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On the L^p-boundedness of pseudo-differential operators with non-regular symbols

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Abstract. In this paper, we consider the continuity property of pseudo-differential operators with symbols whose Fourier transforms have compact support. As applications, we obtain the L^p-boundedness for symbols in Besov spaces and in modulation spaces.

1. Introduction

The continuity property of pseudo-differential operators has been well investigated. Among many works, Calderón–Vaillancourt [3] first treated the boundedness for the class $S^0_{0,0}$, which means that the boundedness of all the derivatives of a symbol assures the L^2-boundedness of the corresponding operator. It should be mentioned that the boundedness of all the derivatives of a symbol is not necessary in their proof.

Authors such as Coifman–Meyer [4], Cordes [5], Miyachi [11] and Muramatu [12], improved the result of [3] along this line, that is, investigated the minimal assumption on the regularity of a symbol. In fact, they proved that the boundedness of the derivatives of a symbol up to a certain order, which exceeds $n/2$, assures the boundedness on $L^2(\mathbb{R}^n)$ of the corresponding operator. In particular, Sugimoto [16] showed that symbols $\sigma(x,\xi)$ in the Besov space $B^{\infty,1}_{(n/2,n/2)}(\mathbb{R}^n\times\mathbb{R}^n)$ imply the L^2-boundedness. This assumption means that $\sigma(x,\xi)$ belongs to $B^{\infty,1}_{n/2}(\mathbb{R}^n)$ with respect to both x and ξ. [16, Theorem 2.1.2] also stated that the class $B^{\infty,1}_{(n/2,n/2)}(\mathbb{R}^n\times\mathbb{R}^n)$ can be replaced by a wider one $B^{\infty,1}_{(1/2,\dots,1/2)}(\mathbb{R}\times\dots\times\mathbb{R})$, which means that $\sigma(x,\xi)=\sigma(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$ belongs to $B^{\infty,1}_{1/2}(\mathbb{R})$ with respect to x_j and ξ_k for all $j,k=1,\dots,n$. This is equivalent to the result of Boulkhemair [1, Corollary 6].

Theorem 1.1. *There exists a constant* $C>0$ *such that*

$$\|\sigma(X,D)f\|_{L^2} \le C(R_1 \dots R_{2n})^{1/2} \|\sigma(x,\xi)\|_{L^\infty_{x,\xi}} \|f\|_{L^2}$$

for all $\sigma(x,\xi)$ *with* $\operatorname{supp}\hat{\sigma} \subset \prod_{j=1}^{2n}[-R_j, R_j]$, $R_j \ge 1$, $j=1,\dots,2n$, *and all* $f \in L^2(\mathbb{R}^n)$. *Here* $\hat{\sigma} = \mathcal{F}_{1,2}\sigma$ *and* $C>0$ *is independent of* $R_j \ge 1$, $j=1,\dots,2n$.

The objective of this paper is to induce a similar estimate for the L^p-boundedness, $p \ne 2$. In this direction, Miyachi [11] proved that symbols $\sigma(x,\xi)$ in the weighted Besov space $B^{\infty,\infty}_{(s_1,s_2),m(p)}(\mathbb{R}^n \times \mathbb{R}^n)$, $s_1 > \min\{n/p, n/2\}$, $s_2 > \max\{n/p, n/2\}$, $m(p) = n|1/p - 1/2|$ imply the L^p-boundedness, $1<p<\infty$. This assumption means that fractional derivatives of $\sigma(x,\xi)$ in x (resp. ξ) higher than $\min\{n/p, n/2\}$ (resp. $\max\{n/p, n/2\}$) are bounded by $C\langle\xi\rangle^{-m(p)}$. Sugimoto [17] treated the critical case $s_1 = n/2$ and $s_2 = n/p$ as in the argument for the L^2-boundedness, and showed that symbols $\sigma(x,\xi)$ in the weighted Besov space $B^{\infty,1}_{(n/2,n/p),m(p)}(\mathbb{R}^n \times \mathbb{R}^n)$ imply the L^p-boundedness for $1<p\le 2$. The following main theorem of this paper extends this result to the case $p>2$. Furthermore, in the case $p<2$, it replaces the symbol class $B^{\infty,1}_{(n/2,n/p),m(p)}(\mathbb{R}^n \times \mathbb{R}^n)$ by the wider one $B^{\infty,1}_{(1/2,\dots,1/2,1/p,\dots,1/p),m(p)}(\mathbb{R} \times \dots \times \mathbb{R})$ as in the case of L^2-boundedness.

Theorem 1.2. *Let* $0<p<\infty$. *Then there exists a constant* $C>0$ *such that*

$$\|\sigma(X,D)f\|_{L^p} \le C(R_1 \dots R_n)^{\min\{1/p,1/2\}} (R_{n+1} \dots R_{2n})^{\max\{1/p,1/2\}} \times \|\sigma(x,\xi)\langle\xi\rangle^{n|1/p-1/2|}\|_{L^\infty_{x,\xi}} \|f\|_{H^p}$$

for all $\sigma(x,\xi)$ *with* $\operatorname{supp}\hat{\sigma} \subset \prod_{j=1}^{2n}[-R_j, R_j]$, $R_j \ge 1$, $j=1,\dots,2n$, *and all* $f \in H^p(\mathbb{R}^n)$. *Here* $H^p(\mathbb{R}^n)$ *is the Hardy space and* $C>0$ *is independent of* $R_j \ge 1$, $j=1,\dots,2n$.

The statement for symbols in the class $B^{\infty,1}_{(1/2,\dots,1/2,1/p,\dots,1/p),m(p)}$ with $p<2$ (resp. $B^{\infty,1}_{(1/p,\dots,1/p,1/2,\dots,1/2),m(p)}$ with $p>2$) can be obtained from Theorem 1.2 as a corollary (see Corollary 5.2(2)). We remark that symbols in the weighted modulation space $M^{\infty,1}_{m(p)}$ also induce the L^p-boundedness. Such a result is just a corollary of Theorem 1.2 again (see Corollary 5.2(3)). The L^2-boundedness with symbols in the modulation space $M^{\infty,1}_0$ was studied by Sjöstrand [13], Boulkhemair [1], Gröchenig–Heil [8] and Toft [18].

Finally, we explain the organization of this paper. Section 2 is for the preliminaries, including the notation and definitions of function spaces. Sections 3 and 4 are devoted to the proofs of Theorem 1.2 with $p\le 2$ and $p\ge 2$, respectively. In Section 5, we state several results induced from Theorem 1.2.

2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the *Fourier transform* $\mathcal{F}f$ and the *inverse Fourier transform* $\mathcal{F}^{-1}f$ of $f\in\mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi)=\hat{f}(\xi)=\int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx \quad \text{and} \quad \mathcal{F}^{-1}f(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\,d\xi.$$

Let $\sigma(x,\xi)\in\mathcal{S}'(\mathbb{R}^{2n})$, where $x,\xi\in\mathbb{R}^n$. We denote by $\mathcal{F}_1\sigma(y,\xi)$ and $\mathcal{F}_2\sigma(x,\eta)$ the partial Fourier transforms of σ with respect to the first variable and the second variable, respectively. That is, $\mathcal{F}_1\sigma(y,\xi)=\mathcal{F}[\sigma(\,\cdot\,,\xi)](y)$ and $\mathcal{F}_2\sigma(x,\eta)=\mathcal{F}[\sigma(x,\cdot\,)](\eta)$. We also denote by $\mathcal{F}_1^{-1}\sigma$ and $\mathcal{F}_2^{-1}\sigma$ the partial inverse Fourier transforms of σ with respect to the first variable and the second variable, respectively. We write $\mathcal{F}_{1,2}=\mathcal{F}_1\mathcal{F}_2$ and $\mathcal{F}_{1,2}^{-1}=\mathcal{F}_1^{-1}\mathcal{F}_2^{-1}$, and note that $\mathcal{F}_{1,2}$ and $\mathcal{F}_{1,2}^{-1}$ are the usual Fourier transform and inverse Fourier transform of distributions on $\mathbb{R}^n\times\mathbb{R}^n$. For $\sigma\in\mathcal{S}'(\mathbb{R}^{2n})$, the pseudo-differential operator $\sigma(X,D)$ is defined by

$$\sigma(X,D)f(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)\,d\xi \quad \text{for } f\in\mathcal{S}(\mathbb{R}^n).$$

We use the notation $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$, where $\xi\in\mathbb{R}^n$. For $1\le p\le\infty$, p' is the conjugate exponent of p (that is, $1/p+1/p'=1$).

We introduce Hardy spaces based on Stein [14, Chapter 3]. Let $0<p<\infty$, and let $\Phi\in\mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n}\Phi(x)\,dx\neq 0$. Then the *Hardy space* $H^p(\mathbb{R}^n)$ consists of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^p}=\Big\|\sup_{0<t<\infty}|\Phi_t*f|\Big\|_{L^p}<\infty,$$

where $\Phi_t(x)=t^{-n}\Phi(x/t)$. It is known that $H^p(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ if $1<p<\infty$ ([14, Chapter 3, Section 1.2.1]), and the definition of $H^p(\mathbb{R}^n)$ is independent of the choice of $\Phi\in\mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}\Phi(x)\,dx\neq 0$ ([14, Chapter 3, Theorem 1]). Let $0<p\le 1$. We say that a is an H^p-*atom* if a satisfies

$$\text{(2.1)} \qquad \operatorname{supp} a\subset B, \quad \|a\|_{L^\infty}\le|B|^{-1/p} \quad \text{and} \quad \int_{\mathbb{R}^n} x^\beta a(x)\,dx=0$$

for all $|\beta|\le[n/p-n]$, where B is an open ball and $[n/p-n]$ stands for the largest integer $\le n/p-n$. It is also known that every $f\in H^p(\mathbb{R}^n)$ can be written as a sum of H^p-atoms:

$$f=\sum_{j=1}^{\infty}\lambda_j a_j \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

where $\{a_j\}_{j\ge1}$ is a collection of H^p-atoms and $\{\lambda_j\}_{j\ge1}$ is a sequence of complex numbers with $\sum_{j=1}^{\infty}|\lambda_j|^p<\infty$. Moreover,

$$\frac{1}{C}\inf\Big(\sum_{j=1}^{\infty}|\lambda_j|^p\Big)^{1/p}\le\|f\|_{H^p}\le C\inf\Big(\sum_{j=1}^{\infty}|\lambda_j|^p\Big)^{1/p},$$

where the infimum is taken over all representations of f ([14, Chapter 3, Theorem 2]). By virtue of atomic decomposition of H^p, if an L^2-bounded operator T satisfies $\|Ta\|_{L^p}\le C$ for all H^p-atoms a, then T is bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, where $0<p\le1$.

We recall the definitions of Besov and modulation spaces. Assume that $0<q\le\infty$ and $\rho\in\mathbb{R}$. Let $\psi_0,\psi\in\mathcal{S}(\mathbb{R}^N)$ be such that

$$\begin{aligned}&\operatorname{supp}\psi_0\subset\{\xi\in\mathbb{R}^N:|\xi|\le2\},\\ &\operatorname{supp}\psi\subset\{\xi\in\mathbb{R}^N:2^{-1}\le|\xi|\le2\},\\ &\psi_0(\xi)+\sum_{j=1}^{\infty}\psi(2^{-j}\xi)=1\quad\text{for all }\xi\in\mathbb{R}^N,\end{aligned}\tag{2.2}$$

and set $\psi_j(\xi)=\psi(2^{-j}\xi)$ if $j\ge1$. The *weighted Besov space* $B^{\infty,q}_{(s_1,s_2),\rho}(\mathbb{R}^n\times\mathbb{R}^n)$ consists of all $\sigma\in\mathcal{S}'(\mathbb{R}^{2n})$ such that

$$\|\sigma\|_{B^{\infty,q}_{(s_1,s_2)}}=\Big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(2^{js_1}2^{ks_2}\|\psi_j(D_x)\psi_k(D_\xi)\sigma(x,\xi)\langle\xi\rangle^{\rho}\|_{L^\infty_{x,\xi}})^q\Big)^{1/q}<\infty,$$

where $s_1,s_2\in\mathbb{R}$, $\{\psi_j\}_{j\ge0},\{\psi_k\}_{k\ge0}$ are as in (2.2) with $N=n$,

$$\psi_j(D_x)\psi_k(D_\xi)\sigma(x,\xi)=\mathcal{F}^{-1}_{(y,\eta)\to(x,\xi)}[\psi_j(y)\psi_k(\eta)\hat{\sigma}(y,\eta)]$$

and $x,y,\xi,\eta\in\mathbb{R}^n$. Similarly, $B^{\infty,q}_{(s_1,\dots,s_{2n}),\rho}(\mathbb{R}\times\dots\times\mathbb{R})$ is defined by the norm

$$\begin{aligned}\|\sigma\|_{B^{\infty,q}_{(s_1,\dots,s_{2n}),\rho}}=\Big(&\sum_{j_1,\dots,j_n=0}^{\infty}\sum_{k_1,\dots,k_n=0}^{\infty}(2^{j_1s_1+\dots+j_ns_n}2^{k_1s_{n+1}+\dots+k_ns_{2n}}\\ &\times\|\psi_{j_1}(D_{x_1})\dots\psi_{j_n}(D_{x_n})\psi_{k_1}(D_{\xi_1})\dots\psi_{k_n}(D_{\xi_n})\sigma(x,\xi)\langle\xi\rangle^{\rho}\|_{L^\infty_{x,\xi}})^q\Big)^{1/q},\end{aligned}$$

where $s_1, \ldots, s_{2n} \in \mathbb{R}$, $\{\psi_{j_1}\}_{j_1 \geq 0}, \ldots, \{\psi_{k_n}\}_{k_n \geq 0}$ are as in (2.2) with $N=1$,

$$\psi_{j_1}(D_{x_1}) \ldots \psi_{j_n}(D_{x_n}) \psi_{k_1}(D_{\xi_1}) \ldots \psi_{k_n}(D_{\xi_n}) \sigma(x, \xi) = \mathcal{F}^{-1}_{(y,\eta)\to(x,\xi)}[\psi_{j_1}(y_1) \ldots \psi_{j_n}(y_n) \psi_{k_1}(\eta_1) \ldots \psi_{k_n}(\eta_n) \hat{\sigma}(y, \eta)],$$

$y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp} \varphi \subset [-1, 1]^n \quad \text{and} \quad \sum_{k \in \mathbb{Z}^n} \varphi(\xi - k) = 1 \text{ for all } \xi \in \mathbb{R}^n. \tag{2.3}$$

Then the *weighted modulation space* $M^{\infty,q}_\rho(\mathbb{R}^n \times \mathbb{R}^n)$ consists of all $\sigma \in \mathcal{S}'(\mathbb{R}^{2n})$ such that

$$\|\sigma\|_{M^{\infty,q}_\rho} = \left(\sum_{k \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}^n} \|\varphi(D_x - k) \varphi(D_\xi - l) \sigma(x, \xi) \langle \xi \rangle^\rho\|^q_{L^\infty_{x,\xi}} \right)^{1/q} < \infty,$$

where

$$\varphi(D_x - k) \varphi(D_\xi - l) \sigma(x, \xi) = \mathcal{F}^{-1}_{(y,\eta)\to(x,\xi)}[\varphi(y - k) \varphi(\eta - l) \hat{\sigma}(y, \eta)].$$

3. Proof of Theorem 1.2 with $p \leq 2$

In this section we prove Theorem 1.2 with $p \leq 2$.

Lemma 3.1. *Let $s \in \mathbb{R}$. Then there exists a constant $C > 0$ such that*

$$\|\sigma(X, D) \langle D \rangle^s f\|_{L^2} \leq C (R_1 \ldots R_{2n})^{1/2} \|\sigma(x, \xi) \langle \xi \rangle^s\|_{L^\infty_{x,\xi}} \|f\|_{L^2}$$

for all $\sigma(x, \xi)$ with $\operatorname{supp} \hat{\sigma} \subset \prod_{k=1}^{2n} [-R_k, R_k]$, $R_k \geq 1$, $k = 1, \ldots, 2n$, and all $f \in L^2(\mathbb{R}^n)$. Here $C > 0$ is independent of $R_k \geq 1$, $k = 1, \ldots, 2n$.

Proof. Let $\omega_s(\xi) = \langle \xi \rangle^s$, and let $\{\psi_j\}_{j \geq 0}$ be as in (2.2) with $N = n$. Then

$$\sigma(x, \xi) \langle \xi \rangle^s = \sum_{j=0}^\infty \sigma(x, \xi) \psi_j(D) \omega_s(\xi) = \sum_{j=0}^\infty \sigma_j(x, \xi). \tag{3.1}$$

Since $\operatorname{supp} \hat{\sigma} \subset \prod_{k=1}^{2n} [-R_k, R_k]$ and $\operatorname{supp} \widehat{\psi_j(D)\omega_s} \subset \{\xi \in \mathbb{R} : |\xi| \leq 2^{j+1}\}$, we have

$$\operatorname{supp} \widehat{\sigma_j} \subset \left(\prod_{k=1}^n [-R_k, R_k] \right) \times \left(\prod_{k=n+1}^{2n} [-(R_k + 2^{j+1}), R_k + 2^{j+1}] \right) \quad \text{for all } j \geq 0.$$

Hence, by Theorem 1.1,

(3.2) $$\|\sigma_j(X,D)f\|_{L^2} \le C(R_1 \ldots R_n)^{1/2}((R_{n+1}+2^j)\ldots(R_{2n}+2^j))^{1/2}\|\sigma_j(x,\xi)\|_{L^\infty_{x,\xi}}\|f\|_{L^2}$$

for all $f \in L^2(\mathbb{R}^n)$ and $j \ge 0$. If $j=0$ then

$$\begin{aligned}|\sigma_0(x,\xi)| &= \left|\sigma(x,\xi)\int_{\mathbb{R}^n}\Psi_0(\eta)\langle\xi-\eta\rangle^s\,d\eta\right| \\ &\le C|\sigma(x,\xi)|\int_{\mathbb{R}^n}|\Psi_0(\eta)|\langle\xi\rangle^s\langle\eta\rangle^{|s|}\,d\eta = C|\sigma(x,\xi)|\langle\xi\rangle^s,\end{aligned}$$

where $\Psi_0 = \mathcal{F}^{-1}\psi_0$, and consequently

(3.3) $$\|\sigma_0(x,\xi)\|_{L^\infty_{x,\xi}} \le C\|\sigma(x,\xi)\langle\xi\rangle^s\|_{L^\infty_{x,\xi}}.$$

We consider the case $j \ge 1$, and note that $\int_{\mathbb{R}^n}\eta^\beta\Psi(\eta)\,d\eta = i^{|\beta|}\partial^\beta\psi(0) = 0$ for all β, where $\Psi = \mathcal{F}^{-1}\psi$. Let $N = [n/2]$. Since

$$\begin{aligned}\psi_j(D)\omega_s(\xi) &= \int_{\mathbb{R}^n} 2^{jn}\Psi(2^j(\xi-\eta))\left(\omega_s(\eta) - \sum_{|\beta|\le N}\frac{(\eta-\xi)^\beta}{\beta!}\partial^\beta\omega_s(\xi)\right)d\eta \\ &= \int_{\mathbb{R}^n} 2^{jn}\Psi(2^j(\xi-\eta)) \\ &\quad \times\left((N+1)\sum_{|\beta|=N+1}\frac{(\eta-\xi)^\beta}{\beta!}\int_0^1(1-t)^N\partial^\beta\omega_s(\xi+t(\eta-\xi))\,dt\right)d\eta\end{aligned}$$

and $\omega_s \in S^s_{1,0}$ (that is, $|\partial^\beta\omega_s(\xi)| \le C_\beta\langle\xi\rangle^{s-|\beta|}$), we see that

$$\begin{aligned}|\psi_j(D)\omega_s(\xi)| &\le C\int_{\mathbb{R}^n}|\Psi(\eta)|\,|2^{-j}\eta|^{N+1}\left(\int_0^1\langle\xi-2^{-j}t\eta\rangle^{s-(N+1)}\,dt\right)d\eta \\ &\le C\int_{\mathbb{R}^n}|\Psi(\eta)|\,|2^{-j}\eta|^{N+1}\left(\int_0^1\langle\xi\rangle^{s-(N+1)}\langle 2^{-j}t\eta\rangle^{|s-(N+1)|}\,dt\right)d\eta \\ &\le C2^{-j(N+1)}\langle\xi\rangle^{s-(N+1)}\left(\int_{\mathbb{R}^n}|\Psi(\eta)|\,|\eta|^{N+1}\langle\eta\rangle^{|s-(N+1)|}\,d\eta\right)\end{aligned}$$

for all $j \ge 1$. This gives

(3.4) $$\|\sigma_j(x,\xi)\|_{L^\infty_{x,\xi}} = \|\sigma(x,\xi)\psi_j(D)\omega_s(\xi)\|_{L^\infty_{x,\xi}} \le C2^{-j(N+1)}\|\sigma(x,\xi)\langle\xi\rangle^s\|_{L^\infty_{x,\xi}}$$

for all $j\geq 1$. Note that if $a,b\geq 1$ then $a+b\leq 2ab$. Therefore, since $2^j\geq 1$, $j\geq 0$, $R_k\geq 1$, $1\leq k\leq 2n$, and $N+1>n/2$, we have by (3.1)–(3.4),

$$\begin{aligned}
\|\sigma(X,D)\langle D\rangle^s f\|_{L^2} &\leq \sum_{j=0}^{\infty}\|\sigma_j(X,D)f\|_{L^2}\\
&\leq C\sum_{j=0}^{\infty}(R_1...R_n)^{1/2}((R_{n+1}+2^j)...(R_{2n}+2^j))^{1/2}\\
&\quad\times\|\sigma_j(x,\xi)\|_{L^\infty_{x,\xi}}\|f\|_{L^2}\\
&\leq C\sum_{j=0}^{\infty}(R_1...R_n)^{1/2}((2^jR_{n+1})...(2^jR_{2n}))^{1/2}\\
&\quad\times\|\sigma_j(x,\xi)\|_{L^\infty_{x,\xi}}\|f\|_{L^2}\\
&\leq C(R_1...R_{2n})^{1/2}\sum_{j=0}^{\infty}2^{-j((N+1)-n/2)}\|\sigma(x,\xi)\langle\xi\rangle^s\|_{L^\infty_{x,\xi}}\|f\|_{L^2}\\
&= C(R_1...R_{2n})^{1/2}\|\sigma(x,\xi)\langle\xi\rangle^s\|_{L^\infty_{x,\xi}}\|f\|_{L^2}.
\end{aligned}$$

The proof is complete. □

We are now ready to prove Theorem 1.2 with $p\leq 2$.

Proof of Theorem 1.2 with $p\leq 2$. Let $\sigma(x,\xi)$ be such that

$$\operatorname{supp}\hat{\sigma}\subset\prod_{j=1}^{2n}[-R_j,R_j],\quad R_j\geq 1,\ j=1,...,2n.$$

We first consider the case $0<p<1$. By virtue of atomic decomposition of H^p, it is enough to prove that

$$\|\sigma(X,D)f\|_{L^p}\leq C(R_1...R_n)^{1/2}(R_{n+1}...R_{2n})^{1/p}\|\sigma(x,\xi)\langle\xi\rangle^{n(1/p-1/2)}\|_{L^\infty_{x,\xi}} \tag{3.5}$$

for all $f\in H^p(\mathbb{R}^n)$ satisfying

$$\operatorname{supp}f\subset\{x\in\mathbb{R}^n:|x|\leq r\},\quad \|f\|_{L^\infty}\leq r^{-n/p}\quad\text{and}\quad\int_{\mathbb{R}^n}x^\beta f(x)\,dx=0 \tag{3.6}$$

for all $|\beta|\leq[n/p-n]$, where $C>0$ is independent of r. In fact, since $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^p}$ are translation invariant, $\sigma(X,D)(f(\cdot-x_0))(x)=\sigma(X+x_0,D)f(x-x_0)$ and $\operatorname{supp}\mathcal{F}_{(x,\xi)\to(y,\eta)}[\sigma(x+x_0,\xi)]\subset\prod_{j=1}^{2n}[-R_j,R_j]$, if (3.5) holds for all f satisfying (3.6) then (3.5) holds for all f satisfying (2.1) (see Remark 5.1). Note that if f satisfies (3.6) then $\|f\|_{L^2} \le C r^{-n(1/p-1/2)}$.

Let f be as in (3.6) with $r>1$, and split $\|\sigma(X,D)f\|_{L^p}^p$ as

$$\|\sigma(X,D)\|_{L^p}^p$$
$$= \left(\int_{\prod_{j=1}^n [-2rR_{n+j}, 2rR_{n+j}]} + \int_{{}^c\left(\prod_{j=1}^n [-2rR_{n+j}, 2rR_{n+j}]\right)} \right) |\sigma(X,D)f(x)|^p \, dx.$$

Since $\operatorname{supp} \mathcal{F}_2^{-1}\sigma(x,\cdot) \subset \prod_{j=1}^n [-R_{n+j}, R_{n+j}]$ for all $x \in \mathbb{R}^n$, $\operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x| \le r\}$ and

$$\sigma(X,D)f(x) = \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \sigma(x,\xi)\, d\xi \right) f(y)\, dy$$
$$= \int_{\mathbb{R}^n} \mathcal{F}_2^{-1}\sigma(x, x-y) f(y)\, dy, \tag{3.7}$$

we see that

$$\operatorname{supp} \sigma(X,D)f \subset \prod_{j=1}^n [-R_{n+j}, R_{n+j}] + \{x \in \mathbb{R}^n : |x| \le r\} \subset \prod_{j=1}^n [-(R_{n+j}+r), R_{n+j}+r].$$

By Hölder's inequality and Theorem 1.1,

$$\left(\int_{\prod_{j=1}^n [-2rR_{n+j}, 2rR_{n+j}]} |\sigma(X,D)f(x)|^p \, dx \right)^{1/p} \tag{3.8}$$
$$\le \left(\prod_{j=1}^n 4rR_{n+j} \right)^{1/p-1/2} \|\sigma(X,D)f\|_{L^2}$$
$$\le C r^{n(1/p-1/2)} (R_{n+1} \dots R_{2n})^{1/p-1/2} (R_1 \dots R_{2n})^{1/2} \|\sigma(x,\xi)\|_{L^\infty_{x,\xi}} \|f\|_{L^2}$$
$$\le C (R_1 \dots R_n)^{1/2} (R_{n+1} \dots R_{2n})^{1/p} \|\sigma(x,\xi) \langle \xi \rangle^{n(1/p-1/2)}\|_{L^\infty_{x,\xi}},$$

where we have used the fact that $1/p-1/2>0$. On the other hand,

$$\int_{{}^c\left(\prod_{j=1}^n [-2rR_{n+j}, 2rR_{n+j}]\right)} |\sigma(X,D)f(x)|^p \, dx = 0. \tag{3.9}$$

In fact, since $r>1$ and $R_{n+j} \ge 1$, $1 \le j \le n$,

$$\operatorname{supp} \sigma(X,D)f \subset \prod_{j=1}^n [-(R_{n+j}+r), R_{n+j}+r] \subset \prod_{j=1}^n [-2rR_{n+j}, 2rR_{n+j}].$$

Hence, (3.8) and (3.9) give (3.5) for f satisfying (3.6) with $r>1$.

Let f be as in (3.6) with $r\le 1$. In this case, our proof is similar to that of [17], but we give it for the reader's convenience. Since

$$\begin{aligned}\operatorname{supp}\sigma(X,D)f &\subset \prod_{j=1}^{n}[-(R_{n+j}+r),R_{n+j}+r]\\ &\subset \prod_{j=1}^{n}[-(R_{n+j}+1),R_{n+j}+1]\subset \prod_{j=1}^{n}[-2R_{n+j},2R_{n+j}],\end{aligned}$$

we have by Hölder's inequality

$$\|\sigma(X,D)f\|_{L^p}\le C(R_{n+1}...R_{2n})^{1/p-1/2}\|\sigma(X,D)f\|_{L^2}. \tag{3.10}$$

Let $\phi\in\mathcal{S}(\mathbb{R}^n)$ be such that $\phi=1$ on $\{\xi\in\mathbb{R}^n:|\xi|\le 1\}$ and $\operatorname{supp}\phi\subset\{\xi\in\mathbb{R}^n:|\xi|\le 2\}$, and split $\sigma(x,\xi)$ as

$$\sigma(x,\xi)=\sigma(x,\xi)\phi(r\xi)+\sigma(x,\xi)(1-\phi(r\xi))=\sigma^{(1)}(x,\xi)+\sigma^{(2)}(x,\xi).$$

Set $K^{(l)}(x,y)=\mathcal{F}_2^{-1}\sigma^{(l)}(x,y)$, $l=1,2$, and $N=[n(1/p-1)]$. By (3.6) and (3.7),

$$\begin{aligned}\sigma^{(1)}(X,D)f(x) &= \int_{|y|\le r}\bigg(K^{(1)}(x,x-y)-\sum_{|\beta|\le N}\frac{(-y)^\beta}{\beta!}\partial_2^\beta K^{(1)}(x,x)\bigg)f(y)\,dy\\ &= \int_{|y|\le r}\bigg((N+1)\sum_{|\beta|=N+1}\frac{(-y)^\beta}{\beta!}\\ &\qquad\times\int_0^1(1-t)^N\partial_2^\beta K^{(1)}(x,x-ty)\,dt\bigg)f(y)\,dy\end{aligned}$$

and

$$\sigma^{(2)}(X,D)f(x)=\int_{|y|\le r}K^{(2)}(x,x-y)f(y)\,dy,$$

where $\partial_2^\beta K^{(1)}(x,y)=\partial_y^\beta K^{(1)}(x,y)$. Then, by Minkowski's inequality for integrals,

$$\begin{aligned}&\|\sigma^{(1)}(X,D)f\|_{L^2}\le Cr^{N+1-n(1/p-1)}\sup_{\substack{0\le t\le 1\\ |y|\le r\\ |\beta|=N+1}}\|\partial_2^\beta K^{(1)}(x,x-ty)\|_{L_x^2},\\ &\|\sigma^{(2)}(X,D)f\|_{L^2}\le Cr^{-n(1/p-1)}\sup_{|y|\le r}\|K^{(2)}(x,x-y)\|_{L_x^2}.\end{aligned} \tag{3.11}$$

Setting $\mathcal{F}g^{(1)}_{t,y,\beta,r}(\xi)=\phi(r\xi)(i\xi)^\beta e^{-ity\cdot\xi}\langle\xi\rangle^{-n(1/p-1/2)}$, we obtain

$$\begin{aligned}\partial_2^\beta K^{(1)}(x,x-ty) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-ty)\cdot\xi}\sigma(x,\xi)\phi(r\xi)(i\xi)^\beta\, d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\langle\xi\rangle^{n(1/p-1/2)} \\ &\quad \times\phi(r\xi)(i\xi)^\beta e^{-ity\cdot\xi}\langle\xi\rangle^{-n(1/p-1/2)}\, d\xi \\ &= \sigma(X,D)\langle D\rangle^{n(1/p-1/2)} g^{(1)}_{t,y,\beta,r}(x).\end{aligned}$$

Similarly,

$$K^{(2)}(x,x-y) = \sigma(X,D)\langle D\rangle^{n(1/p-1/2)} g^{(2)}_{y,r}(x),$$

where $\mathcal{F}g^{(2)}_{y,r}(\xi) = (1-\phi(r\xi))e^{-iy\cdot\xi}\langle\xi\rangle^{-n(1/p-1/2)}$. Then, by Lemma 3.1,

$$\begin{aligned}\|\partial_2^\beta K^{(1)}(x,x-ty)\|_{L^2_x} &= \|\sigma(X,D)\langle D\rangle^{n(1/p-1/2)} g^{(1)}_{t,y,\beta,r}\|_{L^2} \\ &\le C(R_1\ldots R_{2n})^{1/2}\|\sigma(x,\xi)\langle\xi\rangle^{n(1/p-1/2)}\|_{L^\infty_{x,\xi}}\|g^{(1)}_{t,y,\beta,r}\|_{L^2}\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\|K^{(2)}(x,x-y)\|_{L^2_x} &= \|\sigma(X,D)\langle D\rangle^{n(1/p-1/2)} g^{(2)}_{y,r}\|_{L^2} \\ &\le C(R_1\ldots R_{2n})^{1/2}\|\sigma(x,\xi)\langle\xi\rangle^{n(1/p-1/2)}\|_{L^\infty_{x,\xi}}\|g^{(2)}_{y,r}\|_{L^2}.\end{aligned} \tag{3.13}$$

Since $N+1-n(1/p-1/2) > -n/2$ and $-n(1/p-1/2) < -n/2$, by Plancherel's theorem,

$$\begin{aligned}\sup_{\substack{0\le t\le 1\\ |y|\le r\\ |\beta|=N+1}} \|g^{(1)}_{t,y,\beta,r}\|_{L^2} &= \sup_{\substack{0\le t\le 1\\ |y|\le r\\ |\beta|=N+1}} (2\pi)^{-n/2}\|\mathcal{F}g^{(1)}_{t,y,\beta,r}\|_{L^2} \\ &\le (2\pi)^{-n/2}\|\phi(r\xi)|\xi|^{N+1-n(1/p-1/2)}\|_{L^2_\xi} \\ &= Cr^{-(N+1-n(1/p-1))}\end{aligned} \tag{3.14}$$

and

$$\sup_{|y|\le r}\|g^{(2)}_{y,r}\|_{L^2_x} \le (2\pi)^{-n/2}\|(1-\phi(r\xi))|\xi|^{-n(1/p-1/2)}\|_{L^2_\xi} = Cr^{n(1/p-1)}. \tag{3.15}$$

Combining (3.10)–(3.15), we have (3.5) for f satisfying (3.6) with $r\le 1$. Thus, we obtain Theorem 1.2 with $0<p<1$.

We next consider the case $1\le p<2$ (since we already have the case $p=2$ as Theorem 1.1). We use the interpolation theorem for analytic families of operators (see Stein–Weiss [15] and Calderón–Torchinsky [2]). Since our proof is very similar to that of Sugimoto [17, Section 5] (or Miyachi [11, p. 150]), we shall only indicate the necessary modifications.

Let $0<p_0<1$ and $S=\{z\in\mathbb{C}:0\leq\operatorname{Re} z\leq 1\}$. For $1\leq p<2$, we take $0<\theta<1$ such that $1/p=(1-\theta)/p_0+\theta/2$. Set

$$\sigma_z(x,\xi)=e^{(z-\theta)^2}\sigma(x,\xi)\langle\xi\rangle^{n(1/p_0-1/2)(z-\theta)}\quad\text{for } z\in S,$$

and note that

$$\sigma_\theta(X,D)=\sigma(X,D). \tag{3.16}$$

Let $\omega_{n(1/p_0-1/2)}(\xi)=\langle\xi\rangle^{n(1/p_0-1/2)}$, and let $\{\psi_j\}_{j\geq 0}$ be as in (2.2) with $N=n$. Then

$$\sigma_z(x,\xi)=\sum_{j=0}^{\infty}e^{(z-\theta)^2}\sigma(x,\xi)\psi_j(D)\omega_{n(1/p_0-1/2)}^{z-\theta}(\xi)=\sum_{j=0}^{\infty}\sigma_{z,j}(x,\xi).$$

In the same way as in the proof of Lemma 3.1, we can prove that

$$|\psi_j(D)\omega_{n(1/p_0-1/2)}^{it-\theta}(\xi)|\leq C\max\{1,t\}^{N+1}2^{-j(N+1)}\langle\xi\rangle^{-n\theta(1/p_0-1/2)}$$

for all $t\in\mathbb{R}$ and $j\geq 0$, where $N=[n/p_0]$. Hence, since

$$\operatorname{supp}\hat{\sigma}_{it,j}\subset\left(\prod_{k=1}^{n}[-R_k,R_k]\right)\times\left(\prod_{k=n+1}^{2n}[-(R_k+2^{j+1}),R_k+2^{j+1}]\right)$$

for all $t\in\mathbb{R}$ and $j\geq 0$ (see the proof of Lemma 3.1), we have by Theorem 1.2 with $0<p_0<1$,

$$\begin{aligned}
\|\sigma_{it}(X,D)f\|_{L^{p_0}}^{p_0} &\leq\sum_{j=0}^{\infty}\|\sigma_{it,j}(X,D)f\|_{L^{p_0}}^{p_0}\\
&\leq C\sum_{j=0}^{\infty}(R_1...R_n)^{p_0/2}((R_{n+1}+2^j)...(R_{2n}+2^j))^{p_0/p_0}\\
&\quad\times\|\sigma_{it,j}(x,\xi)\langle\xi\rangle^{n(1/p_0-1/2)}\|_{L^\infty_{x,\xi}}^{p_0}\|f\|_{H^{p_0}}^{p_0}\\
&\leq C(R_1...R_n)^{p_0/2}(R_{n+1}...R_{2n})^{p_0/p_0}\\
&\quad\times\sum_{j=0}^{\infty}(e^{-t^2+\theta^2}\max\{1,t\}^{N+1}2^{-j(N+1-n/p_0)})^{p_0}\\
&\quad\times\|\sigma(x,\xi)\langle\xi\rangle^{-n\theta(1/p_0-1/2)}\langle\xi\rangle^{n(1/p_0-1/2)}\|_{L^\infty_{x,\xi}}^{p_0}\|f\|_{H^{p_0}}^{p_0}\\
&\leq C(R_1...R_n)^{p_0/2}(R_{n+1}...R_{2n})^{p_0/p_0}\\
&\quad\times\|\sigma(x,\xi)\langle\xi\rangle^{n(1/p-1/2)}\|_{L^\infty_{x,\xi}}^{p_0}\|f\|_{H^{p_0}}^{p_0},
\end{aligned}$$

that is,

$$\|\sigma_{it}(X,D)f\|_{L^{p_0}} \le C(R_1 \ldots R_n)^{1/2}(R_{n+1}\ldots R_{2n})^{1/p_0} \times \|\sigma(x,\xi)\langle\xi\rangle^{n(1/p-1/2)}\|_{L^\infty_{x,\xi}}\|f\|_{H^{p_0}} \tag{3.17}$$

for all $f\in\mathcal{S}(\mathbb{R}^n)\cap H^{p_0}(\mathbb{R}^n)$ and $t\in\mathbb{R}$. Similarly,

$$\|\sigma_{1+it}(X,D)f\|_{L^2} \le C(R_1 \ldots R_n)^{1/2}(R_{n+1}\ldots R_{2n})^{1/2} \times \|\sigma(x,\xi)\langle\xi\rangle^{n(1/p-1/2)}\|_{L^\infty_{x,\xi}}\|f\|_{L^2} \tag{3.18}$$

for all $f\in\mathcal{S}(\mathbb{R}^n)$ and $t\in\mathbb{R}$. Therefore, by the interpolation theorem for analytic families of operators with (3.16)–(3.18), we obtain Theorem 1.2 with $1\le p<2$. □

4. Proof of Theorem 1.2 with $p\ge 2$

Hwang–Lee [9] essentially proved the following lemma, and obtained Miyachi's result (see the introduction) as a corollary. In order to prove Theorem 1.2 with $p\ge 2$, we modify their estimate as follows:

Lemma 4.1. *Let $2\le p<\infty$ and $m=-n(1/2-1/p)$. For $\varphi_1,\ldots,\varphi_n,\psi_1,\ldots,\psi_n\in\mathcal{S}(\mathbb{R})$ and $f,g\in\mathcal{S}(\mathbb{R}^n)$, set*

$$\varphi(y)=[\varphi_1\otimes\ldots\otimes\varphi_n](y),$$
$$\psi(\eta)=[\psi_1\otimes\ldots\otimes\psi_n](\eta),$$
$$V(f,g)(y,\eta)=\int_{\mathbb{R}^n} e^{iy\cdot t}f(\eta+t)\overline{g(t)}\,dt,$$
$$W(\varphi,\psi,f,g)(x,\xi)=\iint_{\mathbb{R}^{2n}} e^{-i(x\cdot y+\xi\cdot\eta)}\varphi(y)\psi(\eta)V(f,g)(y,\eta)\,dy\,d\eta,$$

where $[\varphi_1\otimes\ldots\otimes\varphi_n](y)=\varphi_1(y_1)\ldots\varphi_n(y_n)$ and $y=(y_1,\ldots,y_n)\in\mathbb{R}^n$. Then there exists a constant $C>0$ such that

$$\iint_{\mathbb{R}^{2n}}\langle\xi\rangle^m|W(\varphi,\psi,f,g)(x,\xi)|\,dx\,d\xi \le C\Bigg(\prod_{j=1}^n\Big(\|\widehat{\varphi}_j\|_{L^{p'}}+\Big\|\widehat{\varphi_j^{(1)}}\Big\|_{L^{p'}}+\Big\|\widehat{\varphi_j^{(2)}}\Big\|_{L^{p'}}\Big)\Bigg) \times\Bigg(\prod_{k=1}^n\Big(\|\hat{\psi}_k\|_{L^2}+\Big\|\widehat{\psi_k^{(1)}}\Big\|_{L^1}+\Big\|(\widehat{\psi_k^{(1)}})^{(1)}\Big\|_{L^1}\Big)\Bigg)\|f\|_{L^p}\|g\|_{L^{p'}},$$

where $\varphi_j^{(1)}(y_j)=d\varphi_j(y_j)/dy_j$, $\varphi_j^{(2)}(y_j)=d^2\varphi_j(y_j)/dy_j^2$ and $C>0$ is independent of $\varphi_1,\ldots,\varphi_n,\psi_1,\ldots,\psi_n\in\mathcal{S}(\mathbb{R})$ and $f,g\in\mathcal{S}(\mathbb{R}^n)$.

To prove Lemma 4.1, we use the following result.

Lemma 4.2. [9, Lemma 2.4] *Let $2\le p<\infty$ and $p'\le q\le p$. Then there exists a constant $C>0$ such that*

$$\left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^q |\xi|^{-n(1-q/p)}\, d\xi\right)^{1/q} \le C\|f\|_{L^{p'}}.$$

Proof of Lemma 4.1. By the Fourier inversion formula, we have

$$\begin{aligned}
W(\varphi,\psi,f,g)(x,\xi) &= \iint_{\mathbb{R}^{2n}} e^{-i(x\cdot y+\xi\cdot\eta)}\varphi(y)\psi(\eta)V(f,g)(y,\eta)\,dy\,d\eta \\
&= \iint_{\mathbb{R}^{2n}} e^{-i(x\cdot y+\xi\cdot\eta)}\varphi(y)\psi(\eta) \\
&\quad\times\left(\int_{\mathbb{R}^n} e^{iy\cdot t} f(\eta+t)\overline{\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{it\cdot\zeta}\hat{g}(\zeta)\,d\zeta\right)}\,dt\right)dy\,d\eta \\
&= \frac{1}{(2\pi)^n}\iiint_{\mathbb{R}^{3n}} e^{-i(x\cdot y+\xi\cdot\eta)}\varphi(y)\psi(\eta)\overline{\hat{g}(\zeta)} \\
&\quad\times\left(\int_{\mathbb{R}^n} e^{i(y-\zeta)\cdot(t-\eta)} f(t)\,dt\right)dy\,d\eta\,d\zeta \\
&= \frac{1}{(2\pi)^n}\iiint_{\mathbb{R}^{3n}} e^{-ix\cdot y}e^{i(y-\zeta)\cdot t}\varphi(y)f(t)\overline{\hat{g}(\zeta)} \\
&\quad\times\left(\int_{\mathbb{R}^n} e^{-i(y+\xi-\zeta)\cdot\eta}\psi(\eta)\,d\eta\right)dt\,dy\,d\zeta \\
&= \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}} e^{-i\zeta\cdot t} f(t)\overline{\hat{g}(\zeta)} \\
&\quad\times\left(\int_{\mathbb{R}^n} e^{i(t-x)\cdot y}\varphi(y)\hat{\psi}(y+\xi-\zeta)\,dy\right)dt\,d\zeta \\
&= \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}} e^{ix\cdot(\xi-\zeta)}e^{-it\cdot\xi} \\
&\quad\times\left(\int_{\mathbb{R}^n} e^{i(t-x)\cdot y}\varphi(y+\zeta-\xi)\hat{\psi}(y)\,dy\right)f(t)\overline{\hat{g}(\zeta)}\,dt\,d\zeta.
\end{aligned}$$

For the sake of simplicity, we only consider the case $n=2$ (the case $n\ne 2$ can be proved in the same way). This means that $\varphi(y)=\varphi_1(y_1)\varphi_2(y_2)$ and $\psi(\eta)=$

$\psi_1(\eta_1)\psi_2(\eta_2)$, where $y=(y_1,y_2), \eta=(\eta_1,\eta_2)\in\mathbb{R}^2$. By Taylor's formula,

$$\begin{aligned}\varphi(y+\zeta-\xi) &= \varphi_1(y_1+\zeta_1-\xi_1)\varphi_2(y_2+\zeta_2-\xi_2)\\ &= \left(\varphi_1(\zeta_1-\xi_1)+y_1\int_0^1 \varphi_1^{(1)}(\zeta_1-\xi_1+s_1y_1)\,ds_1\right)\\ &\quad\times\left(\varphi_2(\zeta_2-\xi_2)+y_2\int_0^1 \varphi_2^{(1)}(\zeta_2-\xi_2+s_2y_2)\,ds_2\right).\end{aligned}$$

Then

$$W(\varphi,\psi,f,g)(x,\xi)=\sum_{l_1=0}^{1}\sum_{l_2=0}^{1} H_{(l_1,l_2)}(x,\xi),$$

where

$$\begin{aligned}H_{(0,0)}(x,\xi) &= \left(\int_{\mathbb{R}^2} e^{-i(t-x)\cdot\xi}\psi(t-x)f(t)\,dt\right)\left(\int_{\mathbb{R}^2} e^{-ix\cdot\zeta}\varphi(\zeta-\xi)\overline{\widehat{g}(\zeta)}\,d\zeta\right),\\ H_{(1,0)}(x,\xi) &= \frac{1}{2\pi i}\int_0^1\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{ix\cdot(\xi-\zeta)}e^{-it\cdot\xi}\varphi_2(\zeta_2-\xi_2)\psi_2(t_2-x_2)\\ &\quad\times\left(\int_{\mathbb{R}} e^{i(t_1-x_1)y_1}\varphi_1^{(1)}(\zeta_1-\xi_1+s_1y_1)\widehat{\psi_1^{(1)}}(y_1)\,dy_1\right)f(t)\overline{\widehat{g}(\zeta)}\,dt\,d\zeta\,ds_1,\\ H_{(0,1)}(x,\xi) &= \frac{1}{2\pi i}\int_0^1\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{ix\cdot(\xi-\zeta)}e^{-it\cdot\xi}\varphi_1(\zeta_1-\xi_1)\psi_1(t_1-x_1)\\ &\quad\times\left(\int_{\mathbb{R}} e^{i(t_2-x_2)y_2}\varphi_2^{(1)}(\zeta_2-\xi_2+s_2y_2)\widehat{\psi_2^{(1)}}(y_2)\,dy_2\right)f(t)\overline{\widehat{g}(\zeta)}\,dt\,d\zeta\,ds_2,\\ H_{(1,1)}(x,\xi) &= \frac{1}{(2\pi i)^2}\int_{[0,1]^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{ix\cdot(\xi-\zeta)}e^{-it\cdot\xi}\\ &\quad\times\left(\int_{\mathbb{R}^2} e^{i(t-x)\cdot y}[\varphi_1^{(1)}\otimes\varphi_2^{(1)}](\zeta-\xi+sy)\big[\widehat{\psi_1^{(1)}}\otimes\widehat{\psi_2^{(1)}}\big](y)\,dy\right)\\ &\quad\times f(t)\overline{\widehat{g}(\zeta)}\,dt\,d\zeta\,ds,\end{aligned}$$

$s=(s_1,s_2)$ and $sy=(s_1y_1,s_2y_2)$.

Let us estimate $H_{(0,0)}$, and recall that $2\le p<\infty$. Since

$$\begin{aligned}H_{(0,0)}(x,\xi) &= \left(\int_{\mathbb{R}^2} e^{-it\cdot\xi}\psi(t)f(t+x)\,dt\right)\left(\int_{\mathbb{R}^2} e^{-iz\cdot\xi}\widehat{\varphi}(z)\overline{g(x-z)}\,dz\right)\\ &= \mathcal{F}[\psi f(\,\cdot\,+x)](\xi)\mathcal{F}[\widehat{\varphi}\overline{g(x-\,\cdot\,)}](\xi),\end{aligned}$$

it follows from Hölder's inequality, Plancherel's theorem, Lemma 4.2 with $q=2$ and Young's inequality that

$$
\begin{aligned}
(4.1)\quad &\iint_{\mathbb{R}^4} \langle\xi\rangle^m |H_{(0,0)}(x,\xi)|\,dx\,d\xi \\
&\le \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} |\mathcal{F}[\psi f(\,\cdot+x)](\xi)|^2\,d\xi\right)^{1/2}\left(\int_{\mathbb{R}^2} |\mathcal{F}[\widehat{\varphi}\overline{g(x-\,\cdot\,)}](\xi)|^2\langle\xi\rangle^{2m}\,d\xi\right)^{1/2} dx \\
&\le C\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} |\psi(t)f(t+x)|^2\,dt\right)^{1/2}\left(\int_{\mathbb{R}^2} |\widehat{\varphi}(z)g(x-z)|^{p'}\,dz\right)^{1/p'} dx \\
&\le C\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} |\psi(t)f(t+x)|^2\,dt\right)^{p/2} dx\right)^{1/p} \\
&\quad\times\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} |\widehat{\varphi}(z)g(x-z)|^{p'}\,dz\right)^{p'/p'} dx\right)^{1/p'} \\
&\le C\|\widehat{\varphi}\|_{L^{p'}}\|\psi\|_{L^2}\|f\|_{L^p}\|g\|_{L^{p'}} \\
&= C\|\widehat{\varphi}_1\|_{L^{p'}}\|\widehat{\varphi}_2\|_{L^{p'}}\|\hat{\psi}_1\|_{L^2}\|\hat{\psi}_2\|_{L^2}\|f\|_{L^p}\|g\|_{L^{p'}}.
\end{aligned}
$$

We next consider $H_{(1,0)}$ and $H_{(0,1)}$. Using

$$\frac{1}{1+i(t_1-x_1)}(1+\partial_{y_1})e^{i(t_1-x_1)y_1} = e^{i(t_1-x_1)y_1},$$

we have

$$
\begin{aligned}
H_{(1,0)}(x,\xi) &= \frac{1}{2\pi i}\sum_{\substack{j,k=0\\(j,k)\neq(1,1)}}^{1}(-1)^{j+k}\int_0^1\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{ix\cdot(\xi-\zeta)}e^{-it\cdot\xi}\varphi_2(\zeta_2-\xi_2)\psi_2(t_2-x_2) \\
&\quad\times\left(\int_{\mathbb{R}}\frac{e^{i(t_1-x_1)y_1}}{1+i(t_1-x_1)}s_1^j\varphi_1^{(1+j)}(\zeta_1-\xi_1+s_1y_1)\left(\widehat{\psi_1^{(1)}}\right)^{(k)}(y_1)\,dy_1\right) \\
&\quad\times f(t)\overline{\hat{g}(\zeta)}\,dt\,d\zeta\,ds_1 \\
&= \frac{e^{ix\cdot\xi}}{2\pi i}\sum_{\substack{j,k=0\\(j,k)\neq(1,1)}}^{1}(-1)^{j+k}\int_0^1 s_1^j\int_{\mathbb{R}} e^{-ix_1y_1}\left(\widehat{\psi_1^{(1)}}\right)^{(k)}(y_1) \\
&\quad\times\left(\int_{\mathbb{R}^2} e^{-it\cdot(\xi-y_1e_1)}\frac{\psi_2(t_2-x_2)}{1+i(t_1-x_1)}f(t)\,dt\right) \\
&\quad\times\left(\int_{\mathbb{R}^2} e^{-ix\cdot\zeta}\varphi_1^{(1+j)}(\zeta_1-\xi_1+s_1y_1)\varphi_2(\zeta_2-\xi_2)\overline{\hat{g}(\zeta)}\,d\zeta\right)dy_1\,ds_1,
\end{aligned}
$$

where $y_1e_1=(y_1,0)$. Hence, since

$$\int_{\mathbb{R}^2} e^{-ix\cdot\zeta}\varphi_1^{(1+j)}(\zeta_1-\xi_1+s_1y_1)\varphi_2(\zeta_2-\xi_2)\overline{\hat{g}(\zeta)}\,d\zeta = \int_{\mathbb{R}^2} e^{-iz\cdot\xi}e^{is_1y_1z_1}\widehat{\varphi_1^{(1+j)}}(z_1)\widehat{\varphi}_2(z_2)\overline{g(x-z)}\,dz,$$

we have by the same argument as in (4.1),

$$\begin{aligned}
(4.2)\quad & \iint_{\mathbb{R}^4} \langle\xi\rangle^m |H_{(1,0)}(x,\xi)|\,dx\,d\xi \\
&\le C \sum_{\substack{j,k=0 \\ (j,k)\neq(1,1)}}^{1} \int_0^1\int_{\mathbb{R}} \left|\left(\widehat{\psi_1^{(1)}}\right)^{(k)}(y_1)\right| \Bigg(\iint_{\mathbb{R}^4} \left|\int_{\mathbb{R}^2} e^{-it\cdot(\xi-y_1e_1)}\frac{\psi_2(t_2-x_2)}{1+i(t_1-x_1)} f(t)\,dt\right| \\
&\quad \times \left|\int_{\mathbb{R}^2} e^{-ix\cdot\zeta}\varphi_1^{(1+j)}(\zeta_1-\xi_1+s_1y_1)\varphi_2(\zeta_2-\xi_2)\overline{\hat{g}(\zeta)}\,d\zeta\right| \langle\xi\rangle^m\,dx\,d\xi\Bigg)\,dy_1\,ds_1 \\
&\le C \sum_{\substack{j,k=0 \\ (j,k)\neq(1,1)}}^{1} \int_0^1\int_{\mathbb{R}} \left|\left(\widehat{\psi_1^{(1)}}\right)^{(k)}(y_1)\right| \Bigg(\int_{\mathbb{R}^2} \\
&\quad \times \left(\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} e^{-it\cdot\xi}\frac{\psi_2(t_2-x_2)}{1+i(t_1-x_1)} f(t)\,dt\right|^2 d\xi\right)^{1/2} \\
&\quad \times \left(\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} e^{-iz\cdot\xi}e^{is_1y_1z_1}\widehat{\varphi_1^{(1+j)}}(z_1)\widehat{\varphi}_2(z_2)\overline{g(x-z)}\,dz\right|^2 \langle\xi\rangle^{2m} d\xi\right)^{1/2} dx\Bigg)\,dy_1\,ds_1 \\
&\le C \sum_{\substack{j,k=0 \\ (j,k)\neq(1,1)}}^{1} \left\|\widehat{\varphi_1^{(1+j)}}\right\|_{L^{p'}} \|\widehat{\varphi}_2\|_{L^{p'}} \left\|\left(\widehat{\psi_1^{(1)}}\right)^{(k)}\right\|_{L^1} \|\hat{\psi}_2\|_{L^2}\|f\|_{L^p}\|g\|_{L^{p'}}.
\end{aligned}$$

In the same way, we can prove that

$$\begin{aligned}
(4.3)\quad & \iint_{\mathbb{R}^4} \langle\xi\rangle^m |H_{(0,1)}(x,\xi)|\,dx\,d\xi \\
&\le C \sum_{\substack{j,k=0 \\ (j,k)\neq(1,1)}}^{1} \|\widehat{\varphi}_1\|_{L^{p'}} \left\|\widehat{\varphi_2^{(1+j)}}\right\|_{L^{p'}} \|\hat{\psi}_1\|_{L^2} \left\|\left(\widehat{\psi_2^{(1)}}\right)^{(k)}\right\|_{L^1} \|f\|_{L^p}\|g\|_{L^{p'}}.
\end{aligned}$$

Finally, we consider $H_{(1,1)}$. By using

$$\frac{1}{(1+i(t_1-x_1))(1+i(t_2-x_2))}(1+\partial_{y_1})(1+\partial_{y_2})e^{i(t-x)\cdot y} = e^{i(t-x)\cdot y},$$

we see that

$$
\begin{aligned}
H_{(1,1)}(x,\xi) &= \frac{1}{(2\pi i)^2} \sum_{\substack{j_1,j_2,k_1,k_2=0\\(j_1,k_1)\neq(1,1)\\(j_2,k_2)\neq(1,1)}}^{1} (-1)^{j_1+j_2+k_1+k_2} \int_{[0,1]^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{ix\cdot(\xi-\zeta)} e^{-it\cdot\xi} f(t)\overline{\hat{g}(\zeta)} \\
&\quad \times \bigg(\int_{\mathbb{R}^2} \frac{e^{i(t-x)\cdot y}}{(1+i(t_1-x_1))(1+i(t_2-x_2))} s_1^{j_1} s_2^{j_2} \\
&\quad \times [\varphi_1^{(1+j_1)} \otimes \varphi_2^{(1+j_2)}](\zeta-\xi+sy) \Big[\Big(\widehat{\psi_1^{(1)}}\Big)^{(k_1)} \otimes \Big(\widehat{\psi_2^{(1)}}\Big)^{(k_2)} \Big](y)\, dy \bigg)\, dt\, d\zeta\, ds \\
&= \frac{e^{ix\cdot\xi}}{(2\pi i)^2} \sum_{\substack{j_1,j_2,k_1,k_2=0\\(j_1,k_1)\neq(1,1)\\(j_2,k_2)\neq(1,1)}}^{1} (-1)^{j_1+j_2+k_1+k_2} \int_{[0,1]^2} s_1^{j_1} s_2^{j_2} \int_{\mathbb{R}^2} e^{-ix\cdot y} \\
&\quad \times \Big[\Big(\widehat{\psi_1^{(1)}}\Big)^{(k_1)} \otimes \Big(\widehat{\psi_2^{(1)}}\Big)^{(k_2)} \Big](y) \\
&\quad \times \bigg(\int_{\mathbb{R}^2} e^{-it\cdot(\xi-y)} \frac{f(t)}{(1+i(t_1-x_1))(1+i(t_2-x_2))}\, dt \bigg) \\
&\quad \times \bigg(\int_{\mathbb{R}^2} e^{-ix\cdot\zeta} [\varphi_1^{(1+j_1)} \otimes \varphi_2^{(1+j_2)}](\zeta-\xi+sy) \overline{\hat{g}(\zeta)}\, d\zeta \bigg)\, dy\, ds.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_{\mathbb{R}^2} e^{-ix\cdot\zeta} [\varphi_1^{(1+j_1)} \otimes \varphi_2^{(1+j_2)}](\zeta-\xi+sy) \overline{\hat{g}(\zeta)}\, d\zeta \\
&\qquad = \int_{\mathbb{R}^2} e^{-iz\cdot\xi} e^{isy\cdot z} \Big[\widehat{\varphi_1^{(1+j_1)}} \otimes \widehat{\varphi_2^{(1+j_2)}} \Big](z) \overline{g(x-z)}\, dz,
\end{aligned}
$$

by the same argument as in (4.1) and (4.2), we can prove that

$$
\begin{aligned}
(4.4)\quad & \iint_{\mathbb{R}^4} \langle\xi\rangle^m |H_{(1,1)}(x,\xi)|\, dx\, d\xi \\
&\quad \le C \|f\|_{L^p} \|g\|_{L^{p'}} \\
&\qquad \times \sum_{\substack{j_1,j_2,k_1,k_2=0\\(j_1,k_1)\neq(1,1)\\(j_2,k_2)\neq(1,1)}}^{1} \Big\| \widehat{\varphi_1^{(1+j_1)}} \Big\|_{L^{p'}} \Big\| \widehat{\varphi_2^{(1+j_2)}} \Big\|_{L^{p'}} \Big\| \Big(\widehat{\psi_1^{(1)}}\Big)^{(k_1)} \Big\|_{L^1} \Big\| \Big(\widehat{\psi_2^{(1)}}\Big)^{(k_2)} \Big\|_{L^1}.
\end{aligned}
$$

Therefore, by (4.1)–(4.4),

$$\iint_{\mathbb{R}^4} \langle\xi\rangle^m |W(\varphi,\psi,f,g)(x,\xi)|\,dx\,d\xi$$

$$\leq C\Bigg(\prod_{j=1}^{2}\Big(\|\widehat{\varphi}_j\|_{L^{p'}}+\Big\|\widehat{\varphi_j^{(1)}}\Big\|_{L^{p'}}+\Big\|\widehat{\varphi_j^{(2)}}\Big\|_{L^{p'}}\Big)\Bigg)$$

$$\times\Bigg(\prod_{k=1}^{2}\Big(\|\hat{\psi}_k\|_{L^2}+\Big\|\widehat{\psi_k^{(1)}}\Big\|_{L^1}+\Big\|\Big(\widehat{\psi_k^{(1)}}\Big)^{(1)}\Big\|_{L^1}\Big)\Bigg)\|f\|_{L^p}\|g\|_{L^{p'}},$$

and we obtain Lemma 4.1 with $n=2$. □

We are now ready to prove Theorem 1.2 with $p\geq 2$.

Proof of Theorem 1.2 with $p\geq 2$. Let $2\leq p<\infty$, $m=-n(1/2-1/p)$, and let $\sigma\in L^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ and $\varphi\in\mathcal{S}(\mathbb{R})$ be such that $\operatorname{supp}\hat{\sigma}\subset\prod_{j=1}^{2n}[-R_j,R_j]$, $\varphi=1$ on $[-2,2]$ and $\operatorname{supp}\varphi\subset[-4,4]$. We set $\varphi_j(y_j)=\varphi(y_j/R_j)$, $\psi_k(\eta_k)=\varphi(\eta_k/R_{n+k})$, $j,k=1,\dots,n$, $\varphi(y)=[\varphi_1\otimes\dots\otimes\varphi_n](y)$ and $\psi(\eta)=[\psi_1\otimes\dots\otimes\psi_n](\eta)$, and note that we have $\hat{\sigma}(y,\eta)=\varphi(y)\psi(\eta)\hat{\sigma}(y,\eta)$. Then, for $f,g\in\mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\mathbb{R}^n}\sigma(X,D)f(x)\overline{g(x)}\,dx &= \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}}\sigma(x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)\overline{g(x)}\,dx\,d\xi\\ &= \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}}\hat{\sigma}(y,\eta)\mathcal{F}^{-1}_{(x,\xi)\to(y,\eta)}[e^{ix\cdot\xi}\hat{f}(\xi)\overline{g(x)}]\,dy\,d\eta\\ &= \frac{1}{(2\pi)^{2n}}\iint_{\mathbb{R}^{2n}}\hat{\sigma}(y,\eta)\varphi(y)\psi(\eta)V(f,g)(y,\eta)\,dy\,d\eta\\ &= \frac{1}{(2\pi)^{2n}}\iint_{\mathbb{R}^{2n}}\sigma(x,\xi)W(\varphi,\psi,f,g)(x,\xi)\,dx\,d\xi,\end{aligned}$$

where $V(f,g)$ and $W(\varphi,\psi,f,g)$ are as in Lemma 4.1. Hence, by Lemma 4.1,

$$\left|\int_{\mathbb{R}^n}\sigma(X,D)f(x)\overline{g(x)}\,dx\right|$$

$$\leq \|\sigma(x,\xi)\langle\xi\rangle^{-m}\|_{L^\infty_{x,\xi}}\iint_{\mathbb{R}^{2n}}\langle\xi\rangle^m|W(\varphi,\psi,f,g)(x,\xi)|\,dx\,d\xi$$

$$\leq C\|\sigma(x,\xi)\langle\xi\rangle^{-m}\|_{L^\infty_{x,\xi}}\Bigg(\prod_{j=1}^{n}\Big(\|\widehat{\varphi}_j\|_{L^{p'}}+\Big\|\widehat{\varphi_j^{(1)}}\Big\|_{L^{p'}}+\Big\|\widehat{\varphi_j^{(2)}}\Big\|_{L^{p'}}\Big)\Bigg)$$

$$\times\Bigg(\prod_{k=1}^{n}\Big(\|\hat{\psi}_k\|_{L^2}+\Big\|\widehat{\psi_k^{(1)}}\Big\|_{L^1}+\Big\|\Big(\widehat{\psi_k^{(1)}}\Big)^{(1)}\Big\|_{L^1}\Big)\Bigg)\|f\|_{L^p}\|g\|_{L^{p'}}.$$

On the other hand,

$$\|\widehat{\varphi}_j\|_{L^{p'}} = R_j^{1/p}\|\widehat{\varphi}\|_{L^{p'}}, \qquad \|\hat{\psi}_k\|_{L^2} = R_{n+k}^{1/2}\|\widehat{\varphi}\|_{L^2},$$

$$\Big\|\widehat{\varphi_j^{(1)}}\Big\|_{L^{p'}} = R_j^{-1/p'}\|\widehat{\varphi^{(1)}}\|_{L^{p'}}, \qquad \Big\|\widehat{\psi_k^{(1)}}\Big\|_{L^1} = R_{n+k}^{-1}\|\widehat{\varphi^{(1)}}\|_{L^1},$$

$$\Big\|\widehat{\varphi_j^{(2)}}\Big\|_{L^{p'}} = R_j^{-(1+1/p')}\|\widehat{\varphi^{(2)}}\|_{L^{p'}}, \qquad \Big\|\Big(\widehat{\psi_k^{(1)}}\Big)^{(1)}\Big\|_{L^1} = \|(\widehat{\varphi^{(1)}})^{(1)}\|_{L^1}$$

for $1 \le j,k \le n$. Therefore, noting that $R_j \ge 1$, $j=1,...,2n$, we obtain

$$\left|\int_{\mathbb{R}^n} \sigma(X,D)f(x)\overline{g(x)}\,dx\right| \le C\|f\|_{L^p}\|g\|_{L^{p'}}(R_1...R_n)^{1/p}(R_{n+1}...R_{2n})^{1/2}\|\sigma(x,\xi)\langle\xi\rangle^{-m}\|_{L^\infty_{x,\xi}}$$

for all $f,g \in \mathcal{S}(\mathbb{R}^n)$. This completes the proof. □

5. Application of Theorem 1.2 to function spaces

We first give the following remark.

Remark 5.1. It follows from Theorem 1.2 that there exists a constant $C>0$ such that

$$\|\sigma(X,D)f\|_{L^p} \le C(R_1...R_n)^{\min\{1/p,1/2\}}(R_{n+1}...R_{2n})^{\max\{1/p,1/2\}} \times\|\sigma(x,\xi)\langle\xi\rangle^{n|1/p-1/2|}\|_{L^\infty_{x,\xi}}\|f\|_{H^p} \tag{5.1}$$

for all $\sigma(x,\xi)$ with $\operatorname{supp}\hat{\sigma} \subset (y_0,\eta_0)+\prod_{j=1}^{2n}[-R_j,R_j]$, $R_j \ge 1$, $j=1,...,2n$, and $f \in H^p(\mathbb{R}^n)$, where $C>0$ is independent of $(y_0,\eta_0) \in \mathbb{R}^n \times \mathbb{R}^n$ and $R_j \ge 1$, $j=1,...,2n$. This can be proved as follows: Let

$$T_{(y_0,\eta_0)}\sigma(x,\xi) = \sigma(x-y_0,\xi-\eta_0) \quad \text{and} \quad M_{(y_0,\eta_0)}\sigma(x,\xi) = e^{i(y_0\cdot x+\eta_0\cdot\xi)}\sigma(x,\xi).$$

Since

$$\mathcal{F}_{1,2}[T_{(\eta_0,0)}M_{-(y_0,\eta_0)}\sigma](y,\eta) = e^{-i\eta_0\cdot y}\hat{\sigma}(y+y_0,\eta+\eta_0),$$

if $\operatorname{supp}\hat{\sigma} \subset (y_0,\eta_0)+\prod_{j=1}^{2n}[-R_j,R_j]$ then $\operatorname{supp}\mathcal{F}[T_{(\eta_0,0)}M_{-(y_0,\eta_0)}\sigma] \subset \prod_{j=1}^{2n}[-R_j,R_j]$. On the other hand,

$$\begin{aligned}\sigma(X,D)f(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)\,d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{i(y_0\cdot x+\eta_0\cdot\xi)}(M_{-(y_0,\eta_0)}\sigma)(x,\xi)\hat{f}(\xi)\,d\xi\end{aligned}$$

$$= \frac{e^{iy_0\cdot x}}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x+\eta_0)\cdot\xi}(M_{-(y_0,\eta_0)}\sigma)(x+\eta_0-\eta_0,\xi)\hat{f}(\xi)\,d\xi$$
$$= \frac{e^{iy_0\cdot x}}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x+\eta_0)\cdot\xi}(T_{(\eta_0,0)}M_{-(y_0,\eta_0)}\sigma)(x+\eta_0,\xi)\hat{f}(\xi)\,d\xi$$
$$= e^{iy_0\cdot x}(T_{(\eta_0,0)}M_{-(y_0,\eta_0)}\sigma)(X,D)f(x+\eta_0).$$

Hence, by Theorem 1.2,

$$\begin{aligned}\|\sigma(X,D)f\|_{L^p} &= \|(T_{(\eta_0,0)}M_{-(y_0,\eta_0)}\sigma)(X,D)f\|_{L^p}\\ &\le C(R_1\ldots R_n)^{\min\{1/p,1/2\}}(R_{n+1}\ldots R_{2n})^{\max\{1/p,1/2\}}\\ &\quad\times\|(T_{(\eta_0,0)}M_{-(y_0,\eta_0)}\sigma)(x,\xi)\langle\xi\rangle^{n|1/p-1/2|}\|_{L^\infty_{x,\xi}}\|f\|_{H^p}\\ &= C(R_1\ldots R_n)^{\min\{1/p,1/2\}}(R_{n+1}\ldots R_{2n})^{\max\{1/p,1/2\}}\\ &\quad\times\|\sigma(x,\xi)\langle\xi\rangle^{n|1/p-1/2|}\|_{L^\infty_{x,\xi}}\|f\|_{H^p}\end{aligned}$$

for all $\sigma(x,\xi)$ with $\operatorname{supp}\hat{\sigma}\subset(y_0,\eta_0)+\prod_{j=1}^{2n}[-R_j,R_j]$, $R_j\ge 1$, $j=1,\ldots,2n$, and $f\in H^p(\mathbb{R}^n)$.

Corollary 5.2. *Let* $0<p<\infty$ *and* $m(p)=n|1/p-1/2|$. *If* $\sigma\in\mathcal{S}'(\mathbb{R}^{2n})$ *satisfies one of the following conditions*:

(1) $\sigma\in B^{\infty,\min\{p,1\}}_{(\min\{1/p,1/2\}n,\max\{1/p,1/2\}n),m(p)}(\mathbb{R}^n\times\mathbb{R}^n)$,

(2) $\sigma\in B^{\infty,\min\{p,1\}}_{(s_1,\ldots,s_n,s_{n+1},\ldots,s_{2n}),m(p)}(\mathbb{R}\times\ldots\times\mathbb{R})$, $s_1,\ldots,s_n=\min\{1/p,1/2\}$, $s_{n+1},\ldots,s_{2n}=\max\{1/p,1/2\}$,

(3) $\sigma\in M^{\infty,\min\{p,1\}}_{m(p)}(\mathbb{R}^n\times\mathbb{R}^n)$,

then $\sigma(X,D)$ *is bounded from* $H^p(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$.

Proof. It is enough to prove Corollary 5.2 under the assumption (2) or (3), since

$$B^{\infty,\min\{p,1\}}_{(\min\{1/p,1/2\}n,\max\{1/p,1/2\}n),m(p)}(\mathbb{R}^n\times\mathbb{R}^n)$$
$$\subset B^{\infty,\min\{p,1\}}_{(\min\{1/p,1/2\},\ldots,\min\{1/p,1/2\},\max\{1/p,1/2\},\ldots,\max\{1/p,1/2\}),m(p)}(\mathbb{R}\times\ldots\times\mathbb{R})$$

(see [1, Appendix A4(ii)] and [16, Theorem 1.3.9]).

(2) We consider the case $p\le 1$. Let $\{\psi_{j_1}\}_{j_1\ge 0},\dots,\{\psi_{k_n}\}_{k_n\ge 0}$ be as in (2.2) with $N=1$. Since

$$\operatorname{supp}\mathcal{F}[\psi_{j_1}(D_{x_1})\dots\psi_{j_n}(D_{x_n})\psi_{k_1}(D_{\xi_1})\dots\psi_{k_n}(D_{\xi_n})\sigma] \subset \left(\prod_{l=1}^{n}[-2^{j_l+1},2^{j_l+1}]\right)\times\left(\prod_{l=1}^{n}[-2^{k_l+1},2^{k_l+1}]\right),$$

we have by the partition of unity and Theorem 1.2

$$\begin{aligned}
&\|\sigma(X,D)f\|_{L^p}^p \\
&\quad\le \sum_{j_1,\dots,j_n=0}^{\infty}\sum_{k_1,\dots,k_n=0}^{\infty}\|\psi_{j_1}(D_{x_1})\dots\psi_{j_n}(D_{x_n})\psi_{k_1}(D_{\xi_1})\dots\psi_{k_n}(D_{\xi_n})\sigma(X,D)f\|_{L^p}^p \\
&\quad\le C\sum_{j_1,\dots,j_n=0}^{\infty}\sum_{k_1,\dots,k_n=0}^{\infty}(2^{j_1+1}\dots2^{j_n+1})^{p/2}(2^{k_1+1}\dots2^{k_n+1})^{p/p} \\
&\qquad\times\|\psi_{j_1}(D_{x_1})\dots\psi_{j_n}(D_{x_n})\psi_{k_1}(D_{\xi_1})\dots\psi_{k_n}(D_{\xi_n})\sigma(x,\xi)\langle\xi\rangle^{m(p)}\|_{L^\infty_{x,\xi}}^p\|f\|_{H^p}^p \\
&\quad= C\|\sigma\|^p_{B^{\infty,p}_{(1/2,\dots,1/2,1/p,\dots,1/p),m(p)}}\|f\|_{H^p}^p.
\end{aligned}$$

In the same way, we can give a proof for the case $p>1$.

(3) We consider the case $p>1$. Let $\{\varphi(\cdot-k)\}_{k\in\mathbb{Z}^n},\{\varphi(\cdot-l)\}_{l\in\mathbb{Z}^n}$ be as in (2.3). Since

$$\operatorname{supp}\mathcal{F}[\varphi(D_x-k)\varphi(D_\xi-l)\sigma]\subset(k,l)+[-1,1]^n,$$

we have by the partition of unity and Remark 5.1,

$$\begin{aligned}
\|\sigma(X,D)f\|_{L^p} &\le \sum_{k\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n}\|\varphi(D_x-k)\varphi(D_\xi-l)\sigma(X,D)f\|_{L^p} \\
&\le C\sum_{k\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n}\|\varphi(D_x-k)\varphi(D_\xi-l)\sigma(x,\xi)\langle\xi\rangle^{m(p)}\|_{L^\infty_{x,\xi}}\|f\|_{L^p} \\
&= C\|\sigma\|_{M^{\infty,1}_{m(p)}}\|f\|_{L^p}.
\end{aligned}$$

In the same way, we can prove the case $p\le 1$. □

We end this paper by giving the following remark on α-modulation spaces.

Remark 5.3. The α-modulation spaces $M^{p,q}_{s,\alpha}$, a parameterized family of function spaces, were introduced by Gröbner [7] (see also Feichtinger–Gröbner [6]). It is known that they include Besov spaces $B^{p,q}_s$ and modulation spaces $M^{p,q}_s$ as special cases corresponding to $\alpha=1$ and $\alpha=0$. As a corollary of Theorem 1.2, we can obtain a more general result in terms of α-modulation spaces which is an extension of the result of L^2-boundedness in Kobayashi–Sugimoto–Tomita [10].

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