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Perturbation theorems for Hele-Shaw flows and their applications

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Abstract. In this work, we give a perturbation theorem for strong polynomial solutions to the zero surface tension Hele-Shaw equation driven by injection or suction, the so called Polubarinova–Galin equation. This theorem enables us to explore properties of solutions with initial functions close to polynomials. Applications of this theorem are given in the suction and injection cases. In the former case, we show that if the initial domain is close to a disk, most of the fluid will be sucked before the strong solution blows up. In the latter case, we obtain precise large-time rescaling behaviors for large data to Hele-Shaw flows in terms of invariant Richardson complex moments. This rescaling behavior result generalizes a recent result regarding large-time rescaling behavior for small data in terms of moments. As a byproduct of a theorem in this paper, a short proof of existence and uniqueness of strong solutions to the Polubarinova–Galin equation is given.

1. Introduction

This paper deals with classical zero surface tension Hele-Shaw flows. The driving mechanism, injection or suction with a constant rate 2π or -2π at the origin, produces a family of domains $\{\Omega(t)\}_{t\geq 0}$. In two dimensions, Galin and Polubarinova-Kochina reformulated the planar model of Hele-Shaw flows by describing the domains $\{\Omega(t)\}_{t\geq 0}$ by a family of conformal mappings $\{f(\zeta,t)\}_{t\geq 0}$, where $f(\zeta,t)\colon \mathbb{D}\to\Omega(t)$, $f(0,t)=0$ and $f'(0,t)>0$. Here we set

$$f_t(\zeta,t)=\frac{\partial}{\partial t}f(\zeta,t),\quad f'(\zeta,t)=\frac{\partial}{\partial \zeta}f(\zeta,t),\quad \mathbb{D}=\mathbb{D}_1(0)\quad\text{and}\quad \mathbb{D}_r=\mathbb{D}_r(0),$$

where $\mathbb{D}_r(z_0)=\{x\in\mathbb{R}^2\,|\,|x-z_0|<r\}$. Equations for $f(\zeta,t)$, the so called *Polubarinova–Galin equations*, are derived under this reformulation and they are expressed

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in the case of injection and suction as

$$\mathrm{Re}[f_t(\zeta,t)\overline{f'(\zeta,t)\zeta}]=1, \quad \zeta\in\partial\mathbb{D}, \tag{1}$$

and

$$\mathrm{Re}[f_t(\zeta,t)\overline{f'(\zeta,t)\zeta}]=-1, \quad \zeta\in\partial\mathbb{D}, \tag{2}$$

respectively. A solution to (1) or (2) is said to be a *strong solution* for $t\in[0,b)$ if $f(\zeta,t)$ is univalent and analytic in a neighborhood of $\overline{\mathbb{D}}$, $f(0,t)=0$, $f'(0,t)>0$ and $f(\zeta,t)$ is continuously differentiable in $t\in[0,b)$.

For any set $\mathbb{E}$ which contains the origin, we define

$$\mathcal{H}(\mathbb{E})=\{f\mid f \text{ is analytic in a neighborhood of } \mathbb{E}\};$$
$$\mathcal{O}_0(\mathbb{E})=\{f\in\mathcal{H}(\mathbb{E})\mid f'\neq 0, f(0)=0 \text{ and } f'(0)>0\};$$
$$\mathcal{O}(\mathbb{E})=\{f\in\mathcal{O}_0(\mathbb{E})\mid f \text{ is univalent in a neighborhood of } \mathbb{E}\}.$$

The short-time well-posedness of (1) has been thoroughly explored. In Reissig–von Wolfersdorf [7], the authors prove the existence and uniqueness of a short-time strong solution in $\mathcal{O}(\overline{\mathbb{D}})$ if the initial function is in $\mathcal{O}(\overline{\mathbb{D}})$. In Gustafsson [1], the author proves that a strong solution to (1) is a family of polynomials of degree k_0 if its initial function in $\mathcal{O}(\overline{\mathbb{D}})$ is also a polynomial of degree k_0. These results can all be applied to (2) as well even though the authors do not comment on that.

In this paper, we first prove a perturbation theorem for the strong polynomial solutions to the Polubarinova–Galin equations (1) and (2). Many properties for strong polynomial solutions are thoroughly known. This theorem enables us to explore the properties of evolution of perturbed polynomials which are nonpolynomial. We obtain two applications of this theorem in the suction and injection cases.

We first state this perturbation theorem. We define the following norms to describe the evolution of solutions

$$\left|\sum_{j=0}^{\infty}a_j\zeta^j\right|_M=\sum_{j=0}^{\infty}|a_j| \quad\text{and}\quad \left|\sum_{j=0}^{\infty}a_j\zeta^j\right|_{M(r)}=\sum_{j=0}^{\infty}|a_jr^j|.$$

Also, we define the following norm to describe the small perturbation

$$\|v\|_{\rho,m}=\sum_{j=1}^{\infty}|v_j|\rho^j j^{1/2+m}, \quad v=\sum_{j=1}^{\infty}v_j\zeta^j.$$

The perturbation theorem, Theorem 1.1, describes the evolution of small perturbation of polynomials and is stated as follows.

Theorem 1.1. *Given a strong degree-k_0 polynomial solution $f_{k_0}(\zeta,t)$ to* (1) (*or* (2)) *such that for some $T_0>0$ and $r>1$, $f_{k_0}(\zeta,t)\in\mathcal{O}(\overline{\mathbb{D}}_r)$ for all $t\in[0,T_0]$. Then for $\varepsilon>0$, $m\in\mathbb{N}$ and $1<r'<r$, there are $\delta(f_{k_0},T_0,\varepsilon,m,r')>0$ and $\rho(f_{k_0},T_0,\varepsilon,m,r')>1$ such that if $\|f(\cdot,0)-f_{k_0}(\cdot,0)\|_{\rho,m}<\delta$, where $f(0,0)=0$ and $f'(0,0)>0$, then the strong solution to* (1) (*or* (2)) *$f(\zeta,t)$ satisfies*

$$f(\zeta,t)\in\mathcal{C}^1([0,T_0],\mathcal{O}(\overline{\mathbb{D}}_{r'})\cap\mathcal{H}(\mathbb{D}_r)),$$

and for $0\le l\le m$ and $0\le t\le T_0$,

$$\left|f_{k_0}^{(l)}(\cdot,t)-f^{(l)}(\cdot,t)\right|_{M(r)}<\varepsilon.$$

The applications of this theorem and related past results are stated briefly in Sections 1.1 and 1.2 below.

1.1. Here we assume that the driving mechanism is suction. It has been known that strong solutions to (2) must blow up before the fluid is sucked out except for the degree-1 polynomial solutions. However, by taking $k_0=1$ in Theorem 1.1, we prove that if the initial domain is close to a disk, most of the fluid is sucked before the strong solution to (2) blows up.

1.2. Now we assume that the driving mechanism is injection. In Sakai [10] and Gustafsson–Sakai [3], the authors consider solutions of weak formulations and investigate the radius and curvature of two-dimensional moving domains, respectively. For an arbitrary initial shape of the moving domain its asymptote is an expanding disk. Recently, progress regarding this asymptotic behavior has been made by investigating it in terms of conserved quantities, so called Richardson complex moments; see Richardson [8]. In Vondenhoff [11], by restricting multi-dimensional initial domains to be close to balls, the author gives a rescaling behavior of the moving boundaries in terms of conserved moments. In this paper, we aim at generalizing the former result in two-dimensions by assuming a larger set of initial domains.

It is known that there is a general class of polynomials which can give rise to global strong polynomial solutions to (1) and the corresponding initial domains can be quite different from disks; for example, starlike polynomials (e.g., $\zeta+\frac{2}{5}\zeta^2$ and $\zeta/1.1-\frac{15}{14}(\zeta/1.1)^2+\frac{4}{7}(\zeta/1.1)^3-\frac{1}{7}(\zeta/1.1)^4$); see Gustafsson–Prokhorov–Vasil′ev [2]. An arbitrary global strong degree-k_0 polynomial solution to (1), called $f_{k_0}(\zeta,t)$, can have its rescaling behavior precisely described in terms of moments; see Lin [5]. In this paper, as an application of Theorem 1.1, we show that a small perturbation of $f_{k_0}(\zeta,0)$, called $f(\zeta,0)$, can give rise to a global strong solution $f(\zeta,t)$, and a rescaling behavior of the corresponding moving domains, similar to that stated in Vondenhoff [11], is given in terms of moments as well. We can deduce the case when the initial domain is a small perturbation of a disk from this result by letting $k_0=1$. Therefore, this result generalizes the result in Vondenhoff [11]. Lin [5], Vondenhoff [11] and this paper consider different sets of initial data and the rescaling behavior in Lin [5] is different from that in Vondenhoff [11] and this paper. However, geometrically, these rescaling behaviors in the three works all imply that by rescaling the corresponding moving domain $\Omega(t)$, $t\geq 0$, to be a domain $\Omega'(t)$ with area π, the radius and curvature of $\partial\Omega'(t)$ decay to 1 algebraically and the decay is faster if lower moments vanish.

A sketch of the proof of this result is as follows: We first apply Theorem 1.1 and prove the existence of a locally-in-time strong solution $\{f(\zeta,t)\}_{0\leq t\leq T_0}$ where $f(\zeta,T_0)$ is strongly starlike and $f(\mathbb{D},T_0)$ is a small perturbation of a disk, even though $f(\zeta,0)$ can be nonstarlike and $f(\mathbb{D},0)$ is far from a disk. Since starlikeness is a sufficient condition for an initial function to give rise to a global strong solution as shown in Gustafsson–Prokhorov–Vasil′ev [2], and since large-time rescaling behavior for evolution of perturbed disks is shown in Vondenhoff [11] in terms of moments, the solution $f(\zeta,t)$ must be global and a rescaling behavior is given in terms of moments as well.

The structure of this paper is as follows. In Section 2, we prove Theorem 1.1. In Section 3, the application of Theorem 1.1 in the suction case is given. In Section 4, the application of Theorem 1.1 in the injection case is given. As a byproduct of Theorem 2.5, a short proof of existence and uniqueness of strong solutions to (1) is given in Section 5.

2. Proof of Theorem 1.1

The proof of the perturbation theorem in the suction case is almost the same as the proof in the injection case. Therefore, we will just provide the proof of the theorem in the case of injection (1).

As in Gustafsson [1], a reformulation of the Polubarinova–Galin equation (1) is expressed as

$$f_t=\zeta f' P\left[\frac{1}{|f'|^2}\right],\quad \zeta\in\mathbb{D}, \tag{3}$$

where P denotes the Poisson integral which defines the analytic function in the unit disk

$$P[g](\zeta)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}} g(z)\frac{z+\zeta}{z-\zeta}\frac{dz}{z},\quad \zeta\in\mathbb{D}, \tag{4}$$

from boundary data g on $\partial\mathbb{D}$ of its real part. In the mathematical treatment of (3) it makes no difference if $f(\zeta,t)$ is univalent in $\overline{\mathbb{D}}$ or merely locally univalent in $\overline{\mathbb{D}}$; see Gustafsson [1]. To make a distinction, we define a solution to be a strong* solution to (3) as follows.

Definition 2.1. A solution $f(\zeta,t)\in\mathcal{O}_0(\overline{\mathbb{D}})$ is a *strong* solution* to (3) for $0\le t<b$ if $f(\zeta,t)$ is continuously differentiable with respect to $t\in[0,b)$ and satisfies (3).

A univalent strong* solution $f(\zeta,t)$ to (3) must be a strong solution to the Polubarinova–Galin equation (1).

In Section 2.1, we aim to prove a perturbation theorem for strong* polynomial solutions to (3), Theorem 2.5. In Section 2.2, we show that Theorem 1.1 follows directly from Theorem 2.5.

2.1. A perturbation theorem for strong* polynomial solutions

We start with some lemmas before proving the perturbation theorem for strong* polynomial solutions to (3).

Lemma 2.2. *For $1<p<\infty$, there exists $C_p>0$ such that*

$$\|P[g]\|_{L^p([0,2\pi])}\le C_p\|g\|_{L^p([0,2\pi])}$$

for any real-valued function $g\in L^p([0,2\pi])$.

Proof. There exists u which is harmonic in $\mathbb{D}$, continuous in $\overline{\mathbb{D}}$, and satisfies $u=g$ on $\partial\mathbb{D}$. Therefore, by Theorem 17.26 in Rudin [9], it is shown that for $1<p<\infty$, there exists $C_p>0$ such that

$$\|P[u]\|_{L^p([0,2\pi])}\le C_p\|u\|_{L^p([0,2\pi])},$$

which means that

$$\|P[g]\|_{L^p([0,2\pi])}\le C_p\|g\|_{L^p([0,2\pi])}. \quad \square$$

In the proof of the perturbation theorem for strong* polynomial solutions, we use an iterative method. In each iteration, we need to calculate the difference of two polynomial univalent functions h_1 and h_2 which satisfy the assumption of Lemma 2.3. Inequality (7) enables us to estimate $\|h_1'-h_2'\|_{L^2([0,2\pi])}$ locally in time when h_1 and h_2 are both polynomial as shown in the proof of Theorem 2.5.

In the rest of this section, for any function $F(\zeta)$, we define

$$F^*(\zeta)=\overline{F(\zeta^*)},\quad \text{where } \zeta^*=\frac{1}{\bar{\zeta}}.$$

Lemma 2.3. *Let $g(\zeta,t)\in\mathcal{C}^1([0,t_1],\mathcal{O}_0(\overline{\mathbb{D}}_r))$ be a strong[*] solution to (3) and $0<\ell<1$. There is $C(g,t_1,r,\ell)>0$ such that, if $h_1(z,t),h_2(z,t)\in\mathcal{C}^1([0,t_h],\mathcal{O}_0(\overline{\mathbb{D}}_r))$ are two strong[*] solutions to (3), where $0<t_h\le t_1$, and*

$$\max_{[0,t_h]}|h_j'(\,\cdot\,,t)-g'(\,\cdot\,,t)|_{M(r)}\le\ell\min_{\overline{\mathbb{D}}_r\times[0,t_1]}|g'|,\quad j=1,2,\tag{5}$$

then we have

$$\left\|\frac{\partial}{\partial t}[h_1-h_2]\right\|_{L^2([0,2\pi])}\le C\|h_1'-h_2'\|_{L^2([0,2\pi])},\quad 0\le t\le t_h.\tag{6}$$

Furthermore, if h_1 and h_2 are both polynomials of degree $\le n$, then for $0\le t\le t_h$,

$$\|h_1'(\,\cdot\,,t)-h_2'(\,\cdot\,,t)\|^2_{L^2([0,2\pi])}\le e^{2nCt}\|h_1'(\,\cdot\,,0)-h_2'(\,\cdot\,,0)\|^2_{L^2([0,2\pi])}.\tag{7}$$

Proof. (a)

$$\frac{\partial}{\partial t}[h_1-h_2]=\zeta\left[[h_1'-h_2']P\left[\frac{1}{|h_2'|^2}\right]+h_1'P\left[\frac{1}{|h_1'|^2}-\frac{1}{|h_2'|^2}\right]\right].\tag{8}$$

Here, by Lemma 2.2,

$$\left\|P\left[\frac{1}{|h_1'|^2}-\frac{1}{|h_2'|^2}\right]\right\|_{L^2([0,2\pi])}\le C_2\left\|\frac{1}{|h_1'|^2}-\frac{1}{|h_2'|^2}\right\|_{L^2([0,2\pi])}.\tag{9}$$

By taking the L_2 norms of the right-hand and left-hand sides of (8) and then using (9) and Hölder's inequality, we obtain that

$$\begin{aligned}\left\|\frac{\partial}{\partial t}[h_1-h_2]\right\|_{L^2([0,2\pi])}&\le\|h_1'-h_2'\|_{L^2([0,2\pi])}\max_{\partial\mathbb{D}}\left|P\left[\frac{1}{|h_2'|^2}\right]\right|\\&\quad+C_2\|h_1'\|_{L^\infty([0,2\pi])}\left\|\frac{1}{|h_1'|^2}-\frac{1}{|h_2'|^2}\right\|_{L^2([0,2\pi])}\\&\le\left[\max_{\partial\mathbb{D}}\left|P\left[\frac{1}{|h_2'|^2}\right]\right|\right.\\&\quad\left.+C_2\max_{\partial\mathbb{D}}|h_1'|\max_{\partial\mathbb{D}}\frac{|h_1'|+|h_2'|}{|h_1'|^2|h_2'|^2}\right]\|h_1'-h_2'\|_{L^2([0,2\pi])}.\end{aligned}\tag{10}$$

Below we want to bound

$$\max_{\partial\mathbb{D}}|h_1'|\max_{\partial\mathbb{D}}\frac{|h_1'|+|h_2'|}{|h_1'|^2|h_2'|^2}\quad\text{and}\quad\max_{\partial\mathbb{D}}\left|P\left[\frac{1}{|h_2'|^2}\right]\right|$$

in (a1) and (a2), respectively, in terms of g and hereby determine the constant C.

(a1) By assumption (5), for $(z,t)\in\partial\mathbb{D}\times[0,t_h]$,

$$|h_j'(z,t)| \geq |g'(z,t)|-|h_j'(z,t)-g'(z,t)| \geq (1-\ell)|g'(z,t)|, \quad j=1,2, \tag{11}$$

$$|h_j'(z,t)| \leq |g'(z,t)|+|h_j'(z,t)-g'(z,t)| \leq (1+\ell)|g'(z,t)|, \quad j=1,2. \tag{12}$$

Therefore, by (11) and (12), for $0\leq t\leq t_h$,

$$\max_{\partial\mathbb{D}}|h_1'|\max_{\partial\mathbb{D}}\frac{|h_1'|+|h_2'|}{|h_1'|^2|h_2'|^2} \leq 2\frac{(1+\ell)}{(1-\ell)^3}\max_{\partial\mathbb{D}\times[0,t_1]}|g'|\max_{\partial\mathbb{D}\times[0,t_1]}\frac{1}{|g'|^3}. \tag{13}$$

(a2) We start with finding the upper bound of $P[1/|h_2'|^2-1/|g'|^2]$ in terms of g and hereby obtain the upper bound for $P[1/|h_2'|^2]$ in terms of g.

In Gustafsson [1], it is shown that for any $h\in\mathcal{O}_0(\overline{\mathbb{D}}_r)$,

$$P\left[\frac{1}{|h'|^2}\right]=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_r}\frac{1}{h'(z,t)(h')^*(z,t)}\frac{z+\zeta}{z-\zeta}\frac{dz}{z}, \quad \zeta\in\mathbb{D}. \tag{14}$$

By (14), we have for $\zeta\in\mathbb{D}$,

$$P\left[\frac{1}{|h_2'|^2}-\frac{1}{|g'|^2}\right]=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_r}\left(\frac{1}{h_2'(z,t)(h_2')^*(z,t)}-\frac{1}{g'(z,t)(g')^*(z,t)}\right)\frac{z+\zeta}{z-\zeta}\frac{dz}{z}. \tag{15}$$

Therefore,

$$\begin{aligned}\max_{\partial\mathbb{D}}\left|P\left[\frac{1}{|h_2'|^2}-\frac{1}{|g'|^2}\right]\right| &\leq \max_{\partial\mathbb{D}_r}\left|\frac{1}{h_2'(z,t)(h_2')^*(z,t)}-\frac{1}{g'(z,t)(g')^*(z,t)}\right|\frac{r+1}{r-1}\\ &=\max_{\partial\mathbb{D}_r}\left|\frac{h_2'(z,t)-g'(z,t)}{g'(z,t)(g')^*(z,t)h_2'(z,t)}+\frac{(h_2')^*(z,t)-(g')^*(z,t)}{h_2'(z,t)(h_2')^*(z,t)(g')^*(z,t)}\right|\frac{r+1}{r-1}.\end{aligned} \tag{16}$$

By assumption (5), for $(z,t)\in\partial\mathbb{D}_r\times[0,t_h]$,

$$|(h_2')^*(z,t)-(g')^*(z,t)| \leq \ell|(g')^*(z,t)|, \tag{17}$$

$$|(h_2')^*(z,t)| \geq (1-\ell)|(g')^*(z,t)|, \tag{18}$$

$$|h_2'(z,t)-g'(z,t)| \leq \ell|g'(z,t)|, \tag{19}$$

$$|h_2'(z,t)| \geq (1-\ell)|g'(z,t)|. \tag{20}$$

By (17)–(20),

$$\begin{aligned}&\max_{\partial\mathbb{D}_r\times[0,t_h]}\left|\frac{h_2'(z,t)-g'(z,t)}{g'(z,t)(g')^*(z,t)h_2'(z,t)}+\frac{(h_2')^*(z,t)-(g')^*(z,t)}{h_2'(z,t)(h_2')^*(z,t)(g')^*(z,t)}\right|\frac{r+1}{r-1}\\ &\qquad\leq 2\ell\left[\max_{\partial\mathbb{D}_r\times[0,t_1]}\left|\frac{1}{|g'(z,t)|\,|(g')^*(z,t)|(1-\ell)^2}\right|\right]\frac{r+1}{r-1}.\end{aligned}$$

Therefore, by the above inequality and (16), for $0 \le t \le t_h$,

$$\max_{\partial\mathbb{D}}\left|\zeta P\left[\frac{1}{|h_2'|^2}-\frac{1}{|g'|^2}\right]\right| \le 2\ell\left[\max_{\partial\mathbb{D}_r\times[0,t_1]}\left|\frac{1}{|g'(z,t)|\,|(g')^*(z,t)|(1-\ell)^2}\right|\right]\frac{r+1}{r-1}.$$

Hence, for $0 \le t \le t_h$, we have that

$$\begin{aligned}\max_{\partial\mathbb{D}}\left|\zeta P\left[\frac{1}{|h_2'|^2}\right]\right| &\le \max_{\partial\mathbb{D}}\left|\zeta P\left[\frac{1}{|h_2'|^2}-\frac{1}{|g'|^2}\right]\right| + \max_{\partial\mathbb{D}}\left|\zeta P\left[\frac{1}{|g'|^2}\right]\right| \\ &\le 2\ell\left[\max_{\partial\mathbb{D}_r\times[0,t_1]}\left|\frac{1}{|g'(z,t)|\,|(g')^*(z,t)|(1-\ell)^2}\right|\right]\frac{r+1}{r-1} \\ &\quad + \max_{\partial\mathbb{D}\times[0,t_1]}\left|\zeta P\left[\frac{1}{|g'|^2}\right]\right|.\end{aligned}$$

From (a1) and (a2), we prove (6) by choosing C to be

$$\begin{aligned}C = &\max_{\partial\mathbb{D}\times[0,t_1]}\left|\zeta P\left[\frac{1}{|g'|^2}\right]\right| + 2\ell\left[\max_{\partial\mathbb{D}_r\times[0,t_1]}\left|\frac{1}{|g'(z,t)|\,|(g')^*(z,t)|(1-\ell)^2}\right|\right]\frac{r+1}{r-1} \\ &+ C_2\left[2\frac{(1+\ell)}{(1-\ell)^3}\max_{\partial\mathbb{D}\times[0,t_1]}|g'|\max_{\partial\mathbb{D}\times[0,t_1]}\frac{1}{|g'|^3}\right].\end{aligned}$$

(b) Now we assume that h_1 and h_2 are both polynomials of degree $\le n$. Let $h_1 = \sum_{j=1}^{n}\alpha_j(t)\zeta^j$, $h_2 = \sum_{j=1}^{n}\beta_j(t)\zeta^j$ and

$$D(t) = \|h_1' - h_2'\|^2_{L^2([0,2\pi])} = 2\pi\sum_{j=1}^{n}[|\alpha_j(t)-\beta_j(t)|]^2 j^2.$$

Then

$$\begin{aligned}D'(t) &= 2\pi\cdot 2\sum_{j=1}^{n}\mathrm{Re}[(\alpha_j-\beta_j)\overline{(\alpha_j-\beta_j)_t}]j^2 \\ &\le 2\pi\cdot 2n\sum_{j=1}^{n}|(\alpha_j-\beta_j)||(\alpha_j-\beta_j)_t|j \\ &\le 2\pi\cdot 2n\left(\sum_{j=1}^{n}|(\alpha_j-\beta_j)|^2 j^2\right)^{1/2}\left(\sum_{j=1}^{n}|(\alpha_j-\beta_j)_t|^2\right)^{1/2} \\ &= 2n\|[h_1'-h_2']\|_{L^2([0,2\pi])}\left\|\frac{\partial}{\partial t}[h_1-h_2]\right\|_{L^2([0,2\pi])}.\end{aligned}$$

By applying (6) to the above inequality, we conclude that for $0 \le t \le t_h$,

$$D'(t) \le 2nC\|[h_1'-h_2']\|^2_{L^2([0,2\pi])} = 2nCD(t), \tag{21}$$

and therefore

$$D(t) \le D(0)e^{2nCt}, \tag{22}$$

which proves (7). □

The following lemma helps us control the blow-up time of strong* polynomial solutions to (3).

Lemma 2.4. *Given a polynomial mapping $f(\zeta,0)\in\mathcal{O}_0(\overline{\mathbb{D}}_{r_0})$ for some $r_0>1$, there exists a unique strong* polynomial solution $f(\zeta,t)\in\mathcal{O}_0(\overline{\mathbb{D}}_{r_0})$ to (3) at least for a short time. Furthermore, if the strong* polynomial solution ceases to exist at $t=b$, then for any $r>1$,*

$$\liminf_{t\to b}\Big(\min_{\overline{\mathbb{D}}_r}|f'(\zeta,t)|\Big)=0. \tag{23}$$

Proof. The first part follows from Gustafsson [1].

For the second part, assume that (23) does not hold. Then there exists $r>1$ such that

$$\liminf_{t\to b}\Big(\min_{\overline{\mathbb{D}}_r}|f'(\zeta,t)|\Big)>0.$$

This implies that there exist $C>0$ and $1<r'\le r$ such that

$$\min_{\overline{\mathbb{D}}_{r'}}|f'(\zeta,t)|>C, \quad t\in[0,b).$$

Since each coefficient of $f(\zeta,t)$ is bounded for $t\in[0,b)$, there exists $M>0$ such that

$$\sup_{t\in[0,b)}\max_{\overline{\mathbb{D}}_{r'}}|f'(\zeta,t)\zeta|\le M.$$

For $\zeta\in\overline{\mathbb{D}}$,

$$\begin{aligned}\sup_{t\in[0,b)}\left|f'(\zeta,t)\zeta P\left[\frac{1}{|f'|^2}\right]\right| &\le \sup_{t\in[0,b)}\left|\frac{f'(\zeta,t)\zeta}{2\pi i}\int_{\partial\mathbb{D}_{r'}}\frac{1}{f'(z,t)(f')^*(z,t)}\frac{z+\zeta}{z-\zeta}\frac{dz}{z}\right|\\ &\le \sup_{t\in[0,b)}\left(\max_{\overline{\mathbb{D}}}|f'(\zeta,t)\zeta|\cdot\max_{\partial\mathbb{D}_{r'}}\left|\frac{1}{f'(z,t)(f')^*(z,t)}\right|\frac{r'+1}{r'-1}\right)\\ &\le \frac{M}{C^2}\frac{r'+1}{r'-1}.\end{aligned}$$

Hence, for $0\le t_2<t_1<b$ and $\zeta\in\mathbb{D}$,

$$|f(\zeta,t_1)-f(\zeta,t_2)|=\left|\int_{t_2}^{t_1}f'(\zeta,t)\zeta P\left[\frac{1}{|f'|^2}\right]dt\right|\le|t_1-t_2|\frac{M}{C^2}\frac{r'+1}{r'-1}.$$

Therefore $\lim_{t\to b} f(\zeta,t)$ exists and we define it as $f(\zeta,b)$. Note that $f(\zeta,b)$ satisfies $\min_{\overline{\mathbb{D}}_{r'}} |f'(\zeta,b)| \geq C$. Let $f(\zeta,t+b)$ be the strong* solution to (3) with the initial value $f(\zeta,b)$ for $t\in[0,\varepsilon)$. Then $f(\zeta,t)$ is continuous with respect to t for $t\in[0,b+\varepsilon)$ and

$$f(\zeta,t)-f(\zeta,0)=\int_0^t f'(\zeta,s)\zeta P\left[\frac{1}{|f'(\,\cdot\,,s)|^2}\right] ds.$$

This implies that $f(\zeta,t)\in\mathcal{O}_0(\overline{\mathbb{D}})$ is continuously differentiable with respect to t for $t\in[0,b+\varepsilon)$ and satisfies (3). Hence it is impossible that $f(\zeta,t)$ ceases to exist at $t=b$ and therefore for any $r>1$,

$$\liminf_{t\to b}\Big(\min_{\overline{\mathbb{D}}_r} |f'(\zeta,t)|\Big)=0. \qquad \square$$

Theorem 2.5. *Assume that* $f_{k_0}(\zeta,t)\in\mathcal{C}^1([0,t_1],\mathcal{O}_0(\overline{\mathbb{D}}_r))$ *is a strong** *degree-*k_0 *polynomial solution to* (3) *for some* $t_1>0$ *and* $r>1$ *and that* $\rho>r$ *and* $\ell<1$. *If* $f(\zeta,0)$ *satisfies the assumption*

$$\|f(\zeta,0)-f_{k_0}(\zeta,0)\|_{\rho,1}\leq\frac{\ell}{\sqrt{k_0}}\min_{\overline{\mathbb{D}}_r\times[0,t_1]}|f'_{k_0}|,$$

where $f'(0,0)\in\mathbb{R}$ *and* $f(0,0)=0$, *then the following are true*:

(a) *There exists* $C(f_{k_0},t_1,r,\ell)>0$ *such that the strong** *solution* $f(\zeta,t)$ *to* (3) *satisfies*

$$f(\zeta,t)\in\mathcal{C}^1([0,t_0],\mathcal{O}_0(\mathbb{D}_r)\cap\mathcal{C}(\overline{\mathbb{D}}_r)),$$

where $t_0=\min\{(1/Ck_0)\log(\rho/r)t_1\}$. *Moreover,*

$$\max_{[0,t_0]}|f'-f'_{k_0}|_{M(r)}\leq\ell\min_{\overline{\mathbb{D}}_r\times[0,t_1]}|f'_{k_0}|;$$

(b) *Furthermore, if there exist* $\delta>0$ *and* $l\in\mathbb{N}$ *such that*

$$\|f(\,\cdot\,,0)-f_{k_0}(\,\cdot\,,0)\|_{\rho,l}\leq\delta,$$

then there exists $c(l,k_0)>0$ *such that*

$$\max_{[0,t_0]}|f^{(l)}-f_{k_0}^{(l)}|_{M(r)}\leq c(l,k_0)\delta.$$

Remark 2.6. The strong* solution $f(\zeta,t)$ is obtained as a limit of polynomial strong* solutions to (3).

Proof. (a) We take the constant $C(f_{k_0},t_1,r,\ell)$ in (a) to be the same as the one defined in Lemma 2.3. We want to prove (a) by showing that there exists a strong*

solution $f(\zeta,t)\in\mathcal{O}_0(\mathbb{D}_r)$ to (3) for $0\le t\le t_0$, where $f(\zeta,0)=f_{k_0}(\zeta,0)+\sum_{j=1}^{\infty} b_j(0)\zeta^j$ and

$$\sum_{k=1}^{\infty}|b_k(0)|\rho^k k^{3/2}\le\frac{\ell}{\sqrt{k_0}}\min_{\overline{\mathbb{D}}_r\times[0,t_1]}|f'_{k_0}|. \tag{24}$$

Denote the strong* polynomial solution to (3) with the initial value $f_{k_0}(\zeta,0)+\sum_{j=1}^{k} b_j(0)\zeta^j$ by $g_k(\zeta,t)$. The proof for (a) is split into (a1) and (a2). In (a1), we prove that $g_k(\zeta,t), k\ge 1$ exists for $t\in[0,t_0]$. In (a2), we prove that $g_k(\zeta,t)$ converges to the strong* solution $f(\zeta,t)$, as $k\to\infty$, and that $f(\zeta,t)$ exists for $t\in[0,t_0]$.

(a1) By (24), there exists a nonnegative sequence $\{d_k\}_{k\ge 0}$ such that $\sum_{k=0}^{\infty} d_k=1$ and $|b_j(0)|\le M_j\rho^{-j}$ for $j\ge 1$, where

$$M_{k+1}\le\frac{\ell}{\sqrt{k_0}}\frac{d_k}{(k+1)^{3/2}}\min_{\overline{\mathbb{D}}_r\times[0,t_1]}|f'_{k_0}|,\quad k\ge 0.$$

Claim. *Prove that for $k\ge 0$, $g_k(\zeta,t)\in\mathcal{C}^1([0,t_0],\mathcal{O}_0(\overline{\mathbb{D}}_r))$ and*

$$\max_{[0,t_0]}|g'_k-g'_{k+1}|_{M(r)}\le\ell d_k\min_{\overline{\mathbb{D}}_r\times[0,t_1]}|g'_0|.$$

Proof. We prove the claim by induction as follows.

(i) Assume for $0\le k\le n-1$, that

$$\max_{[0,t_0]}|g'_k-g'_{k+1}|_{M(r)}\le\ell d_k\min_{\overline{\mathbb{D}}_r\times[0,t_1]}|g'_0|.$$

(ii) Let us prove that for $t\in[0,t_0]$,

$$|g'_n-g'_{n+1}|_{M(r)}\le\ell d_n\min_{\overline{\mathbb{D}}_r\times[0,t_1]}|g'_0|. \tag{25}$$

Let

$$s_n=\sup\{T\le t_0\,|\,g_{n+1}(\zeta,t)\text{ satisfies (25) for }t\in[0,T]\}.$$

Then $|g'_{n+1}|\ge(1-\ell)|g'_0|$ for $t\in[0,s_n)$. Therefore, by Lemma 2.4,

$$s_n=\max\{T\le t_0\,|\,g_{n+1}(\zeta,t)\text{ satisfies (25) for }t\in[0,T]\}.$$

For $0<t\le s_n$,

$$\max_{[0,t]}|g'_{n+1}-g'_0|_{M(r)}\le\sum_{k=0}^{n}\ell d_k\min_{\overline{\mathbb{D}}_r\times[0,t_1]}|g'_0|\le\ell\min_{\overline{\mathbb{D}}_r\times[0,t_1]}|g'_0|. \tag{26}$$

Also by the assumption in (i), we have that

$$\max_{[0,t_0]} |g_n' - g_0'|_{M(r)} \le \sum_{k=0}^{n-1} \ell d_k \min_{\overline{\mathbb{D}}_r \times [0,t_1]} |g_0'| \le \ell \min_{\overline{\mathbb{D}}_r \times [0,t_1]} |g_0'|. \tag{27}$$

From (26) and (27), g_0, g_n and g_{n+1} satisfy the assumptions on g, h_1 and h_2 in Lemma 2.3, respectively. Let $D(t)=\|g_{n+1}'-g_n'\|^2_{L^2([0,2\pi])}$. From Lemma 2.3, we obtain that for $0\le t\le s_n$,

$$D(t) \le e^{2(n+1)k_0Ct} D(0). \tag{28}$$

We need to show that $s_n = t_0$. Note that if $s_n < t_0$, then the following must hold:

$$|g_n' - g_{n+1}'|_{M(r)} = d_n \min_{\overline{\mathbb{D}}_r \times [0,t_1]} |g_0'| \ell \quad \text{at time } t = s_n. \tag{29}$$

Assume now that $s_n < t_0$. Then for $0 \le t \le s_n$,

$$\begin{aligned} |g_n' - g_{n+1}'|_{M(r)} &\le \sqrt{D(t)(n+1)k_0} r^n \\ &\le \sqrt{(n+1)k_0 D(0) e^{2Ctk_0(n+1)} r^{2n}} \\ &\le \sqrt{(n+1)k_0 D(0) e^{2Cs_nk_0(n+1)} r^{2n}} \\ &< \sqrt{(n+1)k_0 D(0) e^{2Ct_0k_0(n+1)} r^{2n}}. \end{aligned}$$

Since

$$D(0)(n+1)k_0 \le \rho^{-2(n+1)} d_n^2 \min_{\overline{\mathbb{D}}_r \times [0,t_1]} |g_0'|^2 \ell^2,$$

we have that

$$\max_{[0,s_n]} |g_n' - g_{n+1}'|_{M(r)} \le \sqrt{(n+1)k_0 D(0) e^{2Ct_0k_0(n+1)} r^{2n}} < d_n \min_{\overline{\mathbb{D}}_r \times [0,t_1]} |g_0'| \ell,$$

which contradicts (29). Therefore, $s_n = t_0$. □

(a2) By (a1), for $k \ge 1$,

$$\max_{[0,t_0]} |g_k' - g_0'|_{M(r)} \le \ell \sum_{n=0}^{\infty} d_n \min_{\overline{\mathbb{D}}_r \times [0,t_1]} |g_0'| \le \ell \min_{\overline{\mathbb{D}}_r \times [0,t_1]} |g_0'|.$$

There exists $f(\zeta,t) \in \mathcal{C}([0,t_0], \mathcal{O}_0(\mathbb{D}_r) \cap \mathcal{C}(\overline{\mathbb{D}}_r))$ such that $|g_k' - f'|_{M(r)} \to 0$, as $k \to \infty$. Furthermore,

$$\max_{[0,t_0]} |f' - g_0'|_{M(r)} \le \ell \min_{\overline{\mathbb{D}}_r \times [0,t_1]} |g_0'|.$$

Still, we have to show that $f(\zeta,t)$ satisfies (3). Fix $1<r'<r$. Then for $\zeta\in\mathbb{D}_{r'}$ and $0\le t\le t_0$,

$$\frac{\partial}{\partial t}g_k(\zeta,t)=\frac{g_k'(\zeta,t)\zeta}{2\pi i}\int_{\partial\mathbb{D}_{r'}}\frac{1}{g_k'(z,t)(g_k')^*(z,t)}\frac{z+\zeta}{z-\zeta}\frac{dz}{z}. \tag{30}$$

By integrating (30) with respect to t, we have that for $\zeta\in\mathbb{D}_{r'}$ and $0\le t\le t_0$,

$$g_k(\zeta,t)-g_k(\zeta,0)=\int_0^t\frac{g_k'(\zeta,s)\zeta}{2\pi i}\int_{\partial\mathbb{D}_{r'}}\frac{1}{g_k'(z,s)(g_k')^*(z,s)}\frac{z+\zeta}{z-\zeta}\frac{dz}{z}\,ds.$$

Letting $k\to\infty$, we get for ζ in any compact subset of $\mathbb{D}_{r'}$,

$$f(\zeta,t)-f(\zeta,0)=\int_0^t\frac{f'(\zeta,s)\zeta}{2\pi i}\int_{\partial\mathbb{D}_{r'}}\frac{1}{f'(z,s)(f')^*(z,s)}\frac{z+\zeta}{z-\zeta}\frac{dz}{z}\,ds \tag{31}$$

for some $f(\zeta,t)\in\mathcal{C}([0,t_0],\mathcal{O}_0(\mathbb{D}_r)\cap\mathcal{C}(\overline{\mathbb{D}}_r))$. Furthermore, the identity (31) shows that $f(\zeta,t)\in\mathcal{C}^1([0,t_0],\mathcal{H}(\mathbb{D}_r)\cap\mathcal{C}(\overline{\mathbb{D}}_r))$.

(b) Now consider (b). In this case

$$|b_j(0)|\le M_j\rho^{-j},\quad j\ge1,$$

where

$$M_{k+1}\le\frac{1}{(k+1)^{1/2+l}}d_k\delta,\quad k\ge0.$$

First we look at the case $l=2$. Under the assumption in (b), we have that for $n\ge0$,

$$\begin{aligned}\max_{[0,t_0]}|g_n''-g_{n+1}''|_{M(r)}&\le\sqrt{(n+2)^3(k_0+1)^3\tfrac13D(0)e^{2Ct_0k_0(n+1)}r^{n-1}}\\&=\left(\frac{n+2}{n+1}\right)^{3/2}\frac{1}{\sqrt3}(k_0+1)^{3/2}\sqrt{D(0)(n+1)^3e^{2ct_0k_0(n+1)}r^{n-1}}\\&\le\left(\frac{n+2}{n+1}\right)^{3/2}\frac{1}{\sqrt3}(k_0+1)^{3/2}d_n\delta.\end{aligned}$$

Therefore, we have for $n\ge1$ that

$$\max_{[0,t_0]}|g_0''-g_n''|_{M(r)}\le\frac{1}{\sqrt3}2^{3/2}(k_0+1)^{3/2}\delta.$$

Similarly, for $l\in\mathbb{N}$, under the assumption in (b), there exists $c(l,k_0)>0$ such that

$$\max_{[0,t_0]}|g_n^{(l)}-g_{n+1}^{(l)}|_{M(r)}\le c(l,k_0)\sqrt{(n+1)^{2l-1}D(0)e^{2Ct_0k_0(n+1)}}\le c(l,k_0)d_n\delta.$$

Therefore, we have that

$$\max_{[0,t_0]} |g_0^{(l)} - g_n^{(l)}|_{M(r)} \leq c(l,k_0)\delta.$$

Letting $n\to\infty$ shows that

$$\max_{[0,t_0]} |g_0^{(l)} - f^{(l)}|_{M(r)} \leq c(l,k_0)\delta. \quad \square$$

2.2. A perturbation theorem for strong polynomial solutions

In the former subsection, the solutions we considered were locally univalent in $\overline{\mathbb{D}}$. However, the solutions which have physical meaning are required to be univalent in $\overline{\mathbb{D}}$. The following lemma states that these locally univalent solutions are univalent if they are close to a univalent solution.

Lemma 2.7. *Given a function $g(\zeta,t)\in\mathcal{C}^1([0,T_0],\mathcal{O}(\overline{\mathbb{D}}_r))$ and $1<r'<r$, there exists $\eta(g,T_0,r')>0$ such that if*

$$\max_{[0,T_0]} |f'(\,\cdot\,,t) - g'(\,\cdot\,,t)|_{M(r)} \leq \eta,$$

where $f(\zeta,t)\in\mathcal{C}([0,T_0],\mathcal{H}(\mathbb{D}_r)\cap\mathcal{C}(\overline{\mathbb{D}}_r))$, then for $0\leq t\leq T_0$,

$$f(\zeta,t)\in\mathcal{O}(\overline{\mathbb{D}}_{r'}).$$

Proof. The proof is separated into two parts (a) and (b).

(a) First assume that

$$\max_{[0,T_0]} |f'(\,\cdot\,,t) - g'(\,\cdot\,,t)|_{M(r)} \leq \frac{1}{2} \min_{\overline{\mathbb{D}}_r\times[0,T_0]} |g'(z,t)|. \tag{32}$$

We want to show that there exists $r_0>0$ such that for any fixed $z_0\in\overline{\mathbb{D}}_{r'}$,

$$f(\,\cdot\,,t)\colon \overline{\mathbb{D}_{r_0}(z_0)} \to f(\overline{\mathbb{D}_{r_0}(z_0)})$$

is univalent. It is sufficient to prove that

$$\operatorname{Re} \frac{f'(z,t)(z-z_0)}{f(z,t)-f(z_0,t)} \geq \frac{1}{2}, \quad z\in\mathbb{D}_{r_0}(z_0),$$

which means that the function is injective on $\partial\mathbb{D}_{r_0}(z_0)$ and therefore is injective for $z\in\overline{\mathbb{D}_{r_0}(z_0)}$.

Now fix $z_0 \in \overline{\mathbb{D}}_{r'}$. Since $f(z,t)$ is analytic in $\mathbb{D}_r$,

$$f(z,t)=f(z_0,t)+\sum_{n=1}^{\infty}\frac{f^{(n)}(z_0,t)}{n!}(z-z_0)^n, \quad z\in\mathbb{D}_r.$$

Let

$$l=\min\{r',r-r'\}, \quad M=\frac{3}{2}\max_{\overline{\mathbb{D}}_r\times[0,T_0]}|g'| \quad \text{and} \quad m=\frac{1}{2}\min_{\overline{\mathbb{D}}_r\times[0,T_0]}|g'|.$$

By (32), we get that

$$\max_{\overline{\mathbb{D}}_r\times[0,T_0]}|f(z,t)|\le M \quad \text{and} \quad \min_{\overline{\mathbb{D}}_r\times[0,T_0]}|f'(z,t)|\ge m.$$

Note that

$$\left|\frac{f^{(n)}(z_0,t)}{n!}\right|\le Ml^{-n}, \quad n\ge 1.$$

Pick $0<r_0<l$ such that $\sum_{n=2}^{\infty} Ml^{-n}r_0^{n-1}(n-1)\le m/4$. For $|z-z_0|<r_0$, we have

$$\begin{aligned}\left|\frac{f'(z,t)(z-z_0)}{f(z,t)-f(z_0,t)}-1\right| &= \left|\frac{\sum_{n=1}^{\infty} f^{(n)}(z_0,t)(z-z_0)^{n-1}n/n!}{\sum_{n=1}^{\infty} f^{(n)}(z_0,t)(z-z_0)^{n-1}/n!}-1\right| \\ &= \left|\frac{\sum_{n=2}^{\infty} f^{(n)}(z_0,t)(z-z_0)^{n-1}(n-1)/n!}{f'(z_0,t)+\sum_{n=2}^{\infty} f^{(n)}(z_0,t)(z-z_0)^{n-1}/n!}\right| \\ &\le \frac{\sum_{n=2}^{\infty} Ml^{-n}|z-z_0|^{n-1}(n-1)}{m-\sum_{n=2}^{\infty} Ml^{-n}|z-z_0|^{n-1}} \\ &\le \frac{1}{2}.\end{aligned}$$

It follows from the above inequality that

$$\operatorname{Re}\frac{f'(z,t)(z-z_0)}{f(z,t)-f(z_0,t)}\ge\frac{1}{2}, \quad z\in\mathbb{D}_{r_0}(z_0).$$

(b) Assume that there does not exist $\eta>0$ such that the lemma holds. Then there exist η_k, $f^k(\zeta,t)\in\mathcal{C}^1([0,T_0],\mathcal{H}(\mathbb{D}_r)\cap\mathcal{C}(\overline{\mathbb{D}}_r))$ and $\zeta_k^1,\zeta_k^2\in\overline{\mathbb{D}}_{r'}$ where $\zeta_k^1\ne\zeta_k^2$, such that

(b1) $\eta_k\to 0$, as $k\to\infty$;
(b2) $f^k(\zeta_k^1,t_k)=f^k(\zeta_k^2,t_k)$;
(b3) $|f^k(\zeta_k^1,t_k)-g(\zeta_k^1,t_k)|\le\eta_k$ and $|f^k(\zeta_k^2,t_k)-g(\zeta_k^2,t_k)|\le\eta_k$.

Without loss of generality, assume that $t_k \to t_0$, $\zeta_k^1 \to \zeta^1$ and $\zeta_k^2 \to \zeta^2$. Note that $|\zeta^1 - \zeta^2| \geq r_0$. This implies that

$$g(\zeta^1, t_0) = g(\zeta^2, t_0).$$

This contradicts the assumption that $g(\zeta, t_0)$ is univalent in $\overline{\mathbb{D}}_r$. Therefore, there exists $\eta > 0$ such that the lemma holds. □

Proof of Theorem 1.1. (a) By Lemma 2.7, there exists $\eta(f_{k_0}, T_0, r') > 0$ such that if

$$f(\zeta, t) \in \mathcal{C}^1([0, T_0], \mathcal{H}(\mathbb{D}_r)) \quad \text{and} \quad \max_{[0,T_0]} |f'_{k_0}(\,\cdot\,, t) - f'(\,\cdot\,, t)|_{M(r)} \leq \eta,$$

then $f(\zeta, t) \in \mathcal{O}(\overline{\mathbb{D}}_{r'})$ for $t \in [0, T_0]$.

(b) We apply Theorem 2.5 letting $t_1 = T_0$, $\ell = \frac{1}{2}$ and δ be so small that

$$\delta < \min_{1 \leq l \leq m} \frac{\varepsilon}{c(l, k_0)}, \quad \delta < \frac{\ell}{\sqrt{k_0}} \min_{\overline{\mathbb{D}}_r \times [0, T_0]} |f'_{k_0}(\zeta, t)| \quad \text{and} \quad \delta < \min_{1 \leq l \leq m} \frac{\eta}{c(l, k_0)},$$

and $\rho > 1$ be so large that $(\log \rho - \log r)/C k_0 \geq T_0$. We get that for $0 \leq l \leq m$ and $0 \leq t \leq T_0$, the strong* solution $f(\zeta, t)$ to (3) satisfies

$$|f_{k_0}^{(l)}(\,\cdot\,, t) - f^{(l)}(\,\cdot\,, t)|_{M(r)} < \min\{\varepsilon, \eta\}.$$

Therefore $f(\zeta, t) \in \mathcal{O}(\overline{\mathbb{D}}_{r'})$ and hence is a strong solution to (1). □

3. Application: evolution of perturbed disks in the suction case

In this section, we aim at characterizing the evolution of perturbed disks in the suction case.

Lemma 3.1. *Given $f_{k_0}(\zeta, 0) \in \mathcal{O}(\overline{\mathbb{D}})$ which is a polynomial of degree k_0. Let $f_{k_0}(\zeta, t)$ be the strong solution to (2) and assume that the strong solution ceases to exist as $t = b$. Then given $0 < T_0 < b$, there exist $\rho > 1$ and $\delta > 0$ such that, if*

$$\|f(\zeta, 0) - f_{k_0}(\zeta, 0)\|_{\rho,1} < \delta,$$

then the solution $f(\zeta, t)$ to (2) exists for $0 \leq t \leq T_0$.

Proof. (a) There exists $r > 1$ such that $f_{k_0}(\zeta, t) \in \mathcal{O}(\overline{\mathbb{D}}_r)$ for all $0 < t < T_0$.

(b) By Theorem 1.1, the proof is complete. □

Theorem 3.2. *If the initial domain is close to a disk, then most of the fluid is sucked before the corresponding strong solution to* (2) *blows up.*

Proof. Assume that the disk has area π and therefore the conformal mapping is $f_1(\zeta,0)=\zeta$. The strong solution to (2) is $f_1(\zeta,t)=\sqrt{1-2t}\zeta$ and the fluid is sucked out as $t=b=\frac{1}{2}$.

For $T_0<b$, we apply Lemma 3.1 and obtain that there exist $\rho>1$ and $\delta>0$ such that, if $\|f(\zeta,0)-f_1(\zeta,0)\|_{\rho,1}<\delta$, then the solution $f(\zeta,t)$ to (2) exists for $0<t<T_0$. If $b-T_0$ is small, the results show that most of the fluid will be sucked before the strong solution $f(\zeta,t)$ blows up. $\square$

4. Application: large-time rescaling behavior for large data and moments in the injection case

In Richardson [8], given $\Omega(t)$ which solves the Hele-Shaw problem with injection, the Richardson complex moments $\{M_k(t)\}_{k\geq 0}$ are defined by

$$M_k(t)=\frac{1}{\pi}\int_{\Omega(t)} z^k\,dx\,dy,\qquad z=x+iy.$$

The quantity $M_0(t)\pi=\sqrt{2t+M_0(0)}\pi$ is the area of $\Omega(t)$ and $M_k(t), k\geq 1$, are conserved. Let $\Omega'(t)=\{x/\sqrt{2t+M_0(0)}\,|\,x\in\Omega(t)\}$ which always has area π.

Recall the definition of a strongly starlike function as in Gustafsson–Prokhorov–Vasil′ev [2] and Pommerenke [6]. A function $f\in\mathcal{O}(\mathbb{D})$ is said to be *strongly starlike* if there exists $\alpha\in(0,1]$ such that

$$\left|\arg\frac{\zeta f'(\zeta)}{f(\zeta)}\right|<\alpha\frac{\pi}{2},\qquad \zeta\in\mathbb{D}.$$

Such a function is also called a strongly starlike function of order α.

In the case when $\Omega(t)=f(\mathbb{D},t)$, where $f(\zeta,t)$ is a global strong solution which is strongly starlike for $t\geq T_0$, $\partial\Omega'(t)$, $t\geq T_0$, can be expressed by a polar coordinate equation $(1+\bar r_f(t,\theta),\theta)$ for some $\bar r_f(t,\cdot)\colon \mathbb{S}^1\to[-1,\infty)$. The function $\bar r_f(t,\theta)$ satisfies

$$\bar r_f(t,\theta)=\frac{|f(\zeta,t)|}{\sqrt{2t+M_0(0)}}-1,\qquad t\geq T_0,$$

where $\theta=\arg(f(\zeta,t)/|f(\zeta,t)|)$ for ζ on $\partial\mathbb{D}$. The value $\bar r_f(t,\theta)$ is well-defined if the function $f(\zeta,t)$ is strongly starlike.

Define $M_k(f)$, $k\geq 1$, to be the moments corresponding to the moving domain $\Omega(t)=f(\mathbb{D},t)$, where $f(\zeta,t)$ is a strong solution to (1). In this section, we aim at proving the following theorem.

Theorem 4.1. *Given a global strong degree-k_0 polynomial solution*

$$\{f_{k_0}(\zeta,t)\}_{t\geq 0}$$

to (1). *Then the following holds*:

(a) *There exist* $\rho(f_{k_0})>1$, $\delta(f_{k_0})>0$ *and* $T_0(f_{k_0})>0$ *such that if*

$$\|f(\,\cdot\,,0)-f_{k_0}(\,\cdot\,,0)\|_{\rho,3}<\delta,$$

then the strong solution $f(\zeta,t)$ *to* (1) *is global and is a family of strongly starlike functions of order less than* 1 *for* $t\geq T_0$.

(b) *If* $n_0=\min\{k\geq 1|M_k(f)\neq 0\}$, *then*

$$\lim_{t\to\infty}\|\bar r_f(t,\cdot\,)\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}(t)^{\lambda}=0,\quad \lambda\in\left(0,1+\frac{n_0}{2}\right),$$

where $\bar r_f(t,\theta)=|f(\zeta,t)|/\sqrt{2t+M_0(0)}-1$ *and* $\theta=\arg f(\zeta,t)$, *which are well-defined for* $t\geq T_0$.

The proof of Theorem 4.1 is given in Section 4.1. A geometric characterization of the results in Theorem 4.1 is given in Section 4.2.

4.1. Proof of Theorem 4.1

We start with some lemmas before the proof of Theorem 4.1.

Lemma 4.2. *Given a global strong solution* $f(\zeta,t)$ *which is strongly starlike of order less than* 1. *There exists* $\delta'>0$, *such that if* $\|\bar r_f(0,\cdot\,)\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}<\delta'$, *then*

$$\limsup_{t\to\infty}\|\bar r_f(t,\cdot\,)\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}(2t)^{\lambda}=0,\quad \lambda\in\left(0,1+\frac{n_0}{2}\right),$$

where $n_0=\min\{k\geq 1|M_k(f)\neq 0\}$.

Proof. Let $g(\zeta,\tau)=f(\zeta,t)/\sqrt{M_0(0)}$, where $\tau=2\pi t/M_0(0)$. Then

$$\mathrm{Re}[g_\tau\overline{g'\zeta}]=\frac{1}{2\pi},\ \zeta\in\mathbb{D},\quad \text{and}\quad |g(\mathbb{D},0)|=\pi.$$

Since the boundary of $g(\mathbb{D},\tau)$ is analytic, $\bar r_g(\tau,\cdot\,)\in h^{2,\alpha}(\mathbb{S}^1)$, where $h^{2,\alpha}(\mathbb{S}^1)$ is the little Hölder space as defined in Vondenhoff [11]. Then by Theorems 3.3 and 4.3 in Vondenhoff [11], we obtain that there exists $\delta'>0$ such that if $\|\bar r_g(0,\cdot\,)\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}<\delta'$, then

$$\limsup_{\tau\to\infty}\|\bar r_g(\tau,\cdot\,)\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}(2\tau)^{\lambda}=0,\quad \lambda\in\left(0,1+\frac{n_0}{2}\right),$$

where $n_0=\min\{k\geq 1 \mid M_k(f)\neq 0\}=\min\{k\geq 1 \mid M_k(g)\neq 0\}$. Here $\|\bar r_f(t,\cdot)\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}=\|\bar r_g(\tau,\cdot)\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}$. Therefore, we conclude that if $\|\bar r_f(0,\cdot)\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}<\delta'$, then

$$\limsup_{t\to\infty}\|\bar r_f(t,\cdot)\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}(2t)^\lambda=0,\quad \lambda\in\Bigl(0,1+\frac{n_0}{2}\Bigr),$$

where $n_0=\min\{k\geq 1 \mid M_k(f)\neq 0\}$. □

Lemma 4.3. *Given a global strong degree-k_0 polynomial solution $f_{k_0}(\zeta,t)$ to (1), there exists $r>1$ such that for $t\geq 0$,*

$$f_{k_0}(\zeta,t)\in\mathcal{O}(\overline{\mathbb{D}}_r).$$

Also given $\varepsilon>0$, $T_0>0$, $m\in\mathbb{N}$ and $1<r'<r$, there exist $\delta(f_{k_0},T_0,\varepsilon,m,r')>0$ and $\rho(f_{k_0},T_0,\varepsilon,m,r')>1$ such that if $\|f(\cdot,0)-f_{k_0}(\cdot,0)\|_{\rho,m}<\delta$, where $f(0,0)=0$ and $f'(0,0)>0$, then the strong solution $f(\zeta,t)$ to (1) satisfies

$$f(\zeta,t)\in\mathcal{C}^1([0,T_0],\mathcal{O}(\overline{\mathbb{D}}_{r'})\cap\mathcal{H}(\mathbb{D}_r)),$$

and for $0\leq l\leq m$ and $0\leq t\leq T_0$,

$$|f_{k_0}^{(l)}(\cdot,t)-f^{(l)}(\cdot,t)|_{M(r)}<\varepsilon.$$

Proof. (a) There exists $r>1$ such that $f_{k_0}(\zeta,t)\in\mathcal{O}(\overline{\mathbb{D}}_r)$ for all $t>0$.

(b) By Theorem 1.1, the proof is complete. □

Lemma 4.4. *Let $M_0\pi$ be the area of $f(\mathbb{D})$ for some $f(\zeta)=\sum_{j=1}^\infty a_j\zeta^j$ in $\mathcal{O}(\overline{\mathbb{D}})$. Given $\delta'>0$, there exists $\varepsilon'>0$ such that if $|f^{(l)}/a_1|_M<\varepsilon'$ for $2\leq l\leq 3$, then $f(\zeta)$ is strongly starlike of order less than 1 and $\|\bar r_f\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}<\delta'$, where the function $\bar r_f(\theta)=|f(\zeta)|/\sqrt{M_0}-1$ and $\theta=\arg f(\zeta)$.*

Proof. If $\varepsilon'<1$, then $|f''/a_1|_M<1$. This implies that $\sum_{j=2}^\infty j|a_j|<|a_1|$ which is a sufficient condition for coefficients of strongly starlike functions; see Pommerenke [6].

Now we treat the quantity $\|\bar r_f\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}$ by calculating $\max_{\theta\in\mathbb{S}^1}|\partial_\theta^l\bar r_f|$, $0\leq l\leq 3$. Note that $M_0=a_1^2+\sum_{j=2}^\infty j|a_j|^2$. The function $\bar r_f$ satisfies

$$\max_{\theta\in\mathbb{S}^1}|\bar r_f|\leq\left|\frac{a_1}{\sqrt{M_0}}-1\right|+\sum_{j=2}^\infty\left|\frac{a_j}{\sqrt{M_0}}\right|$$

which tends to 0, as $\varepsilon'\to 0$. The function $\partial_\theta\bar r_f$ satisfies

$$\max_{\theta\in\mathbb{S}^1}|\partial_\theta\bar r_f|=\max_{\zeta\in\partial\mathbb{D}}\left|\frac{1}{\mathrm{Re}[f'\zeta/f]}\,\frac{\mathrm{Im}[\zeta f'\bar f]}{|f|\sqrt{M_0}}\right|$$

which tends to 0 as $\varepsilon'\to 0$. Similarly, $\max_{\theta\in\mathbb{S}^1}|\partial_\theta^2\bar{r}_f|$ and $\max_{\theta\in\mathbb{S}^1}|\partial_\theta^3\bar{r}_f|$ tend to 0, as $\varepsilon'\to 0$. We conclude that $\|\bar{r}_f\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}\to 0$, as $\varepsilon'\to 0$.

Finally, there exists $0<\varepsilon'<1$ such that the theorem holds. □

Proof of Theorem 4.1. (a) Let

$$f(\zeta,t)=\sum_{j=1}^{\infty} b_j(t)\zeta^j \quad \text{and} \quad f_{k_0}(\zeta,t)=\sum_{j=1}^{k_0} a_j(t)\zeta^j.$$

Note that $b_1^2(t)\geq b_1^2(0)+2t$ and $a_1^2(t)\geq a_1^2(0)+2t$ as shown in Kuznetsova [4]. We separate the proof for (a) into (a1)–(a5) as follows.

(a1) There exists $\delta'>0$ as stated in Lemma 4.2.

(a2) For such δ', we can find $\varepsilon'>0$ as stated in Lemma 4.4.

(a3) Given $\varepsilon'>0$, there exists $T_0>\frac{1}{2}$ such that for $t\geq T_0$,

$$\left|\frac{f_{k_0}^{(2)}(\,\cdot\,,t)}{a_1(t)}\right|_M<\frac{1}{8}\varepsilon' \quad \text{and} \quad \left|\frac{f_{k_0}^{(3)}(\,\cdot\,,t)}{a_1(t)}\right|_M<\frac{1}{8}\varepsilon' \tag{33}$$

since the coefficients $\{a_j(t)\}_{j\geq 2}$ are bounded and $a_1(t)\geq\sqrt{2t+a_1^2(0)}$ as shown in Kuznetsova [4].

(a4) By Lemma 4.3, for such T_0 and ε', there exist $\rho>1$ and $\delta>0$ such that if $\|f(\,\cdot\,,0)-f_{k_0}(\,\cdot\,,0)\|_{\rho,3}<\delta$, then the strong solution $f(\zeta,t)$ to (1) exists for $t\in[0,T_0]$, and for $0\leq t\leq T_0$, $1\leq l\leq 3$,

$$|f_{k_0}^{(l)}(\,\cdot\,,t)-f^{(l)}(\,\cdot\,,t)|_M<\min\left\{\tfrac{1}{2}a_1(T_0),\,\tfrac{1}{8}\varepsilon'\right\}. \tag{34}$$

From (34) and the fact that $T_0\geq 1$, we also obtain that $b_1(T_0)\geq\max\left\{1,\frac{1}{2}a_1(T_0)\right\}$. Therefore, by (33), (34) and the fact that $b_1(T_0)\geq\max\left\{1,\frac{1}{2}a_1(T_0)\right\}$, we have that

$$\left|\frac{f^{(l)}(\,\cdot\,,T_0)}{b_1(T_0)}\right|_M\leq\frac{1}{2}\varepsilon',\quad 2\leq l\leq 3.$$

Due to the fact in (2), $f(\zeta,T_0)$ is strongly starlike of order less than 1 and

$$\|\bar{r}_f(T_0,\cdot\,)\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}<\delta',$$

where $\bar{r}_f(t,\theta)=|f(\zeta,t)|/\sqrt{M_0(t)}-1$ and $\theta=\arg f(\zeta,t)$.

(a5) By (a1)–(a4), we conclude that there exist $T_0>0$, $\rho>1$ and $\delta>0$ such that if $\|f(\,\cdot\,,0)-f_{k_0}(\,\cdot\,,0)\|_{\rho,3}<\delta$, then the strong solution $f(\zeta,t)$ exists for $t\in[0,T_0]$, $f(\zeta,T_0)\in\mathcal{O}(\overline{\mathbb{D}})$ is a strongly starlike function of order less than 1, and

$$\|\bar{r}_f(T_0,\cdot\,)\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}<\delta'.$$

By Theorem 2.1 in Gustafsson–Prokhorov–Vasil′ev [2], the solution $f(\zeta,t)$ must be global and $f(\zeta,t), t\geq T_0$, has strictly decreasing strongly starlike order $\alpha(t)$ since $f(\zeta,T_0)\in\mathcal{O}(\overline{\mathbb{D}})$ and it is a strongly starlike function. This also implies that $\bar{r}_f(t,\cdot)$ is well defined for $t\geq T_0$.

(b) By (a5), the assumptions in Lemma 4.2 are satisfied and we obtain that

$$\limsup_{t\to\infty}\|\bar{r}_f(t,\cdot)\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}(2t)^{\lambda}=0,\quad \lambda\in\Big(0,1+\frac{n_0}{2}\Big).\qquad\square$$

4.2. Geometric meaning of the rescaling behavior in Theorem 4.1

The initial domains we consider in this section are

$$\{f_{k_0}(\mathbb{D},0)\,|\,f_{k_0}(\zeta,t) \text{ is a global strong polynomial solution of degree } k_0\in\mathbb{N}\}$$

and small perturbations of them. Theorem 4.1 demonstrates that starting with an initial domain $\Omega(0)$ as above, we can obtain a global solution $\Omega(t)$ which is simply connected, has a real-analytic boundary, and a rescaling behavior given in terms of moments. Here we aim at giving a geometric characterization for this rescaling behavior by carrying out an explicit calculation.

Theorem 4.5. *Given a global strong solution $f(\zeta,t)$, where $f(\zeta,0)$ satisfies the assumption of Theorem 4.1 and $\Omega(t)=f(\mathbb{D},t)$, we show that the rescaled domain $\Omega'(t)=\{x\,|\,x\sqrt{|\Omega(t)|/\pi}\in\Omega(t)\}$ has radius satisfying*

$$\max_{z\in\partial\Omega'(t)}\big||z|-1\big|=o\Big(\frac{1}{t}\Big)^{\lambda},\quad \lambda\in\Big(0,1+\frac{n_0}{2}\Big)$$

and curvature $\varkappa(t,z)$, $z\in\Omega'(t)$, satisfying

$$\max_{z\in\Omega'(t)}|\varkappa(t,z)-1|=o\Big(\frac{1}{t}\Big)^{\lambda},\quad \lambda\in\Big(0,1+\frac{n_0}{2}\Big),$$

where $n_0=\min\{k\geq1|M_k(f)\neq0\}$.

Proof. Let $f(\zeta,t)$ be a global strong solution satisfying Theorem 4.1. There exists $T_0>0$ such that $\bar{r}_f(t,\theta), t\geq T_0$, is well defined. The value $|\varkappa(t,z)-1|$ satisfies

$$(35)\qquad |\varkappa-1|=\left|\frac{(1+\bar{r}_f)^2+2(\bar{r}_f')^2-\bar{r}_f''(1+\bar{r}_f)}{[(1+\bar{r}_f)^2+(\bar{r}_f')^2]^{3/2}}-1\right|=O(\|\bar{r}_f\|_{\mathcal{C}^2(\mathbb{S}^1)}),$$

as $\|\bar{r}_f\|_{\mathcal{C}^2(\mathbb{S}^1)} \to 0$. Since $\|\bar{r}_f\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)} = o(1/t)^\lambda$, $\lambda \in (0, 1+n_0/2)$, by the results in Theorem 4.1, we obtain from (35) that

$$\max_{z\in\Omega'(t)} |\varkappa(t,z)-1| = o\left(\frac{1}{t}\right)^\lambda, \quad \lambda \in \left(0, 1+\frac{n_0}{2}\right).$$

Similarly, since $\|\bar{r}_f\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)} = o(1/t)^\lambda$, $\lambda \in (0, 1+n_0/2)$, by the results in Theorem 4.1, we obtain that the radius satisfies

$$\max_{z\in\partial\Omega'(t)} \big||z|-1\big| = o\left(\frac{1}{t}\right)^\lambda, \quad \lambda \in \left(0, 1+\frac{n_0}{2}\right). \qquad \square$$

5. Existence and uniqueness for the Polubarinova–Galin equation

In this section, we assume the short-time well-posedness of strong* polynomial solutions as shown in Gustafsson [1] and we give a shorter proof of short-time well-posedness for strong solutions in the injection case. In particular, the proof of short-time existence of strong solutions is an application of Theorem 2.5 and this proof implies that every strong solution can be approximated by strong* polynomial solutions locally in time. The uniqueness proof is given separately.

5.1. Existence

Theorem 5.1. *Given a function $f(\zeta,0) \in \mathcal{O}_0(\overline{\mathbb{D}}_r) \cap \mathcal{H}(\overline{\mathbb{D}}_{\rho_0})$, where $\rho_0 > r > 1$, there exist $t_0 > 0$ and a strong* solution $f(\zeta,t) \in \mathcal{C}^1([0,t_0], \mathcal{O}_0(\mathbb{D}_r))$ to (3) with initial value $f(\zeta,0)$.*

Proof. (a) For $f(\zeta,0) = \sum_{j=1}^\infty a_j(0)\zeta^j \in \mathcal{H}(\overline{\mathbb{D}}_{\rho_0})$, there exists $M > 0$ such that

$$|\,a_j(0)\,| \le M\rho_0^{-j}.$$

Define $f_n(\zeta,0) = \sum_{j=1}^n a_j(0)\zeta^j$. Then

$$\left|\min_{\overline{\mathbb{D}}_r} |f'(\cdot,0)| - \min_{\overline{\mathbb{D}}_r} |f_n'(\cdot,0)|\right| \le \sum_{j=n+1}^\infty j|a_j(0)|r^j \le \sum_{j=n+1}^\infty jM\left(\frac{\rho_0}{r}\right)^{-j},$$

where $\sum_{j=n+1}^\infty jM(\rho_0/r)^{-j} \to 0$, as $n \to \infty$. Therefore there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{2}\min_{\overline{\mathbb{D}}_r} |f'(\cdot,0)| \le \min_{\overline{\mathbb{D}}_r} |f_n'(\cdot,0)|, \quad n \ge n_0,$$

and $f_n(\zeta,0)\in\mathcal{O}_0(\overline{\mathbb{D}}_r)$. By Gustafsson [1], there exists a strong* polynomial solution $f_n(\zeta,t)\in\mathcal{O}_0(\overline{\mathbb{D}}_r)$ at least for a short time.

(b) Given $1<r_0<\rho_0/r$, there exists $k_0\ge n_0$ such that

$$\sum_{k=k_0+1}^{\infty}|a_k(0)|\left(\frac{\rho_0}{r_0}\right)^k k^{3/2}\le\frac{1}{\sqrt{k_0}}\frac{1}{8}\min_{\overline{\mathbb{D}}_r}|f'(\,\cdot\,,0)|. \tag{36}$$

(c) There exists $t_1>0$ such that the strong* solution $f_{k_0}(\zeta,t)$ to (3) exists for $0\le t\le t_1$ and

$$\min_{\overline{\mathbb{D}}_r\times[0,t_1]}|f'_{k_0}|\ge\frac{1}{4}\min_{\overline{\mathbb{D}}_r}|f'(\,\cdot\,,0)|.$$

By the above, (36) implies that

$$\sum_{k=k_0+1}^{\infty}|a_k(0)|\left(\frac{\rho_0}{r_0}\right)^k k^{3/2}\le\frac{1}{2\sqrt{k_0}}\min_{\overline{\mathbb{D}}_r\times[0,t_1]}|f'_{k_0}|. \tag{37}$$

The inequality (37) implies that

$$\|f(\,\cdot\,,0)-f_{k_0}(\,\cdot\,,0)\|_{\rho_0/r_0,1}\le\frac{1}{2\sqrt{k_0}}\min_{\overline{\mathbb{D}}_r\times[0,t_1]}|f'_{k_0}|.$$

(d) By letting $\rho=\rho_0/r_0$ and $\ell=\frac{1}{2}$, we can see that the assumption in Theorem 2.5 is satisfied by (c). By applying Theorem 2.5, the short-time existence is proven. □

Remark 5.2. The proof can also be applied to the suction case.

If we assume that $f(\zeta,0)$ is univalent, then $f(\zeta,t)$ obtained in Theorem 5.1 is also univalent for a short time. Therefore, we obtain the following results.

Theorem 5.3. *Given $f(\zeta,0)\in\mathcal{O}(\overline{\mathbb{D}}_r)\cap\mathcal{H}(\overline{\mathbb{D}}_{\rho_0})$, where $\rho_0>r>1$, for $1<r'<r$, there exists $b>0$ and a strong solution $f(\zeta,t)\in\mathcal{C}^1([0,b],\mathcal{O}(\overline{\mathbb{D}}_{r'}))$ to (1) with initial value $f(\zeta,0)$.*

As for a given $f(\zeta,0)\in\mathcal{O}(\overline{\mathbb{D}})$, there is $1<r<\rho_0$ so that $f(\zeta,0)\in\mathcal{H}(\overline{\mathbb{D}}_{\rho_0})\cap\mathcal{O}(\overline{\mathbb{D}}_r)$, Theorem 5.3 implies the following directly.

Theorem 5.4. *Given $f(\zeta,0)\in\mathcal{O}(\overline{\mathbb{D}})$, there exists a strong solution $f(\zeta,t)$ to (1) locally in time.*

5.2. Uniqueness

Theorem 5.5. *Strong solutions to* (1) *are unique.*

Proof. (a) Let $f(\zeta,0)\in\mathcal{O}(\overline{\mathbb{D}})$. Assume that there are two strong solutions f_1 and f_2 with the same initial value $f(\zeta,0)$. There exist $r'>1$ and $b>0$ such that $f_1(\zeta,t), f_2(\zeta,t)\in\mathcal{C}([0,b],\mathcal{O}(\overline{\mathbb{D}}_{r'}))$. Let

$$M^2=\max_{j=1,2}\max_{t\in[0,b]}\int_{\partial\mathbb{D}_{r'}}|f_j'|^2\,d\theta.$$

Then

$$|\alpha_j(t)|\le\frac{M}{j}(r')^{-j}\quad\text{and}\quad|\beta_j(t)|\le\frac{M}{j}(r')^{-j}$$

if we write $f_1(\zeta,t)=\sum_{j=1}^{\infty}\alpha_j(t)\zeta^j$ and $f_2(\zeta,t)=\sum_{j=1}^{\infty}\beta_j(t)\zeta^j$.

(b) By (10),

$$\left\|\frac{d}{dt}[f_1-f_2]\right\|_{L^2([0,2\pi])}\le\left[\max_{\partial\mathbb{D}}\left|P\left[\frac{1}{|f_2'|^2}\right]\right|+C_2\max_{\partial\mathbb{D}}|f_1'|\max_{\partial\mathbb{D}}\frac{|f_1'|+|f_2'|}{|f_1'|^2|f_2'|^2}\right]\times\|f_1'-f_2'\|_{L^2([0,2\pi])}.\tag{38}$$

Therefore, by (38), there exists $C>0$ such that, for $t\in[0,b]$,

$$\begin{aligned}\sum_{j=1}^{\infty}|(\alpha_j-\beta_j)_t|^2&\le C\sum_{j=1}^{\infty}[|(\alpha_j-\beta_j)|j]^2\\&\le C\sum_{j=1}^{k}[|(\alpha_j-\beta_j)|j]^2+\sum_{j=k+1}^{\infty}(2M)^2(r')^{-2j}\\&\le C\sum_{j=1}^{k}[|(\alpha_j-\beta_j)|j]^2+4M^2\left(\frac{(r')^{-2(k+1)}}{1-(r')^{-2}}\right).\end{aligned}$$

(c) Let $D_k(t)=\sum_{j=1}^{k}[|(\alpha_j-\beta_j)|j]^2$. Then

$$\begin{aligned}D_k'(t)&=\sum_{j=1}^{k}2\,\mathrm{Re}[(\alpha_j-\beta_j)\overline{(\alpha_j-\beta_j)_t}]j^2\\&\le 2k\left[\sum_{j=1}^{k}[|(\alpha_j-\beta_j)|j]^2\right]^{1/2}\left[\sum_{j=1}^{k}|(\alpha_j-\beta_j)_t|^2\right]^{1/2}\\&\le 2kCD_k^{1/2}(t)\left[D_k(t)+4M^2\left(\frac{(r')^{-2(k+1)}}{1-(r')^{-2}}\right)\right]^{1/2}\end{aligned}$$

$$\leq 2kCD_k^{1/2}(t)\left[D_k^{1/2}(t)+2M\left(\frac{(r')^{-(k+1)}}{(1-(r')^{-2})^{1/2}}\right)\right]$$
$$\leq 2kCD_k(t)+4kMCD_k^{1/2}(t)\left(\frac{(r')^{-(k+1)}}{(1-(r')^{-2})^{1/2}}\right).$$

Note that $|\Omega(t)|=\pi\sum_{j=1}^{\infty} j|\alpha_j(t)|^2=\pi\sum_{j=1}^{\infty} j|\beta_j(t)|^2\leq|\Omega(0)|+2\pi b$, where $|\Omega(t)|$ is the area of the moving domain at time t. So we have that

$$D_k(t)\leq 4k\frac{|\Omega(t)|}{\pi}\leq 4k\frac{|\Omega(0)|+2\pi b}{\pi}=2kA$$

for some $A>0$. Therefore

$$D_k'(t)\leq 2kCD_k(t)+4MC(2A)^{1/2}k^{3/2}\frac{(r')^{-(k+1)}}{(1-(r')^{-2})^{1/2}}.$$

Let $C_0=(2A)^{1/2}(4MC)/(1-(r')^{-2})^{1/2}$. Then

$$D_k'(t)\leq 2kCD_k(t)+C_0(r')^{-(k+1)}k^{3/2},$$
$$(D_k(t)e^{-2kCt})'\leq e^{-2kCt}C_0(r')^{-(k+1)}k^{3/2},$$
$$D_k(t)e^{-2Ckt}\leq\frac{1-e^{-2kCt}}{2kC}C_0(r')^{-(k+1)}k^{3/2}$$

and

$$D_k(t)\leq\frac{1}{2kC}e^{2kCt}C_0(r')^{-(k+1)}k^{3/2}=\frac{1}{2r'C}(e^{2Ct}(r')^{-1})^k k^{1/2}C_0. \tag{39}$$

For $0\leq t<(1/2C)\log r'$, in (39) we let $k\to\infty$, then $D_k(t)\to 0$ since

$$\frac{1}{2C}(e^{2Ct}(r')^{-1})^k k^{1/2}C_0\to 0.$$

Therefore $f_1(\zeta,t)=f_2(\zeta,t)$ for $t\in[0,T)$, where $T=\min\{(1/2C)\log r',b\}$.

(d) Hence, the short-time uniqueness is proven. □

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