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Research Article

Relations between Sequences and Selection Properties

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We consider the set $\mathbb{S}$ of sequences of positive real numbers and show that some subclasses of $\mathbb{S}$ have certain nice selection and game theoretic properties.

1. Introduction

In his famous, influential 1930 paper [1], Karamata initiated the subject nowadays known as regular variation (see [2] and also [3–7]). His motivation was Tauberian theory, and the first triumph of regular variation was a spectacular simplification of the work of Hardy and Littlewood on Tauberian theorems for Laplace transforms; this resulted in what is now called the Hardy-Littlewood-Karamata theorem (see [8], [2, Chapter 1], [9, Chapter 4]). In what follows, we consider both regular variation and rapid variation (see [2, Section 2.4] and references cited there).

However, the theory also was developed to some other directions. Recently, the authors found in [10] (see also [11, 12]) that there is a nice connection between asymptotic analysis of divergent processes (Karamata theory, the theory of rapid variability) and the theory of selection principles, a quickly growing field of mathematics, as well as game theory and Ramsey theory. (We refer the reader to the book [13] for more information about infinite games.) In this paper, we will further demonstrate that certain subclasses of the set $\mathbb{S}$ of sequences of positive real numbers, which are defined in terms of relationships between sequences from $\mathbb{S}$, satisfy some selection principles and game-theoretic conditions. We believe that new techniques that we use in the proofs could be applied to other constructions in the area of selection principles.

Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose members are families of subsets of an infinite set X. Then (see [14, 15]): $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection principle: For each sequence $(A_n : n \in \mathbb{N})$

of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

Recently, in [16], new selection principles $\alpha_i(\mathcal{A},\mathcal{B})$ were introduced and studied (see also [17]).

The basic object in this paper will be

$$\mathbb{S} = \{c = (c_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : c_n > 0 \text{ for each } n \in \mathbb{N}\}, \tag{1.1}$$

the set of sequences of positive real numbers, so that $\mathcal{A}$ and $\mathcal{B}$ will be certain subfamilies of $\mathbb{S}$.

For a sequence $(c_n)_{n\in\mathbb{N}} \in \mathbb{S}$ denote by $\text{Im}(c_n)$ the set of elements appearing in the sequence.

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be subfamilies of $\mathbb{S}$. The symbol $\alpha_i(\mathcal{A},\mathcal{B})$, $i = 1,2,3,4$, denotes the following selection hypothesis.

For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is an element $B \in \mathcal{B}$ such that:
(1) $\alpha_1(\mathcal{A},\mathcal{B})$: for each $n \in \mathbb{N}$ the set $\text{Im}(A_n)\setminus\text{Im}(B)$ is finite;
(2) $\alpha_2(\mathcal{A},\mathcal{B})$: for each $n \in \mathbb{N}$ the set $\text{Im}(A_n) \cap \text{Im}(B)$ is infinite;
(3) $\alpha_3(\mathcal{A},\mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $\text{Im}(A_n) \cap \text{Im}(B)$ is infinite;
(4) $\alpha_4(\mathcal{A},\mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $\text{Im}(A_n) \cap \text{Im}(B)$ is nonempty.

Evidently for arbitrary subclasses $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{S}$, we have

$$\begin{gathered}\alpha_1(\mathcal{A},\mathcal{B}) \Longrightarrow \alpha_2(\mathcal{A},\mathcal{B}) \Longrightarrow \alpha_3(\mathcal{A},\mathcal{B}) \Longrightarrow \alpha_4(\mathcal{A},\mathcal{B}),\\ \mathsf{S}_1(\mathcal{A},\mathcal{B}) \Longrightarrow \alpha_4(\mathcal{A},\mathcal{B}).\end{gathered} \tag{1.2}$$

2. Results

Definition 2.1. Let $a = (a_n)_{n\in\mathbb{N}} \in \mathbb{S}$, and $\mu > 0$ be fixed. A sequence $b = (b_n)_{n\in\mathbb{N}} \in \mathbb{S}$ *μ-dominates a* if there is $n_0 = n_0(\mu)$ such that $a_n < \mu\cdot b_n$ for all $n > n_0$.

Denote by $\{a\}_\mu$ the set of all sequences in $\mathbb{S}$ which μ-dominate a.

Evidently, for $0 < \mu < \nu$, we have $\{a\}_\mu \subsetneq \{a\}_\upsilon$.

Further, let

$$\{a\} = \bigcup_{\mu>0}\{a\}_\mu. \tag{2.1}$$

For $b = (b_n)_{n\in\mathbb{N}} \in \{a\}$, we write

$$a_n = O(b_n), \quad n \longrightarrow \infty, \tag{2.2}$$

and say that a is *subordinated to b*.

Theorem 2.2. *Let $a = (a_j)_{j\in\mathbb{N}} \in \mathbb{S}$ and $\mu > 0$ be fixed. Then $\alpha_2(\{a\}_\mu,\{a\}_\mu)$ holds.*

Proof. Let $(x_i : i \in \mathbb{N})$ be a sequence of elements from $\{a\}_\mu$ and suppose that for each i, we have $x_i = (b_{i,j})_{j\in\mathbb{N}}$. Construct a new sequence $(y_i : i \in \mathbb{N})$ in the following way. There exists j_1 such that $b_{1,j} \geq (1/\mu)a_j$ for all $j \geq j_1$. Consider the sequence $y_1 = (b_{1,j})_{j\geq 1}$. Suppose $i \geq 2$ and that the sequences y_k and numbers j_k have been defined for every $k \leq i-1$.

Put

$$j_i^* = \min\left\{ j \in N : b_{ij} \geq \frac{i}{\mu} a_j \right\},$$
$$j_1 = \begin{cases} j_{i-1}, & \text{if } j_i^* < j_{i-1}; \\ \min\limits_{k \in N} \{ j_{i-1} + k \cdot 2^{i-1} : j_{i-1} + k \cdot 2^{i-1} > j_i^* \}, & \text{if } j_i^* > j_{i-1}. \end{cases} \tag{2.3}$$

Form the sequence y_i in such a way that in the sequence y_{i-1}, we replace each 2^ith element beginning withj_ith by the corresponding elements (of the same indices) of the sequence x_i. Suppose that $y_i = (h_{i,j})_{j \in \mathbb{N}}$.

Let $k_j = \limsup_{i \to +\infty}(h_{i,j})$. Then $k_j \geq (1/\mu)a_j > 0$, for$j \geq j_1$ and $k_j = b_{1,j} > 0$ for$j \in \{1, \ldots, j_1 - 1\}$. If for some $j \geq j_1$, we have $k_j = +\infty$, then we replace k_j with $b_{1,j}$. In this way, we generate the sequence $z = (k_j)_{j \in \mathbb{N}}$ which, by construction, belongs to $\{a\}_\mu$ and has infinitely many common elements with each of the sequences x_i; for x_i, $i > 1$, each 2^{i+1}th element of x_i beginning from $b_{i, j_{i+1}+2^i}$ is such an element.

□

Definition 2.3. Let $a = (a_n)_{n \in \mathbb{N}} \in \mathbb{S}$ and $\mu > 0$ and $\nu > 0$ with $\mu \cdot \nu \geq 1$ be fixed. A sequence $b = (b_n)_{n \in \mathbb{N}} \in \mathbb{S}$ is said to be (μ, v)-*weakly asymptotically equivalent with a* if both $b \in \{a\}_\mu$ and $a \in \{b\}_\nu$ hold, or equivalently, $1/\nu \cdot b_n < a_n < \mu \cdot b_n$ for all but finitely many n.

Remark 2.4. The relation of (μ, ν)-weak asymptotic equivalence is not an equivalence relation on $\mathbb{S}$, except the case $\mu = \nu = 1$.

Denote by

$$\{a\}_{\mu,\nu} := \{ b \in \mathbb{S} : b \text{ is } (\mu, \nu)\text{-weakly asymptotically equivalent to } a \}. \tag{2.4}$$

We say that a sequence $b = (b_n) \in S$ is *weakly asymptotically equivalent* to a, if $b \in \{a\}$ and $a \in \{b\}$(i.e., if $a_n = O(b_n)$, $n \to +\infty$, and $b_n = O(a_n)$, $n \to +\infty$).

The relation of weak asymptotic equivalence is an equivalence relation on the set $\mathbb{S}$. The usual notation for this relation is $a \in \Theta(b)$ (or $b \in \Theta(a)$).

Theorem 2.5. *Let* $a = (a_j)_{j \in \mathbb{N}} \in \mathbb{S}$ *and* $\mu > 0$, $\nu > 0$ *such that* $\mu \cdot \nu \geq 1$ *be fixed. Then* $\alpha_2(\{a\}_{\mu,\nu}, \{a\}_{\mu,\nu})$ *holds.*

Proof. Let $(x_i = (b_{i,j})_{j \in \mathbb{N}} : i \in \mathbb{N})$ be a sequence of elements from $\{a\}_{\mu,\nu}$. Consider now a sequence $y_1 = (b_{1,j})_{j \geq 1}$ (where $(1/\nu)b_{1,j} \leq a_j \leq \mu \cdot b_{1,j}$ for $j \geq j_1$ for some $j_1 \in \mathbb{N}$). Inductively, for each $i \geq 2$ form a sequence y_i as follows. Suppose the sequences $y_1, y_2, \ldots, y_{i-1}$ and natural numbers $j_1, j_2, \ldots, j_{i-1}$ have been already defined. Let

$$j_i^* = \min\left\{ j \in \mathbb{N} : \frac{1}{\nu} b_{i,j} \leq a_j \leq \mu \cdot b_{i,j} \right\}. \tag{2.5}$$

Define

$$j_i = \begin{cases} j_{i-1} & \text{if } j_i^* \le j_{i-1}; \\ \min_{k\in N}\{j_{i-1}+k\cdot 2^{i-1} : j_{i-1}+k\cdot 2^{i-1} \ge j_i^*\} & \text{if } j_i^* > j_{i-1}. \end{cases} \tag{2.6}$$

The sequence y_i will be defined in such a way that in the sequence y_{i-1}, we replace each 2^ith element, beginning from j_ith with the corresponding element (of the same index) from the sequence x_i. Let $y_i = (h_{i,j})_{j\in\mathbb{N}}$, $i \in \mathbb{N}$.

Let $k_j = \limsup_{i\to+\infty}(h_{i,j})$. Then we have $(1/\mu)a_j \le \liminf_{i\to+\infty}(h_{i,j}) \le k_j \le \nu\cdot a_j$ for $j \ge j_1$. Also, we have $k_j = b_{1,j} > 0$ for $j \in \{1,\dots,j_1-1\}$. So, by construction, the sequence $y = (k_j)_{j\in\mathbb{N}}$ belongs to the class $\{a\}_{\mu,\nu}$ and has infinitely many common elements with each of sequences x_i, $i \ge 1$; surely, for x_i, $i > 1$, each 2^{i+1}th element of x_i beginning from $b_{i,j_{i+1}+2^i}$ is such a common element. □

Remark 2.6. Notice that in the proofs of Theorems 2.2 and 2.5 one could replace 2^i with $m^{\psi(i)}$, $i \in \mathbb{N}$, where $m \in \mathbb{N}$ and $m \ge 2$, and $\psi : \mathbb{N}\to\mathbb{N}$ is a strictly increasing function. (So, we used $m = 2$ and $\psi = id_{\mathbb{N}}$.)

Definition 2.7. A sequence $a = (a_n)_{n\in\mathbb{N}} \in \mathbb{S}$ is said to be negligible with respect to a sequence $b = (b_n)_{n\in\mathbb{N}}$ from $\mathbb{S}$ if for every $\epsilon > 0$ there is $n_0 = n_0(\epsilon)$ such that $a_n \le \epsilon\cdot b_n$ whenever $n \ge n_0$.

Denote by $\nabla(a)$ the set of all sequences b in $\mathbb{S}$ such that a is negligible with respect to b. For $b = (b_n)_{n\in\mathbb{N}} \in \nabla(a)$, we use the notation

$$a_n = o(b_n), \quad n \longrightarrow +\infty. \tag{2.7}$$

Observe that $\nabla(a) = \cap_{\mu>0}\{a\}_\mu$.

Let $\mathcal{A}$ and $\mathcal{B}$ be subclasses of $\mathbb{S}$. The symbol $\mathsf{G}(\mathcal{A},\mathcal{B})$ denotes the infinitely long game for two players, ONE and TWO, who play a round for each positive integer. In the nth round ONE chooses a sequence $s_n \in \mathcal{A}$, and TWO responds by choosing an infinite set T_n from $\mathrm{Im}(s_n)$. TWO wins a play $(s_1,T_1;\dots;s_n,T_n;\dots)$ if $\cup_{n\in\mathbb{N}} T_n$ can be arranged in a sequence from $\mathcal{B}$; otherwise, ONE wins.

Evidently, if TWO has a winning strategy in the game $\mathsf{G}(\mathcal{A},\mathcal{B})$ (or even if ONE does not have a winning strategy in $\mathsf{G}(\mathcal{A},\mathcal{B})$), then the selection hypothesis $\alpha_2(\mathcal{A},\mathcal{B})$ is true.

THEOREM 2.8. *Let $a = (a_j)_{j\in\mathbb{N}} \in \mathbb{S}$. The player TWO has a winning strategy in the game $G(\nabla(a),\nabla(a))$.*

Proof. We describe a winning strategy for the player TWO.

Round I. Suppose ONE chooses a sequence $x_1 = (x_{1,j})_{j\in\mathbb{N}}$ from $\nabla(a)$. Then TWO picks a prime number p_1 and a position $j_{p_1} = j_1$ in the sequence x_1 such that $a_j/x_{1,j} \le 1/P_1$ for $j \ge j_{p_1}$, and fix elements x_{1,p_1^k}, $k \in \mathbb{N}$ (for the set $T_1 = \{x_{1,p_1^k} : k \in \mathbb{N}\}$), so that $p_1^k \ge j_{p_1} = j_1$ holds.

Round II. ONE chooses a sequence $x_2 = (x_{2,j})_{j\in\mathbb{N}}$ from $\nabla(a)$. TWO picks a prime number $p_2 > p_1$, finds a position j_{p_2} in the sequence x_2 such that $a_j/x_{2,j} \le 1/P_2$ for $j \ge j_{p_2}$ and puts $j_2 = \max\{j_1, j_{p_2}\}$. In the sequence x_1 TWO finds now elements x_{1,p_2^k}, $k \in \mathbb{N}$, with $p_2^k \ge j_2$ and replaces them by elements x_{2,p_2^k}, $k \in \mathbb{N}$ (so, $T_2 = \{x_{2,p_2^k} : k \in \mathbb{N}\}$).

Round III. ($i \ge 3$): ONE takes a sequence $x_i = (x_{i,j})_{j\in\mathbb{N}}$ from $\nabla(a)$. TWO first chooses a prime number p_i,$p_1 < p_2 < \cdots < p_i$, and then consider a position j_{p_i} such that $a_j/x_{i,j} \ge 1/p_i$ for $j \ge j_{p_i}$ and takes $j_i = \max\{j_{i-1}, j_{p_i}\}$. Now, in the sequence obtained by this procedure in the step $i-1$, one replaces elements x_{1,p_i^k}, $k \in \mathbb{N}$, with $p_i^k \ge j_i$ by elements x_{i,p_i^k}, $k \in \mathbb{N}$ (hence, $T_i = \{x_{i,p_i^k} : k \in \mathbb{N}\}$).

This procedure leads to the sequence $y = (y_j)_{j\in\mathbb{N}}$, where $y_j = x_{i,j}$, if there are $k \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $j = p_i^k$ and $j \ge j_i$, and $y_j = x_{1,j}$ otherwise. The sequence y belongs to $\mathbb{S}$ and, by construction, has infinitely many common elements with every sequence x_i.

We prove that $y \in \nabla(a)$, that is, that $\limsup_{j\to+\infty}(a_j/y_j) = 0$. Suppose, on the contrary, that $\limsup_{j\to+\infty}(a_j/y_j) = A > 0$. This means that there is a subsequence $(a_{j(s)}/y_{j(s)})_{s\in\mathbb{N}}$ of the sequence $(a_j/y_j)_{j\in\mathbb{N}}$ such that

$$\lim_{s\to+\infty} \frac{a_{j(s)}}{y_{j(s)}} = A. \tag{2.8}$$

In other words, there is $s_0 = s_0(A)$ such that for $s \ge s_0$ (so $j \ge j_0 = j(s_0)$) we have $a_{j(s)}/y_{j(s)} \ge A/2 > 0$.

Observe that among elements $y_{j(s)}$, $s \in \mathbb{N}$, which occur in the subsequence $(a_{j(s)}/y_{j(s)})_{s\in\mathbb{N}}$, there do not exist countably many elements from x_i, for each $i \in \mathbb{N}$. Otherwise, those elements would form a subsequence of $(y_{j(s)})_{s\in\mathbb{N}}$ which would contradict to condition (2.8). So, $(y_{j(s)})_{s\in\mathbb{N}}$ may contain only finitely many elements $x_{i,j}$ from x_i for each $i \in \mathbb{N}$.

Choose $i \in \mathbb{N}$, so that $A/3 \ge 1/P_i$ and denote by $j(s_1)$, $s_1 \in \mathbb{N}$, the greatest index of elements from $(y_{j(s)})$ satisfying the condition: elements from sequences $(x_{1,j}),\dots,(x_{i-1,j})$ occur in $(y_{j(s)})$. (There are finitely many such elements and thus $j(s_1) \in \mathbb{N}$ is well defined.) Then, by construction, we have $a_{j(s)}/y_{j(s)} \le 1/p_i \le A/2$, for $s \ge s_1$, which is a contradiction. So, $A = 0$, that is, $y \in \nabla(a)$.

□

Corollary 2.9. *Let $a = (a_j)_{j\in\mathbb{N}} \in \mathbb{S}$. Then the selection property $\alpha_2(\nabla(a), \nabla(a))$ is satisfied (and thus $\alpha_3(\nabla(a), \nabla(a))$ and $\alpha_4(\nabla(a), \nabla(a))$ are also satisfied).*

Remark 2.10. Note that Theorems 2.2 and 2.5 can be formulated and shown in game-theoretic terms.

Let

$$\mathbb{S}_\infty := \{a = (a_n)_{n\in\mathbb{N}} \in \mathbb{S} : \lim_{n\to+\infty} a_n = +\infty\}. \tag{2.9}$$

Corollary 2.11. *$\mathbb{S}_\infty$ has the selection property $\alpha_2(\mathbb{S}_\infty, \mathbb{S}_\infty)$.*

Proof. Let $a = (a_n)_{n\in\mathbb{N}}$ be the constant sequence with $a_n = 1$ for each $n \in \mathbb{N}$ (or $a_n = c > 0$, $n \in \mathbb{N}$). Then $\mathbb{S}_\infty = \triangledown(a)$. By Theorem 2.8, $\alpha_2(\triangledown(a), \triangledown(a))$ is true. Thus we have that $\alpha_2(\mathbb{S}_\infty, \mathbb{S}_\infty)$ is also satisfied. □

Let $a = (a_n)_{n\in\mathbb{N}} \in \mathbb{S}$ be fixed. A sequence $b = (b_n)_{n\in\mathbb{N}}$ in $\mathbb{S}$ is said to be *strongly asymptotically equivalent to a* if for every $\mu > 1$ the conditions $b \in \{a\}_\mu$ and $a \in \{b\}_\mu$ are satisfied.

This is equivalent to the fact that for every $\mu > 1$ there exists $n_0 = n_0(\mu) \in \mathbb{N}$ such that $(1/\mu)\cdot b_n \le a_n \le \mu\cdot b_n$ for all $n \ge n_0$, or to the fact $\lim_{n\to+\infty}(a_n/b_n) = 1$ if each b_n is nonzero.

This relation is an equivalence relation on $\mathbb{S}$ and is also known as the *weak asymptotic equality*.

For a fixed $a \in \mathbb{S}$ denote by $[a]$ the set of all sequences from $\mathbb{S}$ which are strongly asymptotically equivalent to a.

Theorem 2.12. *Let $a \in \mathbb{S}$ be given. Then $\alpha_2([a],[a])$ is true.*

Proof. The proof is quite similar to the proof of Theorem 2.8. □

Corollary 2.13. *Let $a \in \mathbb{S}$ be a constant sequence $a_n = c > 0$ for each $n \in \mathbb{N}$. Then $[a]$ satisfies $\mathsf{S}_1([a],[a])$.*

Corollary 2.14. *Let $a = (a_n)_{n\in\mathbb{N}} \in \mathbb{S}$ be a constant sequence with $a_n = c$, $c > 0$, $n \in \mathbb{N}$. Then $[a] = [c] = \{b \in \mathbb{S} : \lim_{n\to+\infty} b_n = c\}$ satisfies the selection principles $\alpha_k([a],[a])$, $k = \{2,3,4\}$.*

Notice, that under assumptions of Corollary 2.14, the selection principle $\alpha_1([a],[a])$ is also satisfied.

Indeed, let $(b_n : n \in \mathbb{N})$ be a sequence of elements from $[a]$ and let for each n, $b_n = (b_{n,m})_{m\in\mathbb{N}}$. Take an arbitrary $i \in \mathbb{N}$ and set $U_i = (c - 1/i, c + 1/i)$. For each $n \in \mathbb{N}$ there is $m_n \in \mathbb{N}$ such that $b_{n,m} \in U_i$ for each $m \ge m_n$. Put $M = \cup\{\mathbb{N}\setminus\{1,\dots,m_n\} : n \in \mathbb{N}\}$ and let $\varphi : \mathbb{N}\to M$ be any bijection. Then the sequence $(b_{\varphi(n)})_{n\in\mathbb{N}}$ is contained in U_i. Since $i \in \mathbb{N}$ was arbitrary, we conclude that $\alpha_1([a],[a])$ holds.

We end the paper with a result closely related to the considered material.

Let $A \in [0,+\infty)$ and let $a = (a_n)_{n\in\mathbb{N}}$ be the sequence such that $a_n = A$ for each $n \in \mathbb{N}$. A sequence $b = (b_n)_{n\in\mathbb{N}} \in S$ belonging to $[a]$ is said to *converge rapidly* to A if the Landau sequence of b defined by

$$w_n(b) = \sup\{\,|\,b_m - b_k\,| : m \ge n, k \ge n\}, \quad n \in \mathbb{N} \tag{2.10}$$

belongs to de Haan's class $R_{-\infty,s}$ of rapidly varying sequences of index of variability $-\infty$, that is, for each $\lambda > 1$ the following asymptotic condition is satisfied:

$$\lim_{n\to+\infty} \frac{w_{[\lambda n]}}{w_n} = 0. \tag{2.11}$$

Or equivalently,

$$\lim_{n\to+\infty} \frac{w_{[\lambda n]}}{w_n} = +\infty, \quad 0 < \lambda < 1. \tag{2.12}$$

If $a = (a_n)_{n\in\mathbb{N}} \in \mathbb{S}$, then

$$[a]_{R_{-\infty,s}} = \{b = (b_n)_{n\in\mathbb{N}} \in [a] : (w_n(b))_{n\in\mathbb{N}} \in R_{-\infty,s}\}. \tag{2.13}$$

In [11], we showed the following result.

Theorem 2.15. *Let $A \in (0, +\infty)$ and let $a = (a_n)_{n\in\mathbb{N}}$, where $a_n = A$ for each $n \in \mathbb{N}$, be the constant sequence. Then the selection principle $\mathsf{S}_1([a]_{R_{-\infty,s}}, [a]_{R_{-\infty,s}})$ is satisfied.*

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