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A natural map in local cohomology

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Abstract. Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R, M an R-module and n a non-negative integer. In this paper we first study the finiteness properties of the kernel and the cokernel of the natural map $f\colon \operatorname{Ext}^n_R(R/\mathfrak{a}, M)\to\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}^n_{\mathfrak{a}}(M))$, under some conditions on the previous local cohomology modules. Then we get some corollaries about the associated primes and Artinianness of local cohomology modules. Finally we will study the asymptotic behavior of the kernel and the cokernel of the natural map in the graded case.

1. Introduction

Throughout R is a commutative Noetherian ring. Our terminology follows the book [5] on local cohomology. Huneke formulated and discussed several problems in [12] about local cohomology modules $\operatorname{H}^n_{\mathfrak{a}}(M)$ of a finite (by which we mean finitely generated) R-module M. For example when do they have just finitely many associated prime ideals or when are they Artinian? Furthermore the following conjecture was made by Grothendieck in [10].

Conjecture 1.1. For every ideal $\mathfrak{a}\subset R$ and each finite R-module M, the module $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}^n_{\mathfrak{a}}(M))$ is finite for all $n\geq 0$.

This was thought of as a substitute for the Artinianness of $\operatorname{H}^n_{\mathfrak{a}}(M)$, in the case when $\mathfrak{a}$ is the maximal ideal of a local ring. Moreover the finiteness of $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}^n_{\mathfrak{a}}(M))$ implies the finiteness of the set of associated prime ideals of $\operatorname{H}^n_{\mathfrak{a}}(M)$. Although the conjecture is not true in general as was shown by Hartshorne in [11], there are some attempts to show that under some conditions, for some number n, the module $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}^n_{\mathfrak{a}}(M))$ is finite, see [1, Theorem 3], [8, Theorem 6.3.9] and [7, Theorem 2.1].

In Section 3, in view of Grothendieck's conjecture and [8, Theorem 6.3.9], we are led to study finiteness properties of the kernel and the cokernel of the natural homomorphism

$$f\colon \operatorname{Ext}^n_R(R/\mathfrak{a}, M) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M)).$$

We give conditions under which $\operatorname{Ker} f$ (resp. $\operatorname{Coker} f$) belongs to a Serre subcategory $\mathcal{S}$ of the category of R-modules. A class of R-modules closed under taking submodules, quotients and extensions is called a *Serre subcategory*. Examples are given by the class of finitely generated modules, the class of Artinian modules and the class consisting of the zero modules.

To obtain the results about the natural map above we prove a quite general result, Proposition 3.1, in the setting of abelian categories. We were also able to do this using spectral sequence techniques, but we preferred to give a more elementary treatment.

We will apply this to find some relations between the finiteness of the sets of associated primes of $\operatorname{Ext}^n_R(R/\mathfrak{a}, M)$ and $\mathrm{H}^n_{\mathfrak{a}}(M)$, generalizing [13, Proposition 1.1(b)], [2, Lemma 2.5(c)] and [4, Proposition 2.2]. Also an application is given to show that under certain conditions $\operatorname{Ext}^n_R(R/\mathfrak{a}, M)$ is Artinian if and only if $\mathrm{H}^n_{\mathfrak{a}}(M)$ is. One of our main results of this paper is Corollary 4.2.

(a) If $\operatorname{Ext}^{n-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))=0$ for all $j<n$, then f is injective.

(b) If $\operatorname{Ext}^{n+1-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))=0$ for all $j<n$, then f is surjective.

(c) If $\operatorname{Ext}^{t-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))=0$ for $t=n, n+1$ and for all $j<n$, then f is an isomorphism.

In the last section we will study the asymptotic behavior of the kernel and the cokernel of the above mentioned natural map in the graded case.

2. A homological lemma

Let $F\colon \mathcal{A}\to\mathcal{B}$ be an additive covariant functor between abelian categories. Given a morphism $f\colon M\to N$ in $\mathcal{A}$ we get in $\mathcal{B}$ induced morphisms

$$\lambda_f\colon \operatorname{Coker} Ff \longrightarrow F\operatorname{Coker} f \quad \text{and} \quad \varkappa_f\colon F\operatorname{Ker} f \longrightarrow \operatorname{Ker} Ff.$$

Consider the exact sequence $0\to\operatorname{Ker} f\xrightarrow{u}M\xrightarrow{f}N\xrightarrow{v}\operatorname{Coker} f\to 0$ in $\mathcal{A}$. The equalities $Ff\circ Fu=0$ and $Fv\circ Ff=0$, will allow the construction of $\varkappa_f$ and λ_f.

Next we generalize the useful result [14, Lemma 3.1].

Lemma 2.1. *Let $S, T\colon \mathcal{A}\to\mathcal{B}$ be additive covariant functors between abelian categories, such that whenever we have in $\mathcal{A}$ a short exact sequence $0\to X'\xrightarrow{u}X\xrightarrow{v}X''\to 0$ we will have in $\mathcal{B}$ an exact sequence*

$$SX' \xrightarrow{Su} SX \xrightarrow{Sv} SX'' \longrightarrow TX' \xrightarrow{Tu} TX \xrightarrow{Tv} TX''.$$

Let $\mathcal{S}$ be a Serre subcategory of $\mathcal{B}$ and let $f\colon M\to N$ be a morphism in $\mathcal{A}$. Consider the induced morphisms

$$\lambda_f\colon \operatorname{Coker} Sf \longrightarrow S\operatorname{Coker} f \quad \text{and} \quad \varkappa_f\colon T\operatorname{Ker} f \longrightarrow \operatorname{Ker} Tf.$$

(a) *If $T\operatorname{Ker} f$ is in $\mathcal{S}$, then $\operatorname{Ker}\lambda_f$ is also in $\mathcal{S}$.*
(b) *If $S\operatorname{Coker} f$ is in $\mathcal{S}$, then $\operatorname{Coker}\varkappa_f$ is also in $\mathcal{S}$.*
(c) *If $T\operatorname{Im} f$ is in $\mathcal{S}$, then $\operatorname{Coker}\lambda_f$ is also in $\mathcal{S}$.*
(d) *If $S\operatorname{Im} f$ is in $\mathcal{S}$, then $\operatorname{Ker}\varkappa_f$ is also in $\mathcal{S}$.*

Proof. Consider the exact sequences

$$0\longrightarrow K\longrightarrow M\overset{g}{\longrightarrow} I\longrightarrow 0 \quad\text{and}\quad 0\longrightarrow I\overset{h}{\longrightarrow} N\longrightarrow C\longrightarrow 0,$$

where $K=\operatorname{Ker} f$, $I=\operatorname{Im} f$, $C=\operatorname{Coker} f$ and $f=h\circ g$.

There is a commutative diagram with the first row exact:

$$\begin{array}{ccccccc}
\operatorname{Coker} Sg & \longrightarrow & \operatorname{Coker} Sf & \longrightarrow & \operatorname{Coker} Sh & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle\lambda_f} & & \Big\downarrow{\scriptstyle\lambda_h} & & \\
 & & S\operatorname{Coker} f & \overset{\cong}{\longrightarrow} & S\operatorname{Coker} h. & &
\end{array}$$

From the exactness of $SI\overset{Sh}{\to}SN\overset{Sv}{\to}SC$, we get $\operatorname{Ker}\lambda_h=0$. Hence there is an epimorphism $\operatorname{Coker} Sg\to\operatorname{Ker}\lambda_f\to 0$. Furthermore the exactness of $SM\overset{Sg}{\to}SI\to TK$ yields a monomorphism $0\to\operatorname{Coker} Sg\to TK$. Hence if TK is in $\mathcal{S}$ then $\operatorname{Ker}\lambda_f$ is in $\mathcal{S}$, too.

The assertion in (b) follows by dualizing the proof of (a). There is the following commutative diagram with exact bottom row:

$$\begin{array}{ccccccc}
 & & T\operatorname{Ker} g & \overset{\cong}{\longrightarrow} & T\operatorname{Ker} f & & \\
 & & \Big\downarrow{\scriptstyle\varkappa_g} & & \Big\downarrow{\scriptstyle\varkappa_f} & & \\
0 & \longrightarrow & \operatorname{Ker} Tg & \longrightarrow & \operatorname{Ker} Tf & \longrightarrow & \operatorname{Ker} Th.
\end{array}$$

Since $TK\to TM\overset{Tg}{\to}TI$ is exact, $\varkappa_g$ is an epimorphism and therefore there is a monomorphism $0\to\operatorname{Coker}\varkappa_f\to\operatorname{Ker} Th$.

The exactness of $SC\to TI\overset{Th}{\to}TN$ yields an epimorphism $SC\to\operatorname{Ker} Th\to 0$. It follows that if SC is in $\mathcal{S}$, then so is $\operatorname{Coker}\varkappa_f$. This proves (b).

From the exactness of $SI \xrightarrow{Sh} SN \to SC \to TI$, we get an exact sequence $0 \to \operatorname{Coker} Sh \xrightarrow{\lambda_h} SC \to TI$ and finally we get a monomorphism $0 \to \operatorname{Coker} \lambda_h \to TI$. Hence if TI is in $\mathcal{S}$, then $\operatorname{Coker} \lambda_f = \operatorname{Coker} \lambda_h$ is in $\mathcal{S}$. This proves (c).

The exactness of $SI \to TK \xrightarrow{Tg} TI$ yields the exactness of $SI \to T \operatorname{Ker} g \xrightarrow{\varkappa_g} \operatorname{Ker} Tg$. We get an epimorphism $SI \to \operatorname{Ker} \varkappa_g \to 0$. But $\operatorname{Ker} \varkappa_g = \operatorname{Ker} \varkappa_f$. Hence if SI belongs to $\mathcal{S}$ then so does $\operatorname{Ker} \varkappa_f$, and (d) is proved. □

3. A morphism between connected sequences of functors

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories and suppose that both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let $G\colon \mathcal{A} \to \mathcal{B}$ and $F\colon \mathcal{B} \to \mathcal{C}$ be two left exact covariant functors such that for each injective object E in $\mathcal{A}$, $F^n(GE)=0$ for all $n \geq 1$. We denote the nth right derived functors of F, G and their composite FG by F^n, G^n and $(FG)^n$, respectively. Then there is a natural morphism

$$\{\theta_n\}_n \colon \{(FG)^n\}_n \longrightarrow \{FG^n\}_n$$

between connected sequences of functors. Next we will give conditions for the kernel or the cokernel of $\theta_n(M)$ to belong to a given Serre subcategory $\mathcal{S}$ of $\mathcal{C}$. Here M is a fixed object in $\mathcal{C}$ and n is a fixed natural number.

Proposition 3.1.
(a) *If $F^{n-i}G^iM$ belong to $\mathcal{S}$ for all $i<n$, then* $\operatorname{Ker} \theta_n(M)$ *belongs to $\mathcal{S}$.*
(b) *If $F^{n+1-i}G^iM$ belong to $\mathcal{S}$ for all $i<n$, then* $\operatorname{Coker} \theta_n(M)$ *belongs to $\mathcal{S}$.*

Proof. Let first $n=1$ and let $0 \to M \to E \xrightarrow{\eta} N \to 0$ be exact, where E is injective. We get the exact sequences

$$0 \longrightarrow GM \longrightarrow GE \xrightarrow{G\eta} GN \longrightarrow G^1M \longrightarrow 0$$

and

$$0 \longrightarrow FGM \longrightarrow FGE \xrightarrow{FG\eta} FGN \longrightarrow (FG)^1M \longrightarrow 0.$$

Hence $\theta_1(M)\colon (FG)^1M \to FG^1M$ coincides with $\lambda_{G\eta}\colon \operatorname{Coker} FG\eta \to F \operatorname{Coker} G\eta$.

Since $F^1 \operatorname{Ker} G\eta \cong F^1GM$, which belongs to $\mathcal{S}$, we can apply Lemma 2.1 with $S=F$ and $T=F^1$. This settles the case $n=1$ for the statement (a).

The statement (b) for $n=1$ follows from case (c) in Lemma 2.1. Put $I = \operatorname{Im} G_\eta$ and apply F to the exact sequence $0 \to GM \xrightarrow{G_\eta} GE \to I \to 0$. Since $F^1GE=0$ we get $F^1I \cong F^2GM$.

For the induction step we use the commutative diagram ($n \geq 2$):

$$\begin{array}{ccccccc} 0 & \longrightarrow & (FG)^{n-1}N & \longrightarrow & (FG)^n M & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \theta_{n-1}(N)} & & \Big\downarrow{\scriptstyle \theta_n(M)} & & \\ 0 & \longrightarrow & FG^{n-1}(N) & \longrightarrow & FG^n(M) & \longrightarrow & 0. \end{array}$$

Consequently, $\operatorname{Ker}\theta_n(M) \cong \operatorname{Ker}\theta_{n-1}(N)$. When $1 \leq k \leq n-1$ we have isomorphisms $F^{(n-1)-k}G^k N \cong F^{n-(k+1)}G^{k+1}M$. Moreover since $F^{n-1}GE=0$ and $F^nGE=0$, the exactness of the sequences $0 \to GM \to GE \to I \to 0$ and $0 \to I \to GN \to G^1 M \to 0$ implies that $F^{n-1}I \cong F^n GM$ (remember that $n>1$). The second exact sequence, i.e. $0 \to I \to GN \to G^1 M \to 0$, also implies the exactness of $F^{n-1}I \to F^{n-1}GN \to F^{n-1}G^1 M$. Since the endterms are in $\mathcal{S}$, it follows that $F^{n-1}G^k N$ is in $\mathcal{S}$ also for $k=0$.

Consequently we are able to apply the induction hypothesis for (a) to N. The case (b) is treated in the same way (just change n to $n+1$). □

4. Applications to local cohomology

Theorem 4.1. *Let R be a Noetherian ring, $\mathfrak{a}$ be an ideal of R and M be an R-module. Let n be a non-negative integer and $\mathcal{S}$ be a Serre subcategory of the category of R-modules. Consider the natural homomorphism*

$$f\colon \operatorname{Ext}^n_R(R/\mathfrak{a}, M) \to \operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M)).$$

(a) *If $\operatorname{Ext}^{n-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ belongs to $\mathcal{S}$ for all $j<n$, then* $\operatorname{Ker} f$ *belongs to $\mathcal{S}$.*

(b) *If $\operatorname{Ext}^{n+1-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ belongs to $\mathcal{S}$ for all $j<n$, then* $\operatorname{Coker} f$ *belongs to $\mathcal{S}$.*

(c) *If $\operatorname{Ext}^{t-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ belongs to $\mathcal{S}$ for $t=n, n+1$ and for all $j<n$, then* $\operatorname{Ker} f$ *and* $\operatorname{Coker} f$ *both belong to $\mathcal{S}$. Thus $\operatorname{Ext}^n_R(R/\mathfrak{a}, M)$ belongs to $\mathcal{S}$ if and only if $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M))$ belongs to $\mathcal{S}$. (See also* [8, *Theorem* 6.3.9].)

Proof. We apply Proposition 3.1 with $FX=\operatorname{Hom}_R(R/\mathfrak{a}, X)$ and $GX=\Gamma_{\mathfrak{a}}(X)$. Observe that $FG=F$. □

The following corollary is one of our main results of this paper. It is well known, by using a spectral sequence argument as in [13, Proposition 1.1(b)] or another method as in [2, Lemma 2.5(b)], (when M is a finite module and $n=\operatorname{depth}_{\mathfrak{a}} M$, or more generally when the local cohomology vanishes below n, for example use [15, Theorem 11.3]) that the natural map

$$f\colon \operatorname{Ext}^n_R(R/\mathfrak{a}, M) \to \operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M))$$

is an isomorphism. In the following corollary we show that without any condition on M it is not necessary that these local cohomology modules are zero for the natural map f to be an isomorphism.

Corollary 4.2.
(a) *If* $\operatorname{Ext}^{n-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))=0$ *for all* $j<n$, *then* f *is injective.*
(b) *If* $\operatorname{Ext}^{n+1-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))=0$ *for all* $j<n$, *then* f *is surjective.*
(c) *If* $\operatorname{Ext}^{t-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))=0$ *for* $t=n, n+1$ *and for all* $j<n$, *then* f *is an isomorphism.*

Proof. In Proposition 4.1 set $\mathcal{S}=\{0\}$. □

Corollary 4.3. *If* $\mathrm{H}^i_{\mathfrak{a}}(M)=0$ *for all* $i<n$ (*in particular, when* M *is finite and* $n=\operatorname{depth}_{\mathfrak{a}} M$) *then*

$$\operatorname{Ext}^n_R(R/\mathfrak{a}, M) \cong \operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M)).$$

Corollary 4.4. *If* $\operatorname{Ext}^{n-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ *and* $\operatorname{Ext}^{n+1-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ *are finite for all* $j<n$, *then the module* $\operatorname{Ext}^n_R(R/\mathfrak{a}, M)$ *is finite if and only if the module* $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M))$ *is finite.*

Corollary 4.5. *If* $\operatorname{Ext}^{t-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ *is Artinian for* $t=n, n+1$ *and for all* $j<n$, *then* $\operatorname{Ext}^n_R(R/\mathfrak{a}, M)$ *is Artinian if and only if* $\mathrm{H}^n_{\mathfrak{a}}(M)$ *is Artinian.*

Proof. This is immediate by using Theorem 4.1(c) and [5, Theorem 7.1.2]. □

Corollary 4.6.
(a) *If* $\operatorname{Ext}^{n-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))=0$ *for all* $j<n$, *then*

$$\operatorname{Ass}_R(\mathrm{H}^n_{\mathfrak{a}}(M)) \subset \operatorname{Ass}_R(\operatorname{Ext}^n_R(R/\mathfrak{a}, M)) \cup \operatorname{Ass}_R(\operatorname{Coker} f).$$

(b) *If* $\operatorname{Ext}^{n+1-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))=0$ *for all* $j<n$, *then*

$$\operatorname{Ass}_R(\mathrm{H}^n_{\mathfrak{a}}(M)) \subset \operatorname{Ass}_R(\operatorname{Ext}^n_R(R/\mathfrak{a}, M)) \cup \operatorname{Supp}_R(\operatorname{Ker} f).$$

(c) *If* $\operatorname{Ext}^{t-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))=0$ *for* $t=n, n+1$ *and for all* $j<n$, *then*

$$\operatorname{Ass}_R(\operatorname{Ext}^n_R(R/\mathfrak{a}, M)) = \operatorname{Ass}_R(\mathrm{H}^n_{\mathfrak{a}}(M)).$$

For the proof of part (b) in Corollary 4.6 we need the following lemma.

Lemma 4.7. *For any exact sequence* $N\to L\to T\to 0$ *of* R*-modules and* R*-homomorphisms we have*

$$\mathrm{Ass}_R(T)\subset \mathrm{Ass}_R(L)\cup \mathrm{Supp}_R(N).$$

Proof. Let $\mathfrak{p}\in \mathrm{Ass}_R(T)\setminus \mathrm{Supp}_R(N)$. Then $N_\mathfrak{p}=0$ and therefore $L_\mathfrak{p}\cong T_\mathfrak{p}$. Now $\mathfrak{p}R_\mathfrak{p}\in \mathrm{Ass}_{R_\mathfrak{p}}(T_\mathfrak{p})$. Hence $\mathfrak{p}R_\mathfrak{p}\in \mathrm{Ass}_{R_\mathfrak{p}}(L_\mathfrak{p})$ and consequently $\mathfrak{p}\in \mathrm{Ass}_R(L)$. □

Divaani-Aazar and Mafi introduced in [9] weakly Laskerian modules. An R-module M is *weakly Laskerian* if for any submodule N of M the quotient M/N has finitely many associated primes. The weakly Laskerian modules form a Serre subcategory of the category of R-modules. In fact it is the largest Serre subcategory such that each module in it has just finitely many associated prime ideals. In the following corollary we give some conditions for the finiteness of the sets of associated primes of $\mathrm{Ext}^n_R(R/\mathfrak{a},M)$ and $\mathrm{H}^n_\mathfrak{a}(M)$.

Corollary 4.8.

(a) *If* $\mathrm{Ass}_R(\mathrm{H}^n_\mathfrak{a}(M))$ *is a finite set and* $\mathrm{Ext}^{n-j}_R(R/\mathfrak{a},\mathrm{H}^j_\mathfrak{a}(M))$ *is weakly Laskerian for all* $j<n$ (*e.g. if* $\mathrm{H}^j_\mathfrak{a}(M)$ *is weakly Laskerian for all* $j<n$), *then* $\mathrm{Ass}_R(\mathrm{Ext}^n_R(R/\mathfrak{a},M))$ *is a finite set.* (*See also* [8, *Corollary* 6.3.11]).

(b) *If* $\mathrm{Ext}^n_R(R/\mathfrak{a},M)$ *and* $\mathrm{Ext}^{n+1-j}_R(R/\mathfrak{a},\mathrm{H}^j_\mathfrak{a}(M))$ *for all* $j<n$ *are weakly Laskerian* (*e.g. if* M *and* $\mathrm{H}^j_\mathfrak{a}(M)$ *for* $j<n$ *are weakly Laskerian*), *then* $\mathrm{Ass}_R(\mathrm{H}^n_\mathfrak{a}(M))$ *is a finite set.* (*Compare with* [9, *Corollary* 2.7].)

Proof. Note that $\mathrm{Ass}_R(\mathrm{H}^n_\mathfrak{a}(M))=\mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a},\mathrm{H}^n_\mathfrak{a}(M)))$. We use the exact sequence

$$0\longrightarrow \mathrm{Ker}\, f\longrightarrow \mathrm{Ext}^n_R(R/\mathfrak{a},M)\xrightarrow{\ f\ }\mathrm{Hom}_R(R/\mathfrak{a},\mathrm{H}^n_\mathfrak{a}(M))\longrightarrow \mathrm{Coker}\, f\longrightarrow 0.$$

(a) By Theorem 4.1(a), $\mathrm{Ker}\, f$ is weakly Laskerian.
(b) By Theorem 4.1(b), $\mathrm{Coker}\, f$ is weakly Laskerian. □

5. Asymptotic behaviors

We now turn to the graded case. Let $R=\bigoplus_{i\geq 0} R_i$ be a homogeneous graded Noetherian ring and $M=\bigoplus_{i\in\mathbb{Z}} M_i$ be a graded R-module. The module M is said to have

(a) the property of *asymptotic stability of associated primes* if there exists an integer n_0 such that $\operatorname{Ass}_{R_0}(M_n)=\operatorname{Ass}_{R_0}(M_{n_0})$ for all $n\leq n_0$;

(b) the property of *asymptotic stability of supports* if there exists an integer n_0 such that $\operatorname{Supp}_{R_0}(M_n)=\operatorname{Supp}_{R_0}(M_{n_0})$ for all $n\leq n_0$.

(c) the property of *tameness* if $M_i=0$ for all $i\ll 0$ or else $M_i\neq 0$ for all $i\ll 0$.

Note that if M has the property of (a) then M has the property of (b) and if M has the property of (b) then M is tame i.e. has the property of (c).

Let s be a non-negative integer. In order to refine and complete [6, Theorem 4.4] we study the asymptotic behavior of the kernel and cokernel of the natural graded homomorphism between the graded modules $\operatorname{Ext}^s_R(R/R_+,M)$ and $\operatorname{Hom}_R(R/R_+,\mathrm{H}^s_{R_+}(M))$. For more information on the notions in (a), (b) and (c), see [3].

Definition 5.1. A graded module M over a homogeneous graded ring R is called *asymptotically zero* if $M_i=0$ for all $i\ll 0$.

All finite graded R-modules are asymptotically zero.

Theorem 5.2. *Assume that M is a graded R-module and let n be a fixed non-negative integer such that the modules*

$$\operatorname{Ext}^{n-j}_R(R/R_+,\mathrm{H}^j_{R_+}(M)) \quad \textit{and} \quad \operatorname{Ext}^{n-j+1}_R(R/R_+,\mathrm{H}^j_{R_+}(M)), \quad \textit{where } j<s,$$

are asymptotically zero (e.g. they might be finite). Then the kernel and the cokernel of the natural homomorphism

$$f\colon \operatorname{Ext}^s_R(R/R_+,M)\to \operatorname{Hom}_R(R/R_+,\mathrm{H}^s_{R_+}(M))$$

are asymptotically zero.

Therefore $\operatorname{Ext}^s_R(R/R_+,M)$ has one of the properties of (a), (b) *and* (c) *if and only if $\operatorname{Hom}_R(R/R_+,\mathrm{H}^s_{R_+}(M))$ has.*

Proof. Note that the class of graded modules which are asymptotically zero is a Serre subcategory of the category of graded R-modules and we can use Proposition 3.1. □

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