as the solution of the problem

$$\widehat{f}^{\Delta_t}(t,\sigma(s)) = -\widehat{f}^{\Delta_s}(t,s), \quad t,s \in \mathbb{T}, \qquad \widehat{f}(t,t_0) = f(t), \quad t \in \mathbb{T}, \tag{1.5}$$

where $t_0 \in \mathbb{T}$ is fixed and where Δ means the delta differentiation and σ stands for the forward jump operator in $\mathbb{T}$. We will call this problem the *shifting problem*. It can be considered as an initial value problem (with respect to s) with the initial function f at $s = t_0$. The solution $\hat{f}$ of this problem gives a shifting of the function f along the time scale $\mathbb{T}$. Then we define the convolution of two functions $f, g : \mathbb{T} \to \mathbb{R}$ by

$$(f * g)(t) = \int_{t_0}^{t} \hat{f}(t, \sigma(s)) g(s) \Delta s, \quad t \in \mathbb{T}. \tag{1.6}$$

The reasonableness of such a definition is justified by the fact, as we prove in this paper, that the convolution theorem holds for convolutions on time scales defined in this way.

The paper is organized as follows. In Section 2, we introduce shifts and convolutions while the convolution theorem is proved in Section 3. In Section 4, the theory of power series on time scales is developed, and the shifting problem is studied in Section 5. Finally, in Section 6, we investigate the presented concepts in the special case of quantum calculus.

2. Shifts and convolutions

Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = \infty$ and fix $t_0 \in \mathbb{T}$.

Definition 2.1. For a given $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, the solution of the *shifting problem*

$$\begin{gathered} u^{\Delta_t}(t, \sigma(s)) = -u^{\Delta_s}(t, s), \quad t, s \in \mathbb{T},\ t \geq s \geq t_0, \\ u(t, t_0) = f(t), \quad t \in \mathbb{T},\ t \geq t_0, \end{gathered} \tag{2.1}$$

is denoted by $\hat{f}$ and is called the *shift* (or *delay*) of f.

Example 2.2. In the case $\mathbb{T} = \mathbb{R}$, the problem (2.1) takes the form

$$\frac{\partial u(t,s)}{\partial t} = -\frac{\partial u(t,s)}{\partial s}, \quad u(t, t_0) = f(t), \tag{2.2}$$

and its unique solution is $u(t,s) = f(t - s + t_0)$. In the case $\mathbb{T} = \mathbb{Z}$, (2.1) becomes

$$u(t+1, s+1) - u(t, s+1) = -u(t, s+1) + u(t, s), \quad u(t, t_0) = f(t), \tag{2.3}$$

and its unique solution is again $u(t,s) = f(t - s + t_0)$. For the solution of the problem (2.1) in the case $\mathbb{T} = q^{\mathbb{N}_0}$, see Section 6.

Example 2.3. Let $r \in \mathbb{T}$. Then, for any regressive constant λ (see (5.21)),

$$\widehat{e_\lambda(\cdot, r)}(t, s) = e_\lambda(t, s) \quad \forall t, s \in \mathbb{T}, \text{ independent of } r \tag{2.4}$$

and, for $k \in \mathbb{N}_0$ (see (5.4)),

$$\widehat{h_k(\cdot, r)}(t, s) = h_k(t, s) \quad \forall t, s \in \mathbb{T}, \text{ independent of } r. \tag{2.5}$$

Lemma 2.4. *If $\widehat{f}$ is the shift of f, then $\widehat{f}(t,t) = f(t_0)$ for all $t \in \mathbb{T}$.*

Proof. By putting $F(t) = \widehat{f}(t,t)$, we find $F(t_0) = \widehat{f}(t_0,t_0) = f(t_0)$ due to the initial condition in (2.1) and $F^{\Delta}(t) = \widehat{f}^{\Delta_t}(t,\sigma(t)) + \widehat{f}^{\Delta_s}(t,t) = 0$ due to the dynamic equation in (2.1), where we have used [5, Theorem 7.2]. □

In this and the next section, we will assume that the problem (2.1) has a unique solution $\widehat{f}$ for a given initial function f and that the functions f, g, and the complex number z are such that the operations fulfilled in this and the next section are valid. Solvability of the problem (2.1) will be considered in Section 5.

Definition 2.5. For given functions $f,g : \mathbb{T} \to \mathbb{R}$, their *convolution* $f * g$ is defined by

$$(f * g)(t) = \int_{t_0}^{t} \widehat{f}(t,\sigma(s))g(s)\Delta s, \quad t \in \mathbb{T}, \tag{2.6}$$

where $\widehat{f}$ is the shift of f introduced in Definition 2.1.

Theorem 2.6. *The shift of a convolution is given by the formula*

$$(\widehat{f * g})(t,s) = \int_{s}^{t} \widehat{f}(t,\sigma(u))\widehat{g}(u,s)\Delta u. \tag{2.7}$$

Proof. We fix $t_0 \in \mathbb{T}$. Let us put $F(t,s)$ equal to the right-hand side of (2.7). First, we have

$$F(t,t_0) = \int_{t_0}^{t} \widehat{f}(t,\sigma(u))\widehat{g}(u,t_0)\Delta u = \int_{t_0}^{t} \widehat{f}(t,\sigma(u))g(u)\Delta u = (f * g)(t). \tag{2.8}$$

Next, we calculate

$$\begin{aligned}
F^{\Delta_t}(t,\sigma(s)) + F^{\Delta_s}(t,s) &= \int_{\sigma(s)}^{t} \widehat{f}^{\Delta_t}(t,\sigma(u))\widehat{g}(u,\sigma(s))\Delta u + \widehat{f}(\sigma(t),\sigma(t))\widehat{g}(t,\sigma(s)) \\
&\quad + \int_{s}^{t} \widehat{f}(t,\sigma(u))\widehat{g}^{\Delta_s}(u,s)\Delta u - \widehat{f}(t,\sigma(s))\widehat{g}(s,\sigma(s)) \\
&= -\int_{\sigma(s)}^{t} \widehat{f}^{\Delta_s}(t,u)\widehat{g}(u,\sigma(s))\Delta u + \int_{s}^{t} \widehat{f}(t,\sigma(u))\widehat{g}^{\Delta_s}(u,s)\Delta u \\
&\quad + f(t_0)\widehat{g}(t,\sigma(s)) - \widehat{f}(t,\sigma(s))\widehat{g}(s,\sigma(s)).
\end{aligned} \tag{2.9}$$

The first integral after the last equal sign above can be evaluated using integration by parts:

$$\begin{aligned}
&\int_{\sigma(s)}^{t} \widehat{f}^{\Delta_s}(t,u)\widehat{g}(u,\sigma(s))\Delta u \\
&\qquad = \widehat{f}(t,u)\widehat{g}(u,\sigma(s))\,\Big|_{u=\sigma(s)}^{u=t} - \int_{\sigma(s)}^{t} \widehat{f}(t,\sigma(u))\widehat{g}^{\Delta_t}(u,\sigma(s))\Delta u \\
&\qquad = f(t_0)\widehat{g}(t,\sigma(s)) - \widehat{f}(t,\sigma(s))g(t_0) + \int_{\sigma(s)}^{t} \widehat{f}(t,\sigma(u))\widehat{g}^{\Delta_s}(u,s)\Delta u.
\end{aligned} \tag{2.10}$$

Putting these calculations together, we arrive at

$$\begin{aligned}
&F^{\Delta_t}(t,\sigma(s)) + F^{\Delta_s}(t,s) \\
&\quad = \widehat{f}(t,\sigma(s))g(t_0) + \int_{s}^{\sigma(s)} \widehat{f}(t,\sigma(u))\widehat{g}^{\Delta_s}(u,s)\Delta u - \widehat{f}(t,\sigma(s))\widehat{g}(s,\sigma(s)) \\
&\quad = \widehat{f}(t,\sigma(s))g(t_0) + \mu(s)\widehat{f}(t,\sigma(s))\widehat{g}^{\Delta_s}(s,s) - \widehat{f}(t,\sigma(s))\widehat{g}(s,\sigma(s)) \\
&\quad = \widehat{f}(t,\sigma(s))g(t_0) + \widehat{f}(t,\sigma(s))[\widehat{g}(s,\sigma(s)) - \widehat{g}(s,s)] - \widehat{f}(t,\sigma(s))\widehat{g}(s,\sigma(s)) \\
&\quad = 0,
\end{aligned} \tag{2.11}$$

which completes the proof of (2.7). □

Theorem 2.7 (associativity of the convolution). *The convolution is associative, that is,*

$$(f * g) * h = f * (g * h). \tag{2.12}$$

Proof. We use Theorem 2.6. Then

$$\begin{aligned}
&((f * g) * h)(t) \\
&\quad = \int_{t_0}^{t} (\widehat{f * g})(t,\sigma(s))h(s)\Delta s \overset{(2.7)}{=} \int_{t_0}^{t}\int_{\sigma(s)}^{t} \widehat{f}(t,\sigma(u))\widehat{g}(u,\sigma(s))h(s)\Delta u\Delta s \\
&\quad = \int_{t_0}^{t}\int_{t_0}^{u} \widehat{f}(t,\sigma(u))\widehat{g}(u,\sigma(s))h(s)\Delta s\Delta u \\
&\quad = \int_{t_0}^{t} \widehat{f}(t,\sigma(u))(g * h)(u)\Delta u \\
&\quad = (f * (g * h))(t).
\end{aligned} \tag{2.13}$$

Hence the associative property holds. □

Theorem 2.8. *If f is delta differentiable, then*

$$(f*g)^{\Delta} = f^{\Delta}*g + f(t_0)g \tag{2.14}$$

and if g is delta differentiable, then

$$(f*g)^{\Delta} = f*g^{\Delta} + fg(t_0). \tag{2.15}$$

Proof. First note that

$$(f*g)^{\Delta}(t) = \int_{t_0}^{t} \widehat{f}^{\Delta_t}(t,\sigma(s))g(s)\Delta s + \widehat{f}(\sigma(t),\sigma(t))g(t). \tag{2.16}$$

From here, since $\widehat{f}(\sigma(t),\sigma(t)) = f(t_0)$ by Lemma 2.4, and since

$$\widehat{f^{\Delta}}(t,s) = \widehat{f}^{\Delta_t}(t,s), \tag{2.17}$$

the first equal sign in the statement follows. For the second equal sign, we use the definition of $\widehat{f}$ and integration by parts:

$$\begin{aligned}
(f*g)^{\Delta}(t) &= -\int_{t_0}^{t} \widehat{f}^{\Delta_s}(t,s)g(s)\Delta s + f(t_0)g(t) \\
&= -\int_{t_0}^{t} \left(\left(\widehat{f}(t,\cdot)g\right)^{\Delta}(s) - \widehat{f}(t,\sigma(s))g^{\Delta}(s)\right)\Delta s + f(t_0)g(t) \\
&= -\widehat{f}(t,t)g(t) + \widehat{f}(t,t_0)g(t_0) + \int_{t_0}^{t} \widehat{f}(t,\sigma(s))g^{\Delta}(s)\Delta s + f(t_0)g(t) \\
&= (f*g^{\Delta})(t) + f(t)g(t_0).
\end{aligned} \tag{2.18}$$

This completes the proof. □

Corollary 2.9. *The following formula holds:*

$$\int_{t_0}^{t} \widehat{f}(t,\sigma(s))\Delta s = \int_{t_0}^{t} f(s)\Delta s. \tag{2.19}$$

Proof. This follows from Theorem 2.8 by using $g = 1$. □

Theorem 2.10. *If f and g are infinitely often Δ-differentiable, then for all $k \in \mathbb{N}_0$,*

$$\begin{aligned}
(f*g)^{\Delta^k} &= f^{\Delta^k}*g + \sum_{\nu=0}^{k-1} f^{\Delta^\nu}(t_0)g^{\Delta^{k-1-\nu}} \\
&= f*g^{\Delta^k} + \sum_{\nu=0}^{k-1} f^{\Delta^\nu}g^{\Delta^{k-1-\nu}}(t_0),
\end{aligned} \tag{2.20}$$

$$(f*g)^{\Delta^k}(t_0) = \sum_{\nu=0}^{k-1} f^{\Delta^\nu}(t_0)g^{\Delta^{k-1-\nu}}(t_0).$$

Proof. We only prove the first equation as the proof of the second equation is similar and as the third equation clearly follows from the first equation as well as from the second equation. The statement is obviously true for $k = 0$. Assuming it is true for $k \in \mathbb{N}_0$, we use the first equation in Theorem 2.8 to find

$$\begin{aligned}(f * g)^{\Delta^{k+1}} &= \left(f^{\Delta^k} * g\right)^{\Delta} + \sum_{\nu=0}^{k-1} f^{\Delta^\nu}(t_0) g^{\Delta^{k-\nu}} \\ &= f^{\Delta^{k+1}} * g + f^{\Delta^k}(t_0) g + \sum_{\nu=0}^{k-1} f^{\Delta^\nu}(t_0) g^{\Delta^{k-\nu}} \\ &= f^{\Delta^{k+1}} * g + \sum_{\nu=0}^{k} f^{\Delta^\nu}(t_0) g^{\Delta^{k-\nu}},\end{aligned} \tag{2.21}$$

so that the statement is true for $k + 1$. $\square$

We conclude this section with the following extension of Lemma 2.4.

Theorem 2.11. *If $\widehat{f}$ has partial Δ-derivatives of all orders, then for all $k \in \mathbb{N}_0$,*

$$\widehat{f}^{\Delta_t^k}(t,t) = f^{\Delta^k}(t_0), \tag{2.22}$$

where $\widehat{f}^{\Delta_t}$ indicates the Δ-derivative of $\widehat{f}$ with respect to its first variable.

Proof. Let $k \in \mathbb{N}_0$. Our assumptions and the initial condition in (2.1) imply that $f^{\Delta^k}(t) = \widehat{f}^{\Delta_t^k}(t,t_0)$. Hence, by putting $F(t) = \widehat{f}^{\Delta_t^k}(t,t)$, we find $F(t_0) = f^{\Delta^k}(t_0)$ as well as

$$F^{\Delta}(t) = \widehat{f}^{\Delta_t^k \Delta_t}(t,\sigma(t)) + \widehat{f}^{\Delta_t^k \Delta_s}(t,t) = \widehat{f}^{\Delta_t \Delta_t^k}(t,\sigma(t)) + \widehat{f}^{\Delta_s \Delta_t^k}(t,t), \tag{2.23}$$

where we have used [5, Theorem 7.2], the dynamic equation in (2.1), and the equality of mixed partial derivatives (under our assumptions) from [5, Theorem 6.1]. $\square$

3. The convolution theorem

Note that below we assume that $z \in \mathcal{R}$ (the set of regressive functions), that is, $1 + \mu(t)z \neq 0$ for all $t \in \mathbb{T}$ (where μ is the graininess on $\mathbb{T}$). Then $(\ominus z) \in \mathcal{R}$ and therefore $e_{\ominus z}(\cdot,t_0)$ is well defined on $\mathbb{T}$.

Definition 3.1. Assume that $x : \mathbb{T} \to \mathbb{R}$ is a locally Δ-integrable function, that is, it is Δ-integrable over each compact interval of $\mathbb{T}$. Then the *Laplace transform* of x is defined by

$$\mathcal{L}\{x\}(z) = \int_{t_0}^{\infty} x(t) e_{\ominus z}(\sigma(t), t_0) \Delta t \quad \text{for } z \in \mathcal{D}\{x\}, \tag{3.1}$$

where $\mathcal{D}\{x\}$ consists of all complex numbers $z \in \mathcal{R}$ for which the improper integral exists.

Theorem 3.2 (convolution theorem). *Suppose $f, g : \mathbb{T} \to \mathbb{R}$ are locally Δ-integrable functions on $\mathbb{T}$ and their convolution $f * g$ is defined by (2.6). Then,*

$$\mathcal{L}\{f * g\}(z) = \mathcal{L}\{f\}(z) \cdot \mathcal{L}\{g\}(z), \quad z \in \mathcal{D}\{f\} \cap \mathcal{D}\{g\}. \tag{3.2}$$

Proof. We have

$$\begin{aligned} \mathcal{L}\{f * g\} &= \int_{t_0}^{\infty} \frac{(f * g)(t)}{e_z(\sigma(t), t_0)} \Delta t \\ &= \int_{t_0}^{\infty} \frac{1}{e_z(\sigma(t), t_0)} \int_{t_0}^{t} \hat{f}(t, \sigma(s)) g(s) \Delta s \Delta t \\ &= \int_{t_0}^{\infty} \frac{g(s)}{e_z(\sigma(s), t_0)} \left\{ \int_{\sigma(s)}^{\infty} \frac{\hat{f}(t, \sigma(s))}{e_z(\sigma(t), \sigma(s))} \Delta t \right\} \Delta s \\ &= \int_{t_0}^{\infty} \frac{g(s)}{e_z(\sigma(s), t_0)} \Psi(\sigma(s)) \Delta s, \end{aligned} \tag{3.3}$$

where

$$\Psi(s) = \int_{s}^{\infty} \frac{\hat{f}(t, s)}{e_z(\sigma(t), s)} \Delta t. \tag{3.4}$$

According to the following lemma, $\Psi(s)$ is independent of s. Then we can evaluate

$$\Psi(t_0) = \int_{t_0}^{\infty} \frac{\hat{f}(t, t_0)}{e_z(\sigma(t), t_0)} \Delta t = \int_{t_0}^{\infty} \frac{f(t)}{e_z(\sigma(t), t_0)} \Delta t = \mathcal{L}\{f\}, \tag{3.5}$$

and we can conclude that $\mathcal{L}\{f * g\} = \mathcal{L}\{g\} \cdot \mathcal{L}\{f\}$. $\square$

Lemma 3.3. *The function Ψ defined in (3.4) is constant.*

Proof. In order to show that Ψ is independent of s, we will show that $\Psi^{\Delta}(s) \equiv 0$. We use [1, Theorem 1.117(ii)] and Lemma 2.4 to find

$$\begin{aligned}
\Psi^{\Delta}(s) &= \int_s^{\infty} \frac{\hat{f}^{\Delta_s}(t,s)e_z(\sigma(t),s) - (\ominus z)(s)e_z(\sigma(t),s)\hat{f}(t,s)}{e_z(\sigma(t),s)e_z(\sigma(t),\sigma(s))}\Delta t - \frac{\hat{f}(s,\sigma(s))}{e_z(\sigma(s),\sigma(s))} \\
&= \int_s^{\infty} \left\{ \frac{-\hat{f}^{\Delta_t}(t,\sigma(s))}{e_z(\sigma(t),\sigma(s))} + \frac{(z/(1+\mu(s)z))\hat{f}(t,s)}{e_z(\sigma(t),\sigma(s))} \right\}\Delta t - \hat{f}(s,\sigma(s)) \\
&= z\Psi(s) - \int_s^{\infty} \frac{\hat{f}^{\Delta_t}(t,\sigma(s))}{e_z(\sigma(t),\sigma(s))}\Delta t - \hat{f}(s,\sigma(s)) \\
&= z\Psi(s) - \int_s^{\sigma(s)} \frac{\hat{f}^{\Delta_t}(t,\sigma(s))}{e_z(\sigma(t),\sigma(s))}\Delta t - \int_{\sigma(s)}^{\infty} \frac{\hat{f}^{\Delta_t}(t,\sigma(s))}{e_z(\sigma(t),\sigma(s))}\Delta t - \hat{f}(s,\sigma(s)) \\
&= z\Psi(s) - \int_{\sigma(s)}^{\infty} \frac{\hat{f}^{\Delta_t}(t,\sigma(s))}{e_z(\sigma(t),\sigma(s))}\Delta t - f(t_0) \\
&= z\Psi(s) - \int_{\sigma(s)}^{\infty} \hat{f}^{\Delta_t}(t,\sigma(s))e_{\ominus z}(\sigma(t),\sigma(s))\Delta t - f(t_0) \\
&= z\Psi(s) - f(t_0) - \int_{\sigma(s)}^{\infty} \{[\hat{f}(t,\sigma(s))e_{\ominus z}(t,\sigma(s))]^{\Delta_t} \\
&\qquad - (\ominus z)(t)e_{\ominus z}(t,\sigma(s))\hat{f}(t,\sigma(s))\}\Delta t \\
&= z\Psi(s) - z\int_{\sigma(s)}^{\infty} \hat{f}(t,\sigma(s))e_{\ominus z}(\sigma(t),\sigma(s))\Delta t \\
&= z\Psi(s) - z\Psi(\sigma(s)) \\
&= -z\mu(s)\Psi^{\Delta}(s)
\end{aligned} \tag{3.6}$$

so that $(1+\mu(s)z)\Psi^{\Delta}(s) = 0$, and therefore $\Psi^{\Delta}(s) = 0$. □

We may use Lemma 3.3 once again to prove the following important theorem. In there, we make use of the function u_a defined by $u_a(b) = 0$ if $b < a$ and $u_a(b) = 1$ if $b \geq a$.

Theorem 3.4. *Let $s, t_0 \in \mathbb{T}$ with $s \geq t_0$. If $\hat{f}$ is the delay of f, then*

$$\mathscr{L}\{u_s\hat{f}(\cdot,s)\}(z) = e_{\ominus z}(s,t_0)\mathscr{L}\{f\}(z). \tag{3.7}$$

Proof. We have

$$\begin{aligned}
\mathscr{L}\{u_s\hat{f}(\cdot,s)\}(z) &= \int_{t_0}^{\infty} u_s(t)\hat{f}(t,s)e_{\ominus z}(\sigma(t),t_0)\Delta t \\
&= \int_{s}^{\infty} \hat{f}(t,s)e_{\ominus z}(\sigma(t),t_0)\Delta t \\
&= e_{\ominus z}(s,t_0)\int_{s}^{\infty} \hat{f}(t,s)e_{\ominus z}(\sigma(t),s)\Delta t \\
&= e_{\ominus z}(s,t_0)\Psi(s),
\end{aligned} \tag{3.8}$$

where Ψ is defined in (3.4) in Theorem 3.2. In Lemma 3.3, it was shown that Ψ is in fact a constant, namely, $\Psi(t) \equiv \Psi(t_0) = \mathscr{L}\{f\}(z)$, and this concludes the proof. □

4. Power series on time scales

Let $\mathbb{T}$ be a time scale. Following Agarwal and Bohner [6] (see also [1, Section 1.6]), let us introduce the generalized monomials $h_k : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$, $k \in \mathbb{N}_0$, defined recursively by

$$h_0(t,s) = 1, \qquad h_k(t,s) = \int_s^t h_{k-1}(\tau,s)\Delta\tau \quad \forall k \in \mathbb{N},\ s,t \in \mathbb{T}. \tag{4.1}$$

Then

$$h_k^{\Delta_t} = h_{k-1} \quad \forall k \in \mathbb{N}. \tag{4.2}$$

The definition (4.1) obviously implies

$$h_1(t,s) = t - s, \qquad h_2(t,s) = \int_s^t (\tau - s)\Delta\tau \quad \forall t,s \in \mathbb{T}, \tag{4.3}$$

and finding h_k for $k > 1$ is not easy in general. For the case $\mathbb{T} = \mathbb{R}$, we have

$$h_k(t,s) = \frac{(t-s)^k}{k!} \quad \forall k \in \mathbb{N}_0,\ t,s \in \mathbb{R}, \tag{4.4}$$

while for the case $\mathbb{T} = \mathbb{Z}$ we have

$$h_k(t,s) = \frac{(t-s)^{(k)}}{k!} \quad \forall k \in \mathbb{N}_0,\ t,s \in \mathbb{Z}, \tag{4.5}$$

where

$$t^{(0)} = 1, \qquad t^{(k)} = \prod_{i=0}^{k-1}(t-i) \quad \text{for } k \in \mathbb{N}. \tag{4.6}$$

For the functions h_k in the case $\mathbb{T} = q^{\mathbb{N}_0}$ for some $q > 1$ (the quantum calculus case), see Section 6.

Returning to the arbitrary time scale $\mathbb{T}$, we present the following useful property of monomials h_k.

Theorem 4.1. *For all $k,m \in \mathbb{N}_0$,*

$$\int_{t_0}^{t} h_k(t,\sigma(s))h_m(s,t_0)\Delta s = h_{k+m+1}(t,t_0). \tag{4.7}$$

Proof. Setting

$$\varphi(t) = \int_{t_0}^{t} h_k(t,\sigma(s))h_m(s,t_0)\Delta s \tag{4.8}$$

and using the differentiation formula [1, Theorem 1.117]

$$k^{\Delta}(t) = K(\sigma(t),t) + \int_{t_0}^{t} K^{\Delta_t}(t,s)\Delta s, \quad k(t) = \int_{t_0}^{t} K(t,s)\Delta s, \tag{4.9}$$

we easily find that

$$\varphi^{\Delta_t^{k+m+1}}(t) = 1, \quad \varphi(t_0) = \varphi^{\Delta_t}(t_0) = \cdots = \varphi^{\Delta_t^{k+m}}(t_0) = 0. \tag{4.10}$$

Hence the statement follows. □

Now let $n \in \mathbb{N}$ and suppose $f : \mathbb{T} \to \mathbb{C}$ is n times Δ-differentiable on $\mathbb{T}^{\kappa^n}$. Let $\alpha \in \mathbb{T}^{\kappa^{n-1}}$, $t \in \mathbb{T}$, and the functions h_k defined by (4.1). Then we have the formula (Taylor's formula)

$$f(t) = \sum_{k=0}^{n-1} h_k(t,\alpha)f^{\Delta^k}(\alpha) + \int_{\alpha}^{t} h_{n-1}(t,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau. \tag{4.11}$$

In order to get an estimation for the remainder term

$$R_n(t,\alpha) = \int_{\alpha}^{t} h_{n-1}(t,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau \tag{4.12}$$

of Taylor's formula (4.11), we have to estimate the functions h_k. This estimate is taken from Bohner and Lutz [7, Theorem 4.1] and reads as follows.

Theorem 4.2. *For all $k \in \mathbb{N}_0$ and $t,s \in \mathbb{T}$ with $t \geq s$, then*

$$0 \leq h_k(t,s) \leq \frac{(t-s)^k}{k!}. \tag{4.13}$$

Using Theorem 4.2, we can now present the following estimate for R_n.

Theorem 4.3. *For $t \in \mathbb{T}$ with $t \geq \alpha$, set*

$$M_n(t) = \sup\{|f^{\Delta^n}(\tau)| : \tau \in [\alpha,t]_{\mathbb{T}}\}. \tag{4.14}$$

Then

$$|R_n(t,\alpha)| \leq M_n(t)\frac{(t-\alpha)^n}{(n-1)!}. \tag{4.15}$$

Proof. If $\tau \in [\alpha, t)_{\mathbb{T}}$, then $\alpha \leq \sigma(\tau) \leq t$, and applying (4.13) gives

$$0 \leq h_{n-1}(t,\sigma(\tau)) \leq \frac{(t-\sigma(\tau))^{n-1}}{(n-1)!} \leq \frac{(t-\tau)^{n-1}}{(n-1)!} \leq \frac{(t-\alpha)^{n-1}}{(n-1)!}. \tag{4.16}$$

Therefore, we have, from (4.12),

$$|R_n(t,\alpha)| \leq M_n(t)\frac{(t-\alpha)^{n-1}}{(n-1)!}\int_\alpha^t \Delta\tau = M_n(t)\frac{(t-\alpha)^n}{(n-1)!}, \tag{4.17}$$

which completes the proof. □

If a function $f : \mathbb{T} \to \mathbb{R}$ is infinitely often Δ-differentiable at a point $\alpha \in \mathbb{T}^\infty = \bigcap_{n=1}^\infty \mathbb{T}^{\kappa^n}$ (i.e., it has Δ-derivatives at α of all orders), then we can formally write for it the series

$$\sum_{k=0}^\infty h_k(t,\alpha)f^{\Delta^k}(\alpha) = f(\alpha) + h_1(t,\alpha)f^{\Delta}(\alpha) + h_2(t,\alpha)f^{\Delta^2}(\alpha) + \cdots, \tag{4.18}$$

called Taylor's series for the function f at the point α. For given values of α and t, it can be convergent or divergent. The case when Taylor's series for the function f is convergent to that function is of particular importance; in this case, the sum of the series is equal to $f(t)$. Taylor's series (4.18) is convergent to $f(t)$ if and only if the remainder of Taylor's formula

$$f(t) = \sum_{k=0}^{n-1} h_k(t,\alpha)f^{\Delta^k}(\alpha) + R_n(t,\alpha) \tag{4.19}$$

tends to zero as $n \to \infty$, that is, $\lim_{n\to\infty} R_n(t,\alpha) = 0$. It may turn out that with a given function f we can formally associate its Taylor series, at a point α, of the form (4.18) (in other words, this means the Δ-derivatives $f^{\Delta^k}(\alpha)$ make sense for this function for any $k \in \mathbb{N}_0$) and that the series (4.18) is convergent for some values of t but its sum is not equal to $f(t)$.

Let us consider Taylor series expansions for some elementary functions. First we prove the following lemma.

Lemma 4.4. *For all $z \in \mathbb{C}$ and $t \in \mathbb{T}$ with $t \geq \alpha$, the initial value problem*

$$y^\Delta = zy, \qquad y(\alpha) = 1 \tag{4.20}$$

has a unique solution y that is represented in the form

$$y(t) = \sum_{k=0}^\infty z^k h_k(t,\alpha) \tag{4.21}$$

and satisfies the inequality

$$|y(t)| \leq e^{|z|(t-\alpha)}. \tag{4.22}$$

Proof. The initial value problem (4.20) is equivalent to finding a continuous solution of the integral equation

$$y(t) = 1 + z\int_{\alpha}^{t} y(\tau)\Delta\tau. \tag{4.23}$$

We solve (4.23) by the method of successive approximations setting

$$y_0(t) = 1, \qquad y_k(t) = z\int_{\alpha}^{t} y_{k-1}(\tau)\Delta\tau \quad \text{for } k \in \mathbb{N}. \tag{4.24}$$

If the series $\sum_{k=0}^{\infty} y_k(t)$ converges uniformly with respect to $t \in [\alpha, R]_{\mathbb{T}}$, where $R \in \mathbb{T}$ with $R > \alpha$, then its sum will be obviously a continuous solution of (4.23). It follows from (4.24) that

$$y_k(t) = z^k h_k(t,\alpha) \quad \forall k \in \mathbb{N}_0. \tag{4.25}$$

Therefore, using Theorem 4.2, for all $k \in \mathbb{N}_0$ and $t \in \mathbb{T}$ with $t \geq \alpha$, we have

$$|y_k(t)| = |z|^k h_k(t,\alpha) \leq |z|^k \frac{(t-\alpha)^k}{k!}. \tag{4.26}$$

It follows that (4.23) has a continuous solution y satisfying $y(t) = \sum_{k=0}^{\infty} z^k h_k(t,\alpha)$ for all $t \geq \alpha$, and for this solution (4.22) holds.

To prove uniqueness of the solution, assume that (4.23) has two continuous solutions y and x for $t \geq \alpha$. Setting $u = y - x$, we get that

$$u(t) = z\int_{\alpha}^{t} u(\tau)\Delta\tau \quad \text{for } t \in \mathbb{T} \text{ with } t \geq \alpha. \tag{4.27}$$

Next setting

$$M = \sup\{|u(t)| : t \in [\alpha, R]_{\mathbb{T}}\}, \tag{4.28}$$

we have from (4.27)

$$|u(t)| \leq |z| M(t-\alpha) \quad \forall t \in [\alpha, R]_{\mathbb{T}}. \tag{4.29}$$

Using this in the integral in (4.27), we get

$$|u(t)| \leq M|z|^2 \int_{\alpha}^{t} (\tau - \alpha)\Delta\tau = M|z|^2 h_1(t,\alpha). \tag{4.30}$$

Repeating this procedure, we obtain

$$|u(t)| \leq M|z|^k h_k(t,\alpha) \quad \forall t \in [\alpha, R]_{\mathbb{T}},\ k \in \mathbb{N}_0. \tag{4.31}$$

Hence by Theorem 4.2,

$$|u(t)| \leq M|z|^k \frac{(t-\alpha)^k}{k!} \quad \forall t \in [\alpha, R]_{\mathbb{T}},\ k \in \mathbb{N}_0. \tag{4.32}$$

Passing here to the limit as $k \to \infty$, we get $u(t) = 0$ for all $t \in [\alpha, R]_{\mathbb{T}}$. Since R was an arbitrary point in $\mathbb{T}$ with $R > \alpha$, we have that $u(t) = 0$ for all $t \in \mathbb{T}$ with $t \geq \alpha$. □

Note that from (4.20) we have that

$$y^{\Delta^k}(\alpha) = z^k \quad \forall k \in \mathbb{N}_0. \tag{4.33}$$

Therefore, (4.21) is a Taylor series expansion for $y(t)$ when $t \in \mathbb{T}$ and $t \geq \alpha$.

Since $e_z(t,\alpha)$ coincides with the unique solution of the initial value problem (4.20) for $t \in \mathbb{T}$ with $t \geq \alpha$, we get, applying Lemma 4.4, that

$$e_z(t,\alpha) = \sum_{k=0}^{\infty} z^k h_k(t,\alpha) \quad \text{for } t \in \mathbb{T} \text{ with } t \geq \alpha, \tag{4.34}$$

and that

$$\left| e_z(t,\alpha) \right| \leq e^{|z|(t-\alpha)} \quad \text{for } t \in \mathbb{T} \text{ with } t \geq \alpha. \tag{4.35}$$

Using (4.34), for $t \in \mathbb{T}$ with $t \geq \alpha$, we have

$$\begin{aligned}
\cosh_z(t,\alpha) &= \frac{e_z(t,\alpha) + e_{-z}(t,\alpha)}{2} = \sum_{k=0}^{\infty} z^{2k} h_{2k}(t,\alpha), \\
\sinh_z(t,\alpha) &= \frac{e_z(t,\alpha) - e_{-z}(t,\alpha)}{2} = \sum_{k=0}^{\infty} z^{2k+1} h_{2k+1}(t,\alpha), \\
\cos_z(t,\alpha) &= \frac{e_{iz}(t,\alpha) + e_{-iz}(t,\alpha)}{2} = \sum_{k=0}^{\infty} (-1)^k z^{2k} h_{2k}(t,\alpha), \\
\sin_z(t,\alpha) &= \frac{e_{iz}(t,\alpha) - e_{-iz}(t,\alpha)}{2i} = \sum_{k=0}^{\infty} (-1)^k z^{2k+1} h_{2k+1}(t,\alpha).
\end{aligned} \tag{4.36}$$

Definition 4.5. Assume that $\sup \mathbb{T} = \infty$ and $t_0 \in \mathbb{T}$ is fixed. A series of the form

$$\sum_{k=0}^{\infty} a_k h_k(t,t_0) = a_0 + a_1 h_1(t,t_0) + a_2 h_2(t,t_0) + \cdots, \tag{4.37}$$

where a_k are constants for $k \in \mathbb{N}_0$ (which may be complex in the general case) and $t \in \mathbb{T}$, is called a *power series* on the time scale $\mathbb{T}$, the numbers a_k being referred to as its *coefficients*. We denote by $\mathcal{F}$ the set of all functions $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$ of the form

$$f(t) = \sum_{k=0}^{\infty} a_k h_k(t,t_0), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.38}$$

where the coefficients a_k satisfy

$$\left| a_k \right| \leq M R^k \quad \forall k \in \mathbb{N}_0 \tag{4.39}$$

with some constants $M > 0$ and $R > 0$ depending only on the series (4.38).

Note that under the condition (4.39), the series (4.38) converges uniformly on any compact interval $[t_0, L]_{\mathbb{T}}$ of $\mathbb{T}$, where $L \in \mathbb{T}$ with $L > t_0$. Indeed, using Theorem 4.2 and (4.39), we have

$$|a_k h_k(t, t_0)| \leq M \frac{[R(t - t_0)]^k}{k!} \tag{4.40}$$

for all $t \in \mathbb{T}$ with $t \geq t_0$ and $k \in \mathbb{N}_0$. Therefore, the sum $f(t)$ of the series (4.38) satisfies

$$|f(t)| \leq M e^{R(t-t_0)} \quad \forall t \in \mathbb{T} \text{ with } t \geq t_0. \tag{4.41}$$

It is easy to see that $\mathcal{F}$ is a linear space: if $f, g \in \mathcal{F}$, then $\alpha f + \beta g \in \mathcal{F}$ for any constants α and β. Note also that any given function $f \in \mathcal{F}$ can be represented in the form of a power series (4.38) uniquely. Indeed, Δ-differentiating the series (4.38) n times term by term we get, using (4.2),

$$f^{\Delta^n}(t) = a_n + a_{n+1} h_1(t, t_0) + a_{n+2} h_2(t, t_0) + \cdots. \tag{4.42}$$

This series is convergent for $t \in [t_0, \infty)_{\mathbb{T}}$ by (4.13) and (4.39) so that the term-by-term differentiating is valid. Setting $t = t_0$, we find that

$$f^{\Delta^n}(t_0) = a_n. \tag{4.43}$$

Thus the coefficients of the power series (4.38) are defined uniquely by the formula (4.43).

5. Investigation of the shifting problem

For an arbitrary time scale $\mathbb{T}$, we can prove the following theorem.

THEOREM 5.1. *Let $f \in \mathcal{F}$ so that f can be written in the form (4.38) with the coefficients satisfying (4.39) for some constants $M > 0$ and $R > 0$. Then the problem (2.1) has a solution u of the form*

$$u(t, s) = \sum_{k=0}^{\infty} a_k h_k(t, s), \tag{5.1}$$

where a_k are the same coefficients as in the expansion (4.38) of f. This solution is unique in the class of functions u for which

$$A_k(s) := u^{\Delta_t^k}(t, s)|_{t=s}, \quad k \in \mathbb{N}, \tag{5.2}$$

are delta differentiable functions of $s \in \mathbb{T}$ and

$$|A_k(s)| \leq A|s|^k, \qquad |A_k^{\Delta}(s)| \leq B|s|^k \tag{5.3}$$

for all $s \in \mathbb{T}$, $s \geq t_0$ with some constants $A > 0$ and $B > 0$.

Proof. Since (see [6])

$$h_k^{\Delta_t}(t, s) = h_{k-1}(t, s), \qquad h_k^{\Delta_s}(t, s) = -h_{k-1}(t, \sigma(s)), \tag{5.4}$$

we have, from (5.1),

$$u^{\Delta_t}(t,\sigma(s)) = \sum_{k=1}^{\infty} a_k h_{k-1}(t,\sigma(s)), \qquad u^{\Delta_s}(t,s) = -\sum_{k=1}^{\infty} a_k h_{k-1}(t,\sigma(s)). \tag{5.5}$$

Note that the differentiation of the series term by term is valid because the series in (5.5) are convergent for $t \geq s$. From (5.5) it follows that u defined by (5.1) satisfies the dynamic equation in (2.1). We also have

$$u(t,t_0) = \sum_{k=0}^{\infty} a_k h_k(t,t_0) = f(t) \tag{5.6}$$

so that the initial condition in (2.1) is satisfied as well.

To prove the uniqueness of the solution, assume that u is a solution of (2.1) and has the properties (5.2), (5.3). Then we can write for u the Taylor expansion series with respect to the variable t at the point $t = s$ for each fixed s:

$$u(t,s) = \sum_{k=0}^{\infty} A_k(s) h_k(t,s), \tag{5.7}$$

where $A_k(s)$ are the Taylor coefficients defined by (5.2). Substituting (5.7) into the dynamic equation in (2.1), we get

$$\sum_{k=1}^{\infty} A_k(\sigma(s)) h_k^{\Delta_t}(t,\sigma(s)) = -\sum_{k=1}^{\infty} \left[A_k^{\Delta}(s) h_k(t,s) + A_k(\sigma(s)) h_k^{\Delta_s}(t,s)\right], \tag{5.8}$$

where we did not include the terms in the series with $k = 0$ since $h_0(t,s) = 1$ and $A_0(s) = u(s,s) = f(t_0)$ both have zero derivatives as constant functions. Note also that the term by term differentiation of the series is valid due to the conditions (5.3). Next we can use (5.4) to get from (5.8) that

$$\sum_{k=1}^{\infty} A_k(\sigma(s)) h_{k-1}(t,\sigma(s)) = -\sum_{k=1}^{\infty} A_k^{\Delta}(s) h_k(t,s) + \sum_{k=1}^{\infty} A_k(\sigma(s)) h_{k-1}(t,\sigma(s)). \tag{5.9}$$

Hence,

$$\sum_{k=1}^{\infty} A_k^{\Delta}(s) h_k(t,s) = 0, \tag{5.10}$$

and therefore,

$$A_k^{\Delta}(s) = 0 \quad \forall k \in \mathbb{N}_0. \tag{5.11}$$

This means that $A_k(s)$ does not depend on s for any $k \in \mathbb{N}_0$. On the other hand, setting $s = t_0$ in (5.1) and using the initial condition in (2.1) and (4.38), we find that

$$A_k(t_0) = a_k \quad \forall k \in \mathbb{N}_0. \tag{5.12}$$

Consequently, the solution u coincides with (5.1). □

Remark 5.2. We can take f given in (4.38), in particular, to be a finite combination of $h_k(t,t_0)$ for some values $k \in \mathbb{N}_0$. Then the solution u given by (5.1) will be represented by a finite sum.

For convenience we are denoting the solution u of the problem (2.1) by $\widehat{f}$ indicating thus that it depends on the initial function f given in the initial condition in (2.1).

Theorem 5.3. *Let $f,g \in \mathcal{F}$ with*

$$f(t) = \sum_{k=0}^{\infty} a_k h_k(t,t_0), \qquad g(t) = \sum_{m=0}^{\infty} b_m h_m(t,t_0),$$
$$|a_k| \le M_1 R_1^k, \qquad |b_m| \le M_2 R_2^m. \tag{5.13}$$

Then $f * g \in \mathcal{F}$ and

$$(f*g)(t) = \sum_{n=0}^{\infty} c_n h_n(t,t_0), \tag{5.14}$$

where

$$c_0 = 0, \qquad c_n = \sum_{k=0}^{n-1} a_k b_{n-1-k}, \quad n \in \mathbb{N}. \tag{5.15}$$

Proof. Using Theorem 5.1 for $\widehat{f}$, we have

$$\begin{aligned}(f*g)(t) &= \int_{t_0}^{t} \widehat{f}(t,\sigma(s))g(s)\Delta s \\ &= \int_{t_0}^{t} \left\{\sum_{k=0}^{\infty} a_k h_k(t,\sigma(s))\right\}\left\{\sum_{m=0}^{\infty} b_m h_m(s,t_0)\right\}\Delta s \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} a_k b_m \int_{t_0}^{t} h_k(t,\sigma(s))h_m(s,t_0)\Delta s.\end{aligned} \tag{5.16}$$

Hence, making use of Theorem 4.1, we obtain

$$(f*g)(t) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} a_k b_m h_{k+m+1}(t,t_0) = \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n-1} a_k b_{n-k-1}\right\}h_n(t,t_0), \tag{5.17}$$

so that (5.14) and (5.15) are proved.

Next, we have from (5.15) by using (5.13) and setting $R_3 = \max\{R_1, R_2\}$,

$$
\begin{aligned}
|c_n| &\le \sum_{k=0}^{n-1} |a_k| \cdot |b_{n-1-k}| \le \sum_{k=0}^{n-1} M_1 R_1^k M_2 R_2^{n-1-k} \\
&\le M_1 M_2 \sum_{k=0}^{n-1} R_3^k R_3^{n-1-k} = M_1 M_2 R_3^{n-1} \sum_{k=0}^{n-1} 1 \\
&= M_1 M_2 n R_3^{n-1} \le M_\varepsilon (R_3 + \varepsilon)^n, \quad \varepsilon > 0.
\end{aligned}
\tag{5.18}
$$

This shows that $f * g \in \mathcal{F}$. □

Theorem 5.4. *The convolution is commutative and associative, that is, for $f, g, h \in \mathcal{F}$,*

$$f * g = g * f, \quad (f * g) * h = f * (g * h). \tag{5.19}$$

Proof. Since

$$c_n = \sum_{k=0}^{n-1} a_k b_{n-1-k} = \sum_{\nu=0}^{n-1} b_\nu a_{n-1-\nu}, \tag{5.20}$$

it follows from (5.14) and (5.15) that the convolution possesses the commutative property. Associativity was established in Theorem 2.7. □

Theorem 5.1 gives solutions of the problem (2.1) in terms of generalized polynomials h_k, $k \in \mathbb{N}_0$. Noting that for any constant λ, the exponential function e_λ satisfies (see [1, Section 2.2])

$$e_\lambda^{\Delta_t}(t,s) = \lambda e_\lambda(t,s), \qquad e_\lambda^{\Delta_s}(t,s) = -\lambda e_\lambda(t, \sigma(s)) \tag{5.21}$$

so that e_λ satisfies the dynamic equation in (2.1), we can construct some solutions of the problem (2.1) in terms of exponential functions e_λ. For this purpose, let us take a set $\Omega \subset \mathbb{C}$ and denote by $\mathcal{K}$ the set of functions $f : [t_0, \infty)_\mathbb{T} \to \mathbb{C}$ of the form

$$f(t) = \int_\Omega \varphi(\lambda) e_\lambda(t, t_0)\, d\omega(\lambda), \quad t \in [t_0, \infty)_\mathbb{T}, \tag{5.22}$$

where ω is a measure in Ω and $\varphi : \Omega \to \mathbb{C}$ is a function; ω and φ depend on f. Note that the exponential function e_λ is well defined for all complex values of λ if $t \ge t_0$ (see Lemma 4.4). The integral on the right-hand side of (5.22) can be understood to be a Riemann-Stieltjes or Lebesgue-Stieltjes integral. Besides we require that

$$\int_\Omega |\lambda \varphi(\lambda) e_\lambda(t,s)|\, d\omega(\lambda) < \infty \tag{5.23}$$

for all $t, s \in \mathbb{T}$ with $t \ge s \ge t_0$. By choosing various φ and ω, we get according to (5.22) the elements of $\mathcal{K}$. Obviously, $\mathcal{K}$ is a linear space.

If, in particular, ω is a measure concentrated on a finite set $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \mathbb{C}$ with

$$\omega(\{\lambda_k\}) = 1 \quad \forall k \in \{1, 2, \dots, n\} \tag{5.24}$$

and if we denote $\varphi(\lambda_k) = c_k$, then (5.22) yields a function f of the form

$$f(t) = \sum_{k=1}^{n} c_k e_{\lambda_k}(t, t_0). \tag{5.25}$$

Therefore, the space $\mathcal{H}$ contains all exponential, hyperbolic, and trigonometric functions.

THEOREM 5.5. *Suppose that the function $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$ has the form (5.22) and that (5.23) is satisfied. Then the function*

$$u(t,s) = \int_{\Omega} \varphi(\lambda) e_{\lambda}(t,s) d\omega(\lambda) \tag{5.26}$$

has first-order partial delta derivatives for $t \geq s \geq t_0$ and satisfies (2.1).

Proof. We have

$$u(t, t_0) = \int_{\Omega} \varphi(\lambda) e_{\lambda}(t, t_0) d\omega(\lambda) = f(t) \tag{5.27}$$

so that the initial condition in (2.1) is satisfied. Further, we can take first-order partial delta derivatives of u in (5.26), and by the condition (5.23) we can differentiate under the integral sign. Taking into account (5.21), we see that u satisfies the dynamic equation in (2.1) as well. □

Notice that the solution u of the problem (2.1) with the initial function f of the form (5.25) has, according to Theorem 5.5, the form

$$u(t,s) = \sum_{k=1}^{n} c_k e_{\lambda_k}(t,s). \tag{5.28}$$

Remark 5.6. Solutions of the form (5.26) are in fact of the form (5.1). Indeed, we have the expansion (see (4.34))

$$e_{\lambda}(t,s) = \sum_{k=0}^{\infty} \lambda^k h_k(t,s) \tag{5.29}$$

for all $\lambda \in \mathbb{C}$ and $t, s \in \mathbb{T}$ with $t \geq s$. Substituting this into (5.26) and integrating term by term, we get

$$u(t,s) = \sum_{k=0}^{\infty} a_k h_k(t,s), \quad a_k = \int_{\Omega} \lambda^k \varphi(\lambda) d\omega(\lambda). \tag{5.30}$$

The term-by-term integration is valid if we require that

$$\int_{\Omega} |\lambda|^k \, |\varphi(\lambda)| \, d\omega(\lambda) \leq MR^k \tag{5.31}$$

for all $k \in \mathbb{N}_0$ with some constants $M > 0$ and $R > 0$.

Now denote by $\mathcal{F}_0$ the linear space of all generalized polynomials, with complex coefficients, in the variable $t \in \mathbb{T}$. So any element of $\mathcal{F}_0$ has the form

$$f(t) = \sum_{k=0}^{N} a_k h_k(t, t_0) \tag{5.32}$$

for some $N \in \mathbb{N}_0$ and some coefficients $a_k \in \mathbb{C}$, $0 \le k \le N$. Note that $\mathcal{F}_0$ is a subspace of the linear space $\mathcal{F}$ introduced above in Section 4. Further, denote by $\mathcal{D}_k$ the set of all regressive $z \in \mathbb{C}$ satisfying

$$\lim_{t \to \infty} \{h_k(t, t_0) e_{\ominus z}(t, t_0)\} = 0 \tag{5.33}$$

and put $\mathcal{D} = \bigcap_{k=0}^{\infty} \mathcal{D}_k$. We will assume that $\mathcal{D}$ is not empty. It is known [1, 4] that for all $k \in \mathbb{N}_0$ and $z \in \mathcal{D}$, we have

$$\mathcal{L}\{h_k(\cdot, t_0)\}(z) = \frac{1}{z^{k+1}}. \tag{5.34}$$

Therefore, for the generalized polynomial f of the form (5.32), we obtain

$$\mathcal{L}\{f\}(z) = \sum_{k=0}^{N} \frac{a_k}{z^{k+1}}. \tag{5.35}$$

Let us take another element $g \in \mathcal{F}_0$:

$$g(t) = \sum_{m=0}^{M} b_m h_m(t, t_0), \quad \mathcal{L}\{g\}(z) = \sum_{m=0}^{M} \frac{b_m}{z^{m+1}}. \tag{5.36}$$

Proceeding as in the proof of Theorem 5.3, we find that

$$(f * g)(t) = \sum_{k=0}^{N} \sum_{m=0}^{M} a_k b_m h_{k+m+1}(t, t_0). \tag{5.37}$$

The last formula shows that $f * g \in \mathcal{F}_0$ and

$$\begin{aligned}
\mathcal{L}\{f * g\}(z) &= \sum_{k=0}^{N} \sum_{m=0}^{M} a_k b_m \cdot \frac{1}{z^{k+m+2}} \\
&= \left\{ \sum_{k=0}^{N} \frac{a_k}{z^{k+1}} \right\} \left\{ \sum_{m=0}^{M} \frac{b_m}{z^{m+1}} \right\} \\
&= \mathcal{L}\{f\}(z) \cdot \mathcal{L}\{g\}(z).
\end{aligned} \tag{5.38}$$

Thus we have checked the convolution theorem for functions of the class $\mathcal{F}_0$.

6. The quantum calculus case

In this section, we consider $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$. We also let $t_0 = 1$. Calculus on this time scale is called *quantum calculus* [8]. Note that on this time scale we have $\sigma(t) = qt$ and $\mu(t) = (q-1)t$ for all $t \in \mathbb{T}$.

Definition 6.1. We use the notation from [8], in particular

$$[\alpha] = [\alpha]_q = \frac{q^\alpha - 1}{q - 1}, \quad \alpha \in \mathbb{R},$$

$$[n]! = \prod_{k=1}^{n} [k], \quad \begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m]!}{[n]![m-n]!}, \quad m, n \in \mathbb{N}_0, \tag{6.1}$$

$$(t-s)_q^n = \prod_{k=0}^{n-1} (t - q^k s), \quad t, s \in \mathbb{T},\ n \in \mathbb{N}_0.$$

Theorem 6.2. *The quantum calculus monomials are given by*

$$h_n(t,s) = \frac{(t-s)_q^n}{[n]!} \quad \forall n \in \mathbb{N}_0. \tag{6.2}$$

Proof. In [1, Formula (1.19) in Example 1.104] it was shown that

$$h_n(t,s) = \prod_{k=0}^{n-1} \frac{t - q^k s}{\sum_{i=0}^{k} q^i}. \tag{6.3}$$

Using the notation from Definition 6.1 above, the claim follows. □

Corollary 6.3. *The following formula holds:*

$$h_n(q^k t, t) = \begin{bmatrix} k \\ n \end{bmatrix} (\mu(t))^n q^{n(n-1)/2} \quad \forall k, n \in \mathbb{N}_0 \text{ with } k \geq n. \tag{6.4}$$

Proof. We rewrite

$$\begin{aligned} (q^k t - t)_q^n &= (q^k t - t) \cdot (q^k t - qt) \cdot \cdots \cdot (q^k t - q^{n-1} t) \\ &= t^m \cdot q \cdot q^2 \cdot \cdots \cdot q^{n-1} \cdot (q^k - 1) \cdot (q^{k-1} - 1) \cdot \cdots \cdot (q^{k-n+1} - 1) \\ &= (\mu(t))^n \cdot q^{(n-1)n/2} \cdot [k] \cdot [k-1] \cdot \cdots \cdot [k-n+1] \\ &= [n]! \begin{bmatrix} k \\ n \end{bmatrix} (\mu(t))^n q^{n(n-1)/2}. \end{aligned} \tag{6.5}$$

Now an application of Theorem 6.2 completes the proof. □

Theorem 6.4. *If $f : \mathbb{T} \to \mathbb{R}$, then*

$$f^{\Delta^k}(t)\big(\mu(t)\big)^k q^{k(k-1)/2} = \sum_{\nu=0}^{k} (-1)^\nu \begin{bmatrix} k \\ \nu \end{bmatrix} q^{\nu(\nu-1)/2} f^{\sigma^{k-\nu}}(t) \quad \forall k \in \mathbb{N}_0. \tag{6.6}$$

Proof. The statement is true for $k = 0$. Assuming it is true for $k \in \mathbb{N}_0$, we get

$$\begin{aligned}
&f^{\Delta^{k+1}}(t)\big(\mu(t)\big)^{k+1} q^{(k+1)k/2} \\
&= \big(\mu(t) f^{\Delta^{k+1}}(t)\big)\big(\mu(t)\big)^k q^{k(k+1)/2} \\
&= \big[f^{\Delta^k}(\sigma(t)) - f^{\Delta^k}(t)\big]\big(\mu(t)\big)^k q^{k(k+1)/2} \\
&= f^{\Delta^k}(\sigma(t))\big(\mu(\sigma(t))\big)^k q^{-k} q^{k(k+1)/2} - f^{\Delta^k}(t)\big(\mu(t)\big)^k q^{k(k+1)/2} \\
&= f^{\Delta^k}(\sigma(t))\big(\mu(\sigma(t))\big)^k q^{k(k-1)/2} - f^{\Delta^k}(t)\big(\mu(t)\big)^k q^{k(k-1)/2} q^k \\
&= \sum_{\nu=0}^{k} (-1)^\nu \begin{bmatrix} k \\ \nu \end{bmatrix} q^{\nu(\nu-1)/2} f^{\sigma^{k-\nu}}(\sigma(t)) - q^k \sum_{\nu=0}^{k} (-1)^\nu \begin{bmatrix} k \\ \nu \end{bmatrix} q^{\nu(\nu-1)/2} f^{\sigma^{k-\nu}}(t) \\
&= \sum_{\nu=0}^{k} (-1)^\nu \begin{bmatrix} k \\ \nu \end{bmatrix} q^{\nu(\nu-1)/2} f^{\sigma^{k-\nu+1}}(t) + q^k \sum_{\nu=1}^{k+1} (-1)^\nu \begin{bmatrix} k \\ \nu-1 \end{bmatrix} q^{(\nu-1)(\nu-2)/2} f^{\sigma^{k-\nu+1}}(t) \\
&= f^{\sigma^{k+1}}(t) + q^k(-1)^{k+1} q^{k(k-1)/2} f(t) + \sum_{\nu=1}^{k} (-1)^\nu f^{\sigma^{k-\nu+1}}(t) q^{\nu(\nu-1)/2} \left\{ \begin{bmatrix} k \\ \nu \end{bmatrix} + q^{k-\nu+1} \begin{bmatrix} k \\ \nu-1 \end{bmatrix} \right\} \\
&= f^{\sigma^{k+1}}(t) + q^k(-1)^{k+1} q^{k(k-1)/2} f(t) + \sum_{\nu=1}^{k} (-1)^\nu f^{\sigma^{k-\nu+1}}(t) q^{\nu(\nu-1)/2} \begin{bmatrix} k+1 \\ \nu \end{bmatrix} \\
&= \sum_{\nu=0}^{k+1} (-1)^\nu \begin{bmatrix} k+1 \\ \nu \end{bmatrix} q^{\nu(\nu-1)/2} f^{\sigma^{k+1-\nu}}(t),
\end{aligned} \tag{6.7}$$

where we have used [8, Formula (6.3) in Proposition 6.1] to evaluate the expression in the curly braces. Hence the statement is true for $k+1$, which completes the proof. □

Finally, we are in a position to present a formula for the shift of a function defined on the quantum calculus time scale.

Theorem 6.5. *The shift of $f : \mathbb{T} \to \mathbb{R}$ is given by*

$$\hat{f}(q^k t, t) = \sum_{\nu=0}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix} t^\nu (1-t)_q^{k-\nu} f(q^\nu) \quad \forall k \in \mathbb{N}_0. \tag{6.8}$$

Proof. We use the results from this and the previous section to obtain

$$
\begin{aligned}
\widehat{f}(q^k t,t) &= \sum_{m=0}^{k} h_m(q^k t,t) f^{\Delta^m}(1) \\
&= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix} (\mu(t))^m q^{m(m-1)/2} f^{\Delta^m}(1) \\
&= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix} t^m f^{\Delta^m}(1) (\mu(1))^m q^{m(m-1)/2} \\
&= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix} t^m \sum_{\nu=0}^{m} (-1)^\nu \begin{bmatrix} m \\ \nu \end{bmatrix} q^{\nu(\nu-1)/2} f^{\sigma^{m-\nu}}(1) \\
&= \sum_{m=0}^{k} \sum_{\nu=0}^{m} \begin{bmatrix} k \\ m \end{bmatrix} t^m (-1)^{m-\nu} \begin{bmatrix} m \\ m-\nu \end{bmatrix} q^{(m-\nu)(m-\nu-1)/2} f(q^\nu) \\
&= \sum_{\nu=0}^{k} \sum_{m=\nu}^{k} \begin{bmatrix} k \\ m \end{bmatrix} \begin{bmatrix} m \\ m-\nu \end{bmatrix} t^m (-1)^{m-\nu} q^{(m-\nu)(m-\nu-1)/2} f(q^\nu) \\
&= \sum_{\nu=0}^{k} \sum_{m=0}^{k-\nu} \begin{bmatrix} k \\ m+\nu \end{bmatrix} \begin{bmatrix} m+\nu \\ m \end{bmatrix} t^{m+\nu} (-1)^m q^{m(m-1)/2} f(q^\nu) \\
&= \sum_{\nu=0}^{k} \sum_{m=0}^{k-\nu} \begin{bmatrix} k \\ \nu \end{bmatrix} \begin{bmatrix} k-\nu \\ m \end{bmatrix} t^\nu (-t)^m q^{m(m-1)/2} f(q^\nu) \\
&= \sum_{\nu=0}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix} t^\nu \left\{ \sum_{m=0}^{k-\nu} \begin{bmatrix} k-\nu \\ m \end{bmatrix} (-t)^m q^{m(m-1)/2} \right\} f(q^\nu) \\
&= \sum_{\nu=0}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix} t^\nu (1-t)_q^{k-\nu} f(q^\nu),
\end{aligned}
\tag{6.9}
$$

where we have used [8, Formula (5.5)] to evaluate the expression in the curly braces. □

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