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Extremal ω-plurisubharmonic functions as envelopes of disc functionals

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Abstract. For each closed, positive $(1,1)$-current ω on a complex manifold X and each ω-upper semicontinuous function φ on X we associate a disc functional and prove that its envelope is equal to the supremum of all ω-plurisubharmonic functions dominated by φ. This is done by reducing to the case where ω has a global potential. Then the result follows from Poletsky's theorem, which is the special case $\omega=0$. Applications of this result include a formula for the relative extremal function of an open set in X and, in some cases, a description of the ω-polynomial hull of a set.

1. Introduction

If ω is a closed, positive $(1,1)$-current on a connected complex manifold X, then for every point $x_0\in X$ we can find a neighborhood U of x_0 and a plurisubharmonic local potential ψ for ω, i.e., $dd^c\psi=\omega$ on U. Let $u\colon X\to\overline{\mathbb{R}}$ be a function on X with values in the extended real line. If we can locally write $u=v-\psi$, where v is plurisubharmonic, then we say that the function u is *ω-plurisubharmonic*. We denote by $\mathcal{PSH}(X,\omega)$ the set of all ω-plurisubharmonic functions on X which are not identically equal to $-\infty$ in any connected component of X.

If ψ_1 and ψ_2 are two local potentials for ω then their difference is pluriharmonic on their common set of definition. This implies that the singular set, $\operatorname{sing}(\omega)$, of ω is well defined and locally given as $\psi^{-1}(\{-\infty\})$ for a local potential ψ of ω.

We say that a function $\varphi\colon X\to\overline{\mathbb{R}}$ is *ω-upper semicontinuous* if $\varphi+\psi$ is upper semicontinuous on $U\setminus\operatorname{sing}(\omega)$, extends to an upper semicontinuous function on U for every local potential $\psi\colon U\to\mathbb{R}\cup\{-\infty\}$ of ω, and for $a\in\operatorname{sing}(\omega)$ we have $\limsup_{X\setminus\operatorname{sing}(\omega)\ni z\to a} u(z)=u(a)$.

An *analytic disc* is a holomorphic map $f\colon\mathbb{D}\to X$ from the unit disc $\mathbb{D}$ into X. It is said to be *closed* if it can be extended to a holomorphic map in some neighborhood of the closed unit disc. We let $\mathcal{O}(\mathbb{D},X)$ denote the set of all analytic discs and $\mathcal{A}_X$ denote the set of all closed analytic discs in X.

For every analytic disc we have a pullback $f^*\omega$ of ω which is a Borel-measure on $\mathbb{D}$. It is defined locally as the Laplacian of the pullback $f^*\psi$, of a local potential ψ, to an open subset of $\mathbb{D}$. We define $R_{f^*\omega}$ as the Riesz potential of this measure on $\mathbb{D}$.

The main result of this paper is the following theorem.

Theorem 1.1. *Let X be a connected complex manifold, ω be a closed, positive $(1,1)$-current on X, and φ be an ω-upper semicontinuous function on X such that $\{u\in\mathcal{PSH}(X,\omega);u\le\varphi\}$ is nonempty. Then for $x\in X\setminus\operatorname{sing}(\omega)$*

$$\sup\{u(x)\,;u\in\mathcal{PSH}(X,\omega),u\le\varphi\}=\inf\left\{-R_{f^*\omega}(0)+\int_{\mathbb{T}}\varphi\circ f\,d\sigma\,;f\in\mathcal{A}_X,f(0)=x\right\},$$

where σ is the arc length measure on the unit circle $\mathbb{T}$ normalized to 1. Furthermore, if $\{u\in\mathcal{PSH}(X,\omega);u\le\varphi\}$ is empty then the right-hand side is $-\infty$.

This theorem is a generalization of Poletsky's theorem, which is the special case $\omega=0$, see Poletsky [14], Lárusson and Sigurdsson [11], [12], and Rosay [16].

However, if ω has a global potential ψ, i.e. $\psi\in\mathcal{PSH}(X)$ with $dd^c\psi=\omega$, then the formula above becomes

$$\sup\{u(x)\,;u\in\mathcal{PSH}(X,\omega),u\le\varphi\}+\psi(x)=\inf\left\{\int_{\mathbb{T}}(\psi+\varphi)\circ f\,d\sigma\,;f\in\mathcal{A}_X,f(0)=x\right\},$$

which is a direct consequence of Poletsky's theorem since $\psi+\varphi$ is an upper semicontinuous function. This case is handled in Theorem 4.2.

The general case follows from this case and an ω-version of a reduction theorem (Theorem 1.2 in [12]) proved by Lárusson and Sigurdsson, see Theorem 4.5. The reduction theorem states that the right-hand side in Theorem 1.1 is ω-plurisubharmonic on X if all its pullbacks to a manifold with a global potential are, and if we can assume some continuity properties of it with respect to the discs in $\mathcal{A}_X$.

By applying Theorem 1.1 to the characteristic function of the complement of an open set E we get a disc formula for the relative extremal function, which Guedj and Zeriahi introduce in [5], Chapter 4. Our result is the following

$$\begin{aligned}\sup\{u(x)\,;u\in\mathcal{PSH}(X,\omega),u|_E\le 0\text{ and }u\le 1\}\\ =\inf\{-R_{f^*\omega}(0)+\sigma(\mathbb{T}\setminus f^{-1}(E))\,;f\in\mathcal{A}_X\text{ and }f(0)=x\}.\end{aligned}$$

In certain cases this formula can give us a description of the ω-polynomial hull of a set, which is a generalization of the polynomial hull in $\mathbb{C}^n$.

For more information about the recent development of ω-plurisubharmonic functions we refer the reader to Guedj and Zeriahi [5], [6], Harvey and Lawson [7], Kołodziej [10], Dinew [3], and Branker and Stawiska [1]. In these papers X is usually assumed to be a compact Kähler manifold and ω a smooth current on X.

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2. Basic results on ω-plurisubharmonicity

Here we will define ω-plurisubharmonic functions and study their properties analogous to those of plurisubharmonic functions.

Assume X is a complex manifold of dimension n and ω is a closed, positive $(1,1)$-current on X, i.e. ω acts on $(n-1,n-1)$-forms.

It follows from Proposition 1.19, Chapter III in [2], that locally there is a plurisubharmonic function ψ such that $dd^c\psi=\omega$. Here d and d^c are the real differential operators $d=\partial+\bar\partial$ and $d^c=i(\bar\partial-\partial)$. Hence, in $\mathbb{C}$ we have that $dd^c u=\Delta u\, dV$, where Δu is the Laplacian of u and dV is the standard volume form.

Note that the difference of two potentials for ω is a pluriharmonic function, which is thus $\mathcal{C}^\infty$. This implies that the singular set of ω, $\operatorname{sing}(\omega)$, is well defined as the union of all $\psi^{-1}(\{-\infty\})$ for all local potentials ψ of ω.

In the case when ω has continuous local potentials we have no trouble with continuity. If ψ is a continuous local potential for ω then $u+\psi$ is upper semicontinuous if and only if u is. In general this is not always the case, and we do not want to exclude the case when ψ takes the value $-\infty$, e.g., when ω is a current of integration, or if ψ is discontinuous. This however forces us to define the value of $u+\psi$ at points $x\in\operatorname{sing}(\omega)$, where $\psi(x)=-\infty$ and possibly $u(x)=+\infty$. If $u+\psi$ is bounded above on $X\setminus\operatorname{sing}(\omega)$ in a neighborhood of x, then this can be done by taking upper limits of $u+\psi$ as we approach points in $\operatorname{sing}(\omega)$, we therefore make the following definition.

Definition 2.1. A function $u\colon X\to[-\infty,+\infty]$ is called *ω-upper semicontinuous* (ω-usc) if for every $a\in\operatorname{sing}(\omega)$, $\limsup_{X\setminus\operatorname{sing}(\omega)\ni z\to a} u(z)=u(a)$ and for each local potential ψ of ω, defined on an open subset U of X, $u+\psi$ is upper semicontinuous on $U\setminus\operatorname{sing}(\omega)$ and locally bounded above around each point of $\operatorname{sing}(\omega)$.

Equivalently we could say that $\limsup_{X\setminus\mathrm{sing}(\omega)\ni z\to a} u(z)=u(a)$ for every $a\in\mathrm{sing}(\omega)$ and $u+\psi$ extends as

$$\limsup_{U\setminus\mathrm{sing}(\omega)\ni z\to a}(u+\psi)(z)\quad\text{for } a\in\mathrm{sing}(\omega)$$

to an upper semicontinuous function on U with values in $\mathbb{R}\cup\{-\infty\}$. We will denote this extension by $(u+\psi)^\star$.

Note that the question whether $(u+\psi)^\star$ is usc does not depend at all on the values of u at $\mathrm{sing}(\omega)$. The reason for the conditions on u at $\mathrm{sing}(\omega)$ is to ensure that u is Borel measurable and to uniquely determine the function from its values outside of $\mathrm{sing}(\omega)$.

It is easy to see that u is Borel measurable from the fact that $u=(u+\psi)-\psi$ is the difference of two Borel measurable functions on $X\setminus\mathrm{sing}(\omega)$ and that u restricted to the Borel set $\mathrm{sing}(\omega)$ is the increasing limit of usc functions. Hence it is Borel measurable.

Definition 2.2. A function $u\colon X\to[-\infty,+\infty]$ is called *ω-plurisubharmonic* (ω-psh) if it is ω-usc and $(u+\psi)^\star$ is psh on U for every local potential ψ of ω defined on an open subset U of X. We let $\mathcal{PSH}(X,\omega)$ denote the set of all ω-psh functions on X which are not identically equal to $-\infty$ on any connected component of X. When the manifold is one-dimensional we say that these functions are ω-subharmonic and denote the set of $\mathcal{PSH}(X,\omega)$ by $\mathcal{SH}(X,\omega)$.

Note that if $\mathrm{sing}(\omega)$ is closed, an ω-usc function u is ω-psh if and only if $u+\psi$ is psh on $U\setminus\mathrm{sing}(\omega)$ for every local potential ψ, because then $u+\psi$ extends as $(u+\psi)^\star$ to a psh function on U.

We see that the ω-psh functions are locally integrable because outside of the zero set $\mathrm{sing}(\omega)$ they can locally be written as the difference of two functions which are locally integrable on X.

Our approach depends on the fact that we can define the pullback of currents by holomorphic maps. This we can do in two very different cases, first if the map is a submersion and secondly if it is an analytic disc not lying in $\mathrm{sing}(\omega)$.

If $\Phi\colon Y\to X$ is a submersion and ω is a current on X then we can define the inverse image $\Phi^*\omega$ of ω by its action on forms, $\langle\Phi^*\omega,\tau\rangle=\langle\omega,\Phi_*\tau\rangle$, where $\Phi_*\tau$ is the direct image of the form τ. For more details see Demailly [2], Section 2.C.2, Chapter I.

If f is an analytic disc in X with $f(\mathbb{D})\not\subseteq\mathrm{sing}(\omega)$, then we can define a closed, positive $(1,1)$ current $f^*\omega$ on $\mathbb{D}$ in the following way.

Let $a\in\mathbb{D}$ and $\psi\colon U\to\mathbb{R}\cup\{-\infty\}$ be a local potential on an open neighborhood U of $f(a)$, and let V be the connected component of $f^{-1}(U)$ containing a. If

$\psi\circ f\neq-\infty$ on V then we define $f^*\omega=dd^c(\psi\circ f)$. If $\psi\circ f=-\infty$ on V then we define $f^*\omega$ as the measure which sends $\varnothing$ to 0 and E to $+\infty$ for all $E\neq\varnothing$. We denote this measure by $+\infty$. Observe that if $\psi\circ f=-\infty$ on V then the same applies for every other local potential. In fact, if ψ and ψ' are two potentials for ω on open sets U and U', respectively, then on the open set $V\cap V'$ we have two subharmonic functions $f^*\psi$ and $f^*\psi'$ which differ by a harmonic function. Therefore, if one of them is equal to $-\infty$ the other one is also equal to $-\infty$.

Remember that every positive (n,n)-current (of order 0) on an n-dimensional manifold can be given by a positive Radon measure, and conversely every positive Radon measure defines a positive (n,n)-current. So when we pull ω back to $\mathbb{D}$ by an analytic disc it is possible to look at it both as a $(1,1)$-current and as a Radon measure.

We let $R_{f^*\omega}$ be the Riesz potential of the positive measure $f^*\omega$. It is defined by

$$R_{f^*\omega}(z)=\int_{\mathbb{D}} G_{\mathbb{D}}(z,\cdot\,)\,d(f^*\omega), \tag{1}$$

where $G_{\mathbb{D}}$ is the Green function for the unit disc, $G_{\mathbb{D}}(z,w)=\log(|z-w|/|1-z\overline{w}|)$. If $f^*\omega=+\infty$ then $R_{f^*\omega}=-\infty$. The Riesz potential of $f^*\omega$ is not identically $-\infty$ if and only if $f^*\omega$ satisfies the Blaschke condition (see [8], Theorem 3.3.6)

$$\int_{\mathbb{D}}(1-|\zeta|)\,d(f^*\omega)(\zeta)<+\infty.$$

If f is a closed analytic disc not lying in $\operatorname{sing}(\omega)$, then this condition is satisfied since $f^*\omega$ is a Radon measure in a neighborhood of the unit disc, and thus has finite mass on $\mathbb{D}$.

Also if we have a local potential ψ defined in a neighborhood of $\overline{f(\mathbb{D})}$ then the Riesz representation formula, (ibid.), at the point 0 gives

$$\psi(f(0))=R_{f^*\omega}(0)+\int_{\mathbb{T}}\psi\circ f\,d\sigma. \tag{2}$$

Proposition 2.3. *The following are equivalent for a function u on X.*

(i) *u is in $\mathcal{PSH}(X,\omega)$;*

(ii) *u is ω-usc and $h^*u\in\mathcal{SH}(\mathbb{D},h^*\omega)$ for all $h\in\mathcal{A}_X$ such that $h(\mathbb{D})\not\subseteq\operatorname{sing}(\omega)$.*

Proof. Assume $u\in\mathcal{PSH}(X,\omega)$, take $h\in\mathcal{A}_X$, with $h(\mathbb{D})\not\subseteq\operatorname{sing}(\omega)$, and $a\in\mathbb{D}$. Let ψ be a local potential for ω defined in a neighborhood U of $h(a)$. Note that $(u+\psi)^\star\circ h=(u\circ h+\psi\circ h)^\star$, i.e. the extension of $(u+\psi)\circ h$ over $\operatorname{sing}(h^*\omega)$ is the same as the extension of $u+\psi$ over $\operatorname{sing}(\omega)$ pulled back by h, for both functions are subharmonic and equal almost everywhere, and thus the same. Since $(u+\psi)^\star\in\mathcal{PSH}(U)$ and $(u+\psi)^\star\circ h=(u\circ h+\psi\circ h)^\star$ is subharmonic in a neighborhood of a we see that $u\circ h$ is $h^*\omega$-subharmonic.

Assume now that (ii) holds and let $\psi\in\mathcal{PSH}(U)$ be a local potential for ω. Then $(u+\psi)^\star$ is upper semicontinuous, and (ii) implies that $(u+\psi)^\star\circ h\in\mathcal{SH}(\mathbb{D})$ for every $h\in\mathcal{A}_U$. Hence $(u+\psi)^\star\in\mathcal{PSH}(U)$ and we have (i). $\square$

From the proposition we also see that ω-plurisubharmonicity like plurisubharmonicity is a local property, so in condition (ii) it is sufficient to look at $h\in\mathcal{A}_U$ in a neighborhood U of a given point.

If $u_0\neq-\infty$ is psh and φ is an usc function then the family $\{u\in\mathcal{PSH}(X);\ u_0\leq u\leq\varphi\}$ is compact in the L^1_{loc} topology, which implies that

$$\sup\{u(x)\,;u\in\mathcal{PSH}(X)\text{ and }u\leq\varphi\}$$

is plurisubharmonic. We have a similar result for ω-plurisubharmonic functions.

Proposition 2.4. *If* $\varphi\colon X\to[-\infty,+\infty]$ *is* ω*-usc,* $\mathcal{F}_{\omega,\varphi}=\{u\in\mathcal{PSH}(X,\omega);\ u\leq\varphi\}$ *and* $\mathcal{F}_{\omega,\varphi}\neq\varnothing$, *then* $\sup\mathcal{F}_{\omega,\varphi}\in\mathcal{PSH}(X,\omega)$, *and consequently* $\sup\mathcal{F}_{\omega,\varphi}\in\mathcal{F}_{\omega,\varphi}$.

Proof. Define $s_\varphi=\sup\mathcal{F}_{\omega,\varphi}$, by definition $s_\varphi+\psi\leq\varphi+\psi$ outside of $\operatorname{sing}(\omega)$ for every local potential ψ on U. Since φ is ω-usc, the upper semicontinuous regularization $(s_\varphi+\psi)^*$ of $s_\varphi+\psi$ also satisfies

$$(s_\varphi+\psi)^*\leq\varphi+\psi\quad\text{on }U\setminus\operatorname{sing}(\omega).$$

Note that the left-hand side is plurisubharmonic. We define the function S on $X\setminus\operatorname{sing}(\omega)$ by

$$S(x)=(s_\varphi+\psi)^*(x)-\psi(x),$$

where ψ is a local potential for ω in some neighborhood of x. Observe that since the difference between two local potentials is continuous it is clear that the function S is well defined on $X\setminus\operatorname{sing}(\omega)$. We extend S to an ω-usc function on X by taking the lim sup at points in $\operatorname{sing}(\omega)$. Furthermore, it is then obvious that $S\in\mathcal{PSH}(X,\omega)$ and $s_\varphi\leq S\leq\varphi$ follows from the inequality above, so $s_\varphi=S\in\mathcal{F}_{\omega,\varphi}$. $\square$

3. Disc functionals and their envelopes

A *disc functional* H is a function defined on some subset $\mathcal{A}$ of $\mathcal{O}(\mathbb{D},X)$, the set of all analytic discs in a manifold X, with values in $[-\infty,+\infty]$. The *envelope* EH of a disc functional H is then a function defined on the set $X_{\mathcal{A}}=\{x\in X;\ x=f(0)\text{ for some }f\in\mathcal{A}\}$ by the formula

$$EH(x)=\inf\{H(f)\,;f\in\mathcal{A}\text{ and }f(0)=x\},\quad x\in X_{\mathcal{A}}.$$

The *Poisson disc functional* is defined as $H_\varphi(f)=\int_{\mathbb{T}}\varphi\circ f\,d\sigma$, where φ is an usc function and σ is the normalized arc length measure on the unit circle $\mathbb{T}$. If $f\in\mathcal{A}_X$ is a closed analytic disc, $u\in\mathcal{PSH}(X)$ and $u\le\varphi$ then

$$u(f(0))\le\int_{\mathbb{T}}u\circ f\,d\sigma\le H_\varphi(f).$$

The envelope EH_φ of H_φ is plurisubharmonic by the Poletsky theorem [12] and equal to the supremum of all plurisubharmonic functions less than or equal to φ, that is

$$\sup\{u(x)\,;u\in\mathcal{PSH}(X)\text{ and }u\le\varphi\}=\inf\left\{\int_{\mathbb{T}}\varphi\circ f\,d\sigma\,;f\in\mathcal{A}_X\text{ and }f(0)=x\right\}.$$

We will now generalize the definition of the Poisson functional to ω-usc functions and look at the largest ω-psh minorant of φ. This functional will be denoted by $H_{\omega,\varphi}$.

Fix an ω-usc function φ on a complex manifold X and a point $x\in X\setminus\operatorname{sing}(\omega)$, let $f\in\mathcal{A}_X$, $f(0)=x$, and assume there is a function $u\in\mathcal{PSH}(X,\omega)$, $u\le\varphi$. Then, since $f(0)=x\notin\operatorname{sing}(\omega)$, the pullback $f^*\omega$ is a well-defined Radon measure on $\mathbb{D}$. Remember that $R_{f^*\omega}$ is a global potential for $f^*\omega$ on $\mathbb{D}$ and equal to 0 on the boundary, so by Proposition 2.3,

$$u(x)+R_{f^*\omega}(0)\le\int_{\mathbb{T}}u\circ f\,d\sigma+\int_{\mathbb{T}}R_{f^*\omega}\,d\sigma.$$

As $u\le\varphi$ and $R_{f^*\omega}=0$ on $\mathbb{T}$ we see that

$$u(x)\le -R_{f^*\omega}(0)+\int_{\mathbb{T}}\varphi\circ f\,d\sigma. \tag{3}$$

The right-hand side is independent of u, so we define the functional $H_{\omega,\varphi}$ for every $f\in\mathcal{A}_X$ with $f(0)\notin\operatorname{sing}(\omega)$ by

$$H_{\omega,\varphi}(f)=-R_{f^*\omega}(0)+\int_{\mathbb{T}}\varphi\circ f\,d\sigma.$$

Now take supremum on the left-hand side of (3) over all ω-psh functions u satisfying $u\le\varphi$ and infimum on the right over all $f\in\mathcal{A}_X$ such that $f(0)=x$. Then we get the fundamental inequality

$$\sup\mathcal{F}_{\omega,\varphi}\le EH_{\omega,\varphi}\quad\text{on } X\setminus\operatorname{sing}(\omega).$$

Theorem 1.1 states that this is actually an equality.

Since the disc functional is not defined for discs centered at $a\in\operatorname{sing}(\omega)$ we extend the envelope to a function on the whole space X by

$$EH_{\omega,\varphi}(a)=\limsup_{X\setminus\operatorname{sing}(\omega)\ni z\to a} EH_{\omega,\varphi}(z).$$

In the following we let $D_r=\{t\in\mathbb{C};|t|<r\}$ and if x is a point in X then $H(x)$ will denote the value of H at the constant disc $t\mapsto x$, the meaning should always be clear from the context.

Notice now that if we look at the constant discs in X, then $H_{\omega,\varphi}(x)=\varphi(x)$ and consequently $EH_{\omega,\varphi}\le\varphi$. Therefore, if we show that $EH_{\omega,\varphi}$ is ω-psh then it is in $\mathcal{F}_{\omega,\varphi}$ and we have an equality $\sup\mathcal{F}_{\omega,\varphi}=EH_{\omega,\varphi}$.

An immediate corollary of the main theorem is a formula for the *relative extremal function* of a set E in Ω, where Ω is an open subset of X. It is defined as

$$h_{E,\Omega,\omega}(x)=\sup\{u(x)\,;u\in\mathcal{PSH}(\Omega,\omega),u|_E\le 0 \text{ and } u\le 1\}.$$

Now assume E is open and apply Theorem 1.1 to Ω with φ as the characteristic function for the complement of E. For $x\in\Omega\setminus\operatorname{sing}(\omega)$ it gives that

$$h_{E,\Omega,\omega}(x)=\inf\{-R_{f^*\omega}(0)+\sigma(\mathbb{T}\setminus f^{-1}(E))\,;f\in\mathcal{A}_\Omega \text{ and } f(0)=x\}.$$

When $\Omega=X$ we denote this function by $h_{E,\omega}$.

In the local theory, $\omega=0$ and $X\subset\mathbb{C}^n$, the relative extremal function can be used to describe the polynomial hull of a compact set. The result is due to Poletsky [15] and can also be found in the following form in [13], Theorem 2. It states that for a compact set K in $\mathbb{C}^n$, a point a in $\mathbb{C}^n$, and Ω a pseudoconvex neighborhood of K and a, bounded and Runge, the following are equivalent:

(i) a is in the polynomial hull of K;

(ii) for every neighborhood U of K and every $\varepsilon>0$ there is an $f\in\mathcal{A}_\Omega$ with $f(0)=a$ and $\sigma(\mathbb{T}\setminus f^{-1}(U))<\varepsilon$.

If we now wish to use our formula to get a similar result on a general complex manifold we have to start by finding an alternative to the polynomial hull. It follows from Theorem 5.1.7 in [9] that the polynomial hull in $\mathbb{C}^n$ is exactly the hull with respect to the psh functions in $\mathbb{C}^n$ of logarithmic growth. These functions correspond to the ω-psh functions on $\mathbb{P}^n$ if ω is the integration current for the hyperplane at infinity. This motivates the following definition, which is similar to the definition given by Guedj [4] of the ω-polynomial hull.

Definition 3.1. If $K\subset X$ is a compact subset of a complex manifold X and ω is a closed, positive $(1,1)$-current on X, we define the *ω-polynomial hull* of K as

$$\widehat{K}^\omega=\Big\{x\in X\,;u(x)\le\sup_K u \text{ for all } u\in\mathcal{PSH}(X,\omega)\Big\}.$$

Our goal is to use the disc formula above to describe this hull. To make that possible we have to be able to use the relative extremal function to describe the hull, that is there is an $\Omega\subset X$ such that $h^{-1}_{K,\Omega,\omega}(\{0\})=\widehat{K}^\omega$. In the local theory it is sufficient to have $\Omega\subset\mathbb{C}^n$ hyperconvex.

The disc formula only applies to open sets, here we are considering compact sets so we start by showing that it is enough to look at shrinking neighborhoods of compact sets.

Proposition 3.2. *Assume that ω is a closed, positive $(1,1)$-current on a complex manifold X such that ω has continuous local potentials on $X\setminus\operatorname{sing}(\omega)$. Let $K_1\supset K_2\supset\dots$ be sequence of compact subsets of an open set $\Omega\subset X\setminus\operatorname{sing}(\omega)$ and $K=\bigcap_{j=1}^\infty K_j$. Then*

$$\lim_{j\to\infty} h_{K_j,\Omega,\omega}=h_{K,\Omega,\omega}.$$

The proof is the same as in the case $\omega=0$, see Klimek [9], Proposition 4.5.10. We only have to note that the assumptions on ω imply that all ω-psh functions are usc.

Next we derive the result with some assumptions on Ω, below we see that in some cases we can take $\Omega=X$.

Proposition 3.3. *Let K be a compact subset of $\Omega\subset X\setminus\operatorname{sing}(\omega)$, and assume ω has continuous local potentials and Ω satisfies $h^{-1}_{K,\Omega,\omega}(\{0\})=\widehat{K}^\omega$. Then a point $x\in\Omega$ is in $\widehat{K}^\omega$ if and only if for every neighborhood U of K in Ω and every $\varepsilon>0$ there is an analytic disc $f\in\mathcal{A}_\Omega$ such that $f(0)=x$ and*

$$-R_{f^*\omega}(0)+\sigma(\mathbb{T}\setminus f^{-1}(U))<\varepsilon.$$

Proof. Let $x\in\widehat{K}^\omega$. Then $0\le h_{U,\Omega,\omega}(x)\le h_{K,\Omega,\omega}(x)=0$, and by the disc formula for $h_{U,\Omega,\omega}$ there is a disc $f\in\mathcal{A}_\Omega$ such that $f(0)=x$ and

$$-R_{f^*\omega}(0)+\sigma(\mathbb{T}\setminus f^{-1}(U))<\varepsilon.$$

Conversely, if such f's exist then $h_{U,\Omega,\omega}(x)=0$ for every neighborhood U of K. Let $\{K_j\}_{j=1}^\infty$ be a sequence of compact subsets of Ω such that $\bigcap_{j=1}^\infty K_j=K$ and $K_{j+1}\subset\mathring{K}_j$. Then $h_{K_j,\Omega,\omega}(x)=0$ and by Proposition 3.2 we see that $h_{K,\Omega,\omega}(x)=0$. $\square$

Proposition 3.4. *If $\operatorname{sing}(\omega)=\varnothing$ and X is compact then $h^{-1}_{K,\omega}(\{0\})=\widehat{K}^\omega$.*

Proof. Assume $x\in h^{-1}_{K,\omega}(\{0\})$ and let $u\in\mathcal{PSH}(X,\omega)$. Note that if we let $\psi_j: U_j\to\mathbb{R}^+$ be positive local potentials for ω, such that $\bigcup_{j=1}^\infty U_j=X$, then for every j

the function $u+\psi_j$ is usc and locally bounded on U_j, and by a compactness argument we then see that $\sup_X u<+\infty$.

Now let $\tilde{u}=(u-\sup_K u)/(\sup_X u-\sup_K u)$ if $\sup_X u-\sup_K u>1$. Otherwise, if $\sup_X u-\sup_K u\leq 1$, we let $\tilde{u}=u-\sup_K u$. Either way, $\tilde{u}$ is ω-psh, $\tilde{u}\leq 1$ and $\tilde{u}|_K\leq 0$. Therefore $\tilde{u}\leq h_{K,\omega}$ and $\tilde{u}(x)\leq h_{K,\omega}(x)=0$, that is $u(x)\leq \sup_K u$ and $x\in \widehat{K}^\omega$.

Obviously $\widehat{K}^\omega\subset h_{K,\omega}^{-1}(\{0\})$, so $h_{K,\omega}^{-1}(\{0\})=\widehat{K}^\omega$. $\square$

4. Proof of the main result

We start by restricting to the case when ω has a global potential. The general case then follows from the reduction theorem later on.

Lemma 4.1. *Let ω be a closed, positive $(1,1)$-current on a Stein manifold X. If there is a current η such that $d\eta=\omega$, then ω has a global plurisubharmonic potential $\psi\colon X\to\mathbb{R}\cup\{-\infty\}$, so in particular $dd^c\psi=\omega$.*

Proof. Since ω is a positive current it is real, so η can be assumed to be real, $\eta\in\Lambda'_1(X,\mathbb{R})$. Now write $\eta=\eta^{1,0}+\eta^{0,1}$, where $\eta^{1,0}\in\Lambda'_{1,0}(X,\mathbb{C})$ and $\eta^{0,1}\in\Lambda'_{0,1}(X,\mathbb{C})$. Note that $\eta^{0,1}=\overline{\eta^{1,0}}$ since η is real. We see, by counting degrees, that $\bar\partial\eta^{0,1}=\omega^{0,2}=0$. As X is Stein there is a distribution μ on X such that $\bar\partial\mu=\eta^{0,1}$. Then

$$\eta=\overline{\bar\partial\mu}+\bar\partial\mu=\partial\bar\mu+\bar\partial\mu.$$

If we set $\psi=(\mu-\bar\mu)/2i$, then

$$\omega=d\eta=d(\partial\bar\mu+\bar\partial\mu)=(\partial+\bar\partial)(\partial\bar\mu+\bar\partial\mu)=\partial\bar\partial(\mu-\bar\mu)=dd^c\psi.$$

Finally, ψ is a plurisubharmonic function since ω is positive. $\square$

Theorem 4.2. *Let ω be a closed, positive $(1,1)$-current on a manifold X and $\varphi\colon X\to[-\infty,+\infty]$ be an ω-usc function such that $\mathcal{F}_{\omega,\varphi}\neq\varnothing$. If ω has a global potential ψ then $EH_{\omega,\varphi}\in\mathcal{PSH}(X,\omega)$ and consequently $EH_{\omega,\varphi}=\sup\mathcal{F}_{\omega,\varphi}$ on $X\setminus$ $\operatorname{sing}(\omega)$.*

Proof. For $f\in\mathcal{A}_X$, $f(\mathbb{D})\not\subseteq\operatorname{sing}(\omega)$, the Riesz representation (2) of $f^*\psi$ gives

$$H_{\omega,\varphi}(f)+\psi(f(0))=H_{\omega,\varphi}(f)+R_{f^*\omega}(0)+\int_{\mathbb{T}}\psi\circ f\,d\sigma=\int_{\mathbb{T}}(\psi+\varphi)^\star\circ f\,d\sigma=H_{\psi+\varphi}(f).$$

The equality in the middle follows from the fact that $\int_{\mathbb{T}}\varphi\circ f\,d\sigma+\int_{\mathbb{T}}\psi\circ f\,d\sigma=\int_{\mathbb{T}}(\varphi+\psi)^\star\circ f\,d\sigma$ since $\sigma(f^{-1}(\operatorname{sing}(\omega))\cap\mathbb{T})=0$. Therefore

$$EH_{\omega,\varphi}(x)+\psi(x)=\inf\{H_{\omega,\varphi}(f)+\psi(x)\,;f\in\mathcal{A}_X \text{ and } f(0)=x\}=EH_{(\psi+\varphi)^\star}(x).$$

By Poletsky's theorem $EH_{(\psi+\varphi)^\star}$ is psh, and hence $EH_{\omega,\varphi}$ is ω-psh. $\square$

If $\Phi\colon Y\to X$ is a holomorphic map between complex manifolds and H is a disc functional on X, then the pullback Φ^*H is a disc functional on Y defined by $\Phi^*H(f)=H(\Phi\circ f)$ for $f\in\mathcal{A}_Y$. Since $\{\Phi\circ f; f\in\mathcal{A}_Y\}\subset\mathcal{A}_X$ we get the following result.

Lemma 4.3. *$\Phi^*EH\le E\Phi^*H$ and equality holds if every disc in Y is a lifting of a disc in X by Φ.*

Moreover, for the Poisson functional $H_{\omega,\varphi}$ we have the following result.

Lemma 4.4. *Assume that $\Phi\colon Y\to X$ is a holomorphic submersion. Then $\Phi^*H_{\omega,\varphi}=H_{\Phi^*\omega,\Phi^*\varphi}$.*

Proof. By associativity of compositions we have $(\Phi_*f)^*\omega=f^*(\Phi^*\omega)$ for $f\in\mathcal{A}_Y$, $f(\mathbb{D})\not\subseteq\Phi^{-1}(\operatorname{sing}(\omega))$, so

$$\begin{aligned}\Phi^*H_{\omega,\varphi}(f)=H_{\omega,\varphi}(\Phi_*f) &= -R_{(\Phi_*f)^*\omega}(0)+\int_{\mathbb{T}}\varphi\circ\Phi\circ f\,d\sigma\\ &= -R_{f^*(\Phi^*\omega)}(0)+\int_{\mathbb{T}}(\Phi^*\varphi)\circ f\,d\sigma=H_{\Phi^*\omega,\Phi^*\varphi}(f).\end{aligned}$$ □

We will now state the reduction theorem which will enable us to prove Theorem 1.1 using Theorem 4.2.

Theorem 4.5. (Reduction theorem) *Let X be a complex manifold, H be a disc functional on $\mathcal{A}=\{f\in\mathcal{A}_X; f(0)\notin\operatorname{sing}(\omega)\}$ and ω be a positive, closed $(1,1)$-current on X. The envelope EH is ω-plurisubharmonic if it satisfies the following conditions:*

(i) *$E\Phi^*H$ is $\Phi^*\omega$-plurisubharmonic for every holomorphic submersion Φ from a complex manifold where $\Phi^*\omega$ has a global potential and for every $a\in\operatorname{sing}(\omega)$ we have $\limsup_{X\setminus\operatorname{sing}(\omega)\ni z\to a} EH(z)=EH(a)$.*

(ii) *There is an open cover of X by subsets U, with ω-pluripolar subsets $Z\subset U$ and local potentials ψ on U, $\psi^{-1}(\{-\infty\})\subset Z$, such that for every $h\in\mathcal{A}_U$, $h(\mathbb{D})\not\subseteq Z$, the function $t\mapsto(H(h(t))+\psi(h(t)))^\star$ is dominated by an integrable function on $\mathbb{T}$.*

(iii) *If $h\in\mathcal{A}_X$, $h(0)\notin\operatorname{sing}(\omega)$, $t_0\in\mathbb{T}\setminus h^{-1}(\operatorname{sing}(\omega))$ and $\varepsilon>0$, then t_0 has a neighborhood U in $\mathbb{C}$ and there is a local potential ψ in a neighborhood of $h(U)$ such that for all sufficiently small arcs J in $\mathbb{T}$ containing t_0 there is a holomorphic map $F\colon D_r\times U\to X$ so that $F(0,\cdot\,)=h|_U$ and*

$$\frac{1}{\sigma(J)}\int_J(H(F(\,\cdot\,,t))+\psi(F(0,t)))^\star\,d\sigma(t)\le(EH+\psi)(h(t_0))+\varepsilon.$$

Before proving the theorem we show that $H_{\omega,\varphi}$ satisfies the conditions (i)–(iii) and consequently that Theorem 1.1 follows from it.

Proof of Theorem 1.1. Condition (i) follows from Lemma 4.4 since it implies that $E\Phi^* H_{\omega,\varphi} = EH_{\Phi^*\omega,\Phi^*\varphi}$ and by Theorem 4.2, $EH_{\Phi^*\omega,\Phi^*\varphi}$ is $\Phi^*\omega$-psh.

Condition (ii) follows from the fact that $H_{\omega,\varphi}(h(t)) + \psi(h(t)) = \varphi(h(t)) + \psi(h(t))$, which extends to an upper semicontinuous function on $\mathbb{T}$ and is thus dominated by a continuous function.

Assuming h and t_0 as in condition (iii) and $\varepsilon > 0$, set $x = h(t_0)$ and let $f \in \mathcal{A}_X$ be such that $f(0) = x$ and $H_{\omega,\varphi}(f) \leq EH_{\omega,\varphi}(x) + \varepsilon/2$. By Lemma 2.3 in [11] there is an open neighborhood V of x in X, $r > 1$ and a holomorphic function $\widetilde{F}\colon D_r \times V \to X$ such that $\widetilde{F}(\,\cdot\,, x) = f$ on D_r and $\widetilde{F}(0, z) = z$ on V. Shrinking V if necessary, we assume that ψ is a local potential for ω on V. Let $U = h^{-1}(V)$ and define $F\colon D_r \times U \to X$ by $F(s,t) = \widetilde{F}(s, h(t))$. By the Riesz representation (2),

$$(H_{\omega,\varphi}(F(\,\cdot\,, t)) + \psi(F(0,t)))^\star = \int_{\mathbb{T}} (\varphi + \psi)^\star \circ F(s,t)\, d\sigma(s). \tag{4}$$

Since the integrand is usc on $D_r \times U$, it is easily verified that (4) is an usc function of t on U. That allows us, by shrinking U, to assume that

$$(H_{\omega,\psi}(F(\,\cdot\,, t)) + \psi(F(0,t)))^\star \leq H_{\omega,\varphi}(F(\,\cdot\,, t_0)) + \psi(F(0,t_0)) + \frac{\varepsilon}{2}$$

for $t \in U$. Then by the definition of $f = F(\,\cdot\,, t_0)$,

$$(H_{\omega,\varphi}(F(\,\cdot\,, t)) + \psi(F(0,t)))^\star < EH_{\omega,\varphi}(x) + \psi(x) + \varepsilon \quad \text{on } U.$$

Condition (iii) is then satisfied for all arcs J in $\mathbb{T} \cap U$.

Finally, if $\mathcal{F}_{\varphi,\omega} = \varnothing$ then the only function which is both dominated by φ and satisfies the subaverage property is the constant function $-\infty$. We know that $EH_{\varphi,\omega} \leq \varphi$ and the proof of the reduction theorem gives the subaverage property. This implies that $EH_{\varphi,\omega} = -\infty$. $\square$

We now prove that EH is ω-psh if H satisfies the three conditions in Theorem 4.5. The main work is to show that h^*EH satisfies the subaverage property of $h^*\omega$-subharmonic functions for a given analytic disc h. This implies by Proposition 2.3 that EH is ω-psh and that concludes the proof of the reduction theorem.

Lemma 4.6. *Let H be a disc functional on an n-dimensional complex manifold X and ω be a positive, closed $(1,1)$-current on X. If the envelope $E\Phi^*H$ is $\Phi^*\omega$-usc for every holomorphic submersion Φ from an $(n+1)$-dimensional polydisc into X and for every $a \in \operatorname{sing}(\omega)$ we have that $\limsup_{X\setminus \operatorname{sing}(\omega) \ni z \to a} EH(z) = EH(a)$, then EH is ω-usc.*

Proof. First, let U be a coordinate polydisc in X such that there is a potential ψ for ω defined on U. Define $\Phi\colon U\times\mathbb{D}\to U$ as the projection. By assumption and Lemma 4.3,

$$EH(z)+\psi(z)=EH(\Phi(z,t))+\psi(\Phi(z,t))\le E\Phi^*H(z,t)+\psi(\Phi(z,t))<+\infty$$

for every $z\in U\setminus\operatorname{sing}(\omega)$ and $t\in\mathbb{D}$. Then $EH<+\infty$ on $X\setminus\operatorname{sing}(\omega)$ and $EH+\psi$ is bounded above in some neighborhood of every point in $\operatorname{sing}(\omega)\cap U$, where ψ is any local potential for ω.

Let $x\in X\setminus\operatorname{sing}(\omega)$ and $\beta>EH(x)+\psi(x)$. Assume that $f\in\mathcal{A}_X$ is a holomorphic disc defined on D_r such that $f(0)=x$ and $H(f)+\psi(x)<\beta$. By using a theorem of Siu [17] it is shown in the proof of Lemma 2.3 in [11] that for $\tilde r\in]0,r[$ there exists a neighborhood U of the graph $\{(t,f(t));t\in D_r\}$ in $D_r\times X$ and a biholomorphism

$$\Psi\colon U\longrightarrow D_{\tilde r}\times\mathbb{D}^n$$

such that $\Psi(t,f(t))=(t,0)$. Let $\pi\colon\mathbb{C}\times X\to X$ be the projection and define $\Phi=\pi\circ\Psi^{-1}$. Clearly $f=\Phi\circ\tilde f$, where $\tilde f\in A_{D_{\tilde r}\times\mathbb{D}^n}$ is the lifting $\tilde f(t)=(t,0)$. By assumption and the fact that

$$E\Phi^*H(x)+\psi(x)\le\Phi^*H(\tilde f)+\psi(x)=H(f)+\psi(x)<\beta$$

there is a neighborhood W of $0\in D_{\tilde r}\times\mathbb{D}^n$ such that

$$(E\Phi^*H+\Phi^*\psi)^\star<\beta\quad\text{on } W.$$

Then for every z in the open set $\Phi(W)$,

$$(EH(z)+\psi(z))^\star=(\Phi^*EH(\tilde z)+\psi(z))^\star\le(E\Phi^*H(\tilde z)+\Phi^*\psi(\tilde z))^\star<\beta,$$

where $\tilde z\in\Phi^{-1}(\{z\})$. This along with the definition of EH at $\operatorname{sing}(\omega)$ shows that the envelope is ω-usc. ☐

Now we turn to the subaverage property of the envelope.

Proof of Theorem 4.5. We have already shown that the envelope EH is ω-usc, so by Proposition 2.3 we only need to show that for a local potential ψ on an open set $U\subset X$ and every disc $h\in\mathcal{A}_U$ such that $h(0)\notin\operatorname{sing}(\omega)$ we have

$$EH(h(0))+\psi(h(0))\le\int_{\mathbb{T}}(EH\circ h+\psi\circ h)^\star\,d\sigma. \tag{5}$$

Observe that this is automatically satisfied if $EH(h(0))=-\infty$, so we may assume that $EH(h(0))$ is finite. It is sufficient to show that for every $\varepsilon>0$ and every continuous function $v\colon U\to\mathbb{R}$ with $v\geq(EH+\psi)^\star$, there exists $g\in\mathcal{A}_X$ such that $g(0)=h(0)$ and

$$H(g)+\psi(h(0))\leq\int_{\mathbb{T}} v\circ h\,d\sigma+\varepsilon.$$

Then by definition of the envelope $EH(h(0))+\psi(h(0))\leq\int_{\mathbb{T}} v\circ h\,d\sigma+\varepsilon$ for all v and ε, and (5) follows.

We assume that h is holomorphic on D_r, $r>1$ and $h(0)\notin\operatorname{sing}(\omega)$. It is easily verified (see the proof of Theorem 1.2 in [12]) that a function satisfying the subaverage property for all holomorphic discs in X not lying in a pluripolar set Z is plurisubharmonic not only on $X\setminus Z$ but on X. We may therefore assume that $h(\overline{\mathbb{D}})\not\subseteq Z$.

Note that $h(\mathbb{T})\setminus\operatorname{sing}(\omega)$ is dense in $h(\mathbb{T})$ by the subaverage property of $\psi\circ h$ and the fact that $h(0)\notin\operatorname{sing}(\omega)$. Therefore by a compactness argument along with property (iii) we can find a finite number of closed arcs $J_1,...,J_m$ in $\mathbb{T}$, each contained in an open disc U_j centered on $\mathbb{T}$, and holomorphic maps $F_j\colon D_s\times U_j\to X$, $s\in]1,r[$, such that $F_j(0,\cdot\,)=h|_{U_j}$ and, using the continuity of v, such that

$$\underline{\int_{J_j}}(H(F_j(\,\cdot\,,t))+\psi(F(0,t)))^\star\,d\sigma(t)\leq\int_{J_j} v\circ h\,d\sigma+\frac{\varepsilon}{4}\sigma(J_j). \tag{6}$$

We may assume that the discs U_j are relatively compact in D_r and have mutually disjoint closures. By the continuity of v and condition (ii) we may also assume that

$$\int_{\mathbb{T}\setminus\bigcup_{j=1}^m J_j}|v\circ h|\,d\sigma<\frac{\varepsilon}{4} \tag{7}$$

and

$$\overline{\int_{\mathbb{T}\setminus\bigcup_{j=1}^m J_j}}(H(h(w))+\psi(h(w)))^\star\,d\sigma(w)<\frac{\varepsilon}{4}. \tag{8}$$

We now embed the graph of h in $\mathbb{C}^4\times X$ as

$$K_0=\{(w,0,0,0,h(w))\,;w\in\overline{\mathbb{D}}\}$$

and the graphs of the F_j's as

$$K_j=\{(w,z,0,0,F_j(z,w))\,;w\in J_j\text{ and }z\in\overline{\mathbb{D}}\}.$$

Let $\Phi\colon\mathbb{C}^4\times X\to X$ denote the projection. This function restricted to a smaller subset will be our submersion.

What is needed to find the disc g we are looking for is a Stein neighborhood V of the compact set $K=\bigcup_{j=0}^m K_j$ in $\mathbb{C}^4\times X$ where we can solve $d\eta=\Phi^*\omega$. Then we have, by Lemma 4.1, a global potential for the pullback $\Phi^*\omega$, and the $\Phi^*\omega$-plurisubharmonicity of $E\Phi^*H$, given by property (i), then gives the existence of g.

For convenience we let $U_0=D_r$ and $F_0(z,w)=h(z)$. In [12], using Siu's theorem [17] and slightly shrinking the U_j's and the s, Lárusson and Sigurdsson define for $j=0,\dots,m$ a biholomorphism Φ_j from $U_j\times D_s^{n+3}$ onto its image in $\mathbb{C}^4\times X$ satisfying

$$\Phi_j(w,z,0)=(w,z,0,0,F_j(z,w)),\quad w\in U_j,\ z\in D_s,$$

for $j=1,\dots,m$, and

$$\Phi_0(w,0)=(w,0,0,0,h(w)),\quad w\in D_r.$$

The image of each Φ_j is therefore biholomorphic to a $(4+n)$-dimensional polydisc. These extensions of the graphs above are defined such that the first coordinate is the identity map. This tells us that these images are mutually disjoint for $j\geq 1$ and that the intersection of the image of Φ_0 and Φ_j is a subset of $U_j\times\mathbb{C}^3\times X$.

As in [12] we let U_j'' and U_j' be discs concentric with U_j such that

$$J_j\subset U_j''\Subset U_j'\Subset U_j,$$

and we assume that our Φ_0 is the Φ_0 after the modification made in [12] which is necessary to have all but the first coordinate of $\Phi_j^{-1}\circ\Phi_0$ close to the identity. This modification which is done by precomposing Φ_0 with a holomorphic map is necessary for constructing the Stein neighborhood. Importantly for our purpose it does not change the first coordinate.

Now, for each $w\in\overline{U_j'}$ there is an $\varepsilon_w>0$ such that $\Phi_0(w,D_{\varepsilon_w}^{n+3})\subset\Phi_j(w,D_s^{n+3})$. This holds by continuity for $\varepsilon_w/2$ on a neighborhood of w in U_j. By compactness of $\bigcup_{j=1}^m\overline{U_j'}$ there is an ε independent of w such that $\Phi_0(w,D_\varepsilon^{n+3})\subset\Phi_j(w,D_s^{n+3})$ for $w\in\overline{U_j'}$. We now restrict Φ_0 to $D_r\times D_\varepsilon^{n+3}$. Then the intersection of the images $\Phi_0(D_r\times D_\varepsilon^{n+3})$ and $\Phi_j(U_j'\times D_s^{n+3})$ is $\Phi_0(U_j'\times D_\varepsilon^{n+3})$.

We define $V_j=\Phi_j(U_j\times D_s^{n+3})$, $V_0=\Phi_0(U_0\times D_\varepsilon^{n+3})$ and $U=\bigcup_{j=0}^m V_j$. To solve $d\eta=\Phi^*\omega$ on U it is then enough to show that the cohomology $H^2(U)$ is zero. This can be done using the exact Mayer–Vietoris sequence ([18], Chapter 11, Theorem 3),

$$\dots\longrightarrow H^q(M\cup N)\longrightarrow H^q(M)\oplus H^q(N)\longrightarrow H^q(M\cap N)\longrightarrow H^{q+1}(M\cup N)\longrightarrow\dots.$$

We start by letting $M=V_0$ and $N=V_1$, these sets and their intersection are biholomorphic to a polydisc, so they are smoothly contractable and then by Poincaré's lemma $H^2(V_j)=H^1(V_0\cap V_1)=0$. Consequently, we see from the Mayer–Vietoris sequence

$$\dots\longrightarrow H^1(V_0\cap V_1)\longrightarrow H^2(V_0\cup V_1)\longrightarrow H^2(V_0)\oplus H^2(V_1)\longrightarrow\dots$$

that $H^2(V_0 \cup V_1)=0$. Next we let $M=V_0 \cup V_1$ and $N=V_2$, then $H^1((V_0 \cup V_1)\cap V_2)= H^1(V_0 \cap V_2)=0$ since V_1 and V_2 are disjoint. The sequence above tells us then that $H^2(V_0 \cup V_1 \cup V_2)=0$. Iterating this process for all the V_j's we finally see that $H^2(U)=H^2\big(\bigcup_{j=0}^m V_j\big)=0$.

The next step is to find a Stein neighborhood V of K which is a subset of U. Note that K only relies on the holomorphic functions h and F_j. Therefore V can be constructed in exactly the same way as in [12]. It is done by defining a continuous strictly plurisubharmonic exhaustion function ρ on U. This function is positive and satisfies $K\subset\rho^{-1}\big([0,\frac{1}{2}]\big)$. Finally V is defined as $\rho^{-1}([0,1))$.

Then by Lemma 4.1 we have a global potential on V for $\Phi^*\omega$. By property (i) the envelope $E\Phi^*H$ is then $\Phi^*\omega$-psh on V and if $\tilde{h}\colon D_r \to V$ is the lifting $w\mapsto (w,0,0,0,h(w))$ of h, then

$$E\Phi^*H(\tilde{h}(0))+\Phi^*\psi(\tilde{h}(0)) \le \int_{\mathbb{T}} (E\Phi^*H\circ\tilde{h}+\Phi^*\psi\circ\tilde{h})^\star \, d\sigma.$$

Since $EH(h(0))\neq-\infty$ we have that $-\infty<\Phi^*EH(\tilde{h}(0))\le E\Phi^*H(\tilde{h}(0))$ and we may assume there is a disc $\tilde{g}\in\mathcal{A}_V$ such that $\tilde{g}(0)=\tilde{h}(0)$ and $\Phi^*H(\tilde{g})\le E\Phi^*H(\tilde{g}(0))+\varepsilon/4$. Define the disc $g=\Phi\circ\tilde{g}\in\mathcal{A}_X$. Then $g(0)=h(0)$ and since $H(g)=\Phi^*H(\tilde{g})$ and $\Phi^*\psi(\tilde{h})=\psi(h)$,

$$H(g)+\psi(h(0)) \le \int_{\mathbb{T}} (E\Phi^*H\circ\tilde{h}+\psi\circ h)^\star \, d\sigma+\frac{\varepsilon}{4}. \tag{9}$$

For $w\in J_j$, $1\le j\le m$, we have a lifting of $F_j(\,\cdot\,,w)$ by Φ given by

$$z \longmapsto (w,z,0,0,F_j(z,w)).$$

Clearly $0\mapsto\tilde{h}(w)$, so

$$E\Phi^*H(\tilde{h}(w)) \le \Phi^*H(\widetilde{F}_j(\,\cdot\,,w)) = H(F_j(\,\cdot\,,w)).$$

However, if $w\in\mathbb{T}\setminus\bigcup_{j=1}^m J_j$ then $E\Phi^*H(\tilde{h}(w))\le\Phi^*H(\tilde{h}(w))=H(h(w))$. Therefore,

$$\int_{\mathbb{T}} E\Phi^*H\circ\tilde{h}\,d\sigma \le \sum_{j=1}^m \underline{\int_{J_j}} H(F_j(\,\cdot\,,w))\,d\sigma(w) + \overline{\int_{\mathbb{T}\setminus\bigcup_{j=1}^m J_j}} H(h(w))\,d\sigma(w).$$

Adding the integral of $\psi(h)$ to both sides of this inequality and using the inequalities (6) and (8) we see that

$$\int_{\mathbb{T}} (E\Phi^*H\circ\tilde{h}+\psi\circ h)^\star \, d\sigma \le \int_{\bigcup_{j=1}^m J_j} v\circ h\,d\sigma+\frac{\varepsilon}{4}\sigma\bigg(\bigcup_{j=1}^m J_j\bigg)+\frac{\varepsilon}{4}.$$

Then by using first (9) and then (7) we have finally that

$$H(g)+\psi(h(0)) \le \int_{\bigcup_{j=1}^m J_j} v\circ h\,d\sigma+\frac{3}{4}\varepsilon < \int_{\mathbb{T}} v\circ h\,d\sigma+\varepsilon. \qquad \square$$

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