Ark. Mat., **48** (2010), 395–403
DOI: 10.1007/s11512-009-0113-5

On the completeness of certain kernel-defined semi-inner product spaces

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Abstract. Let X be a compact Hausdorff space. A kernel function on $X\times X$, enjoying additional properties, naturally defines a semi-inner product structure on certain subspaces of all finite signed Borel measures on X. This paper discusses the question of completeness of such spaces.

1. Introduction and notation

Let X be a compact Hausdorff space. We denote by $C(X)$ the Banach space of all real-valued continuous functions on X equipped with the usual supremum norm. Further, $\mathcal{M}(X)$ denotes the linear space of all finite signed Borel measures on X and the usual measure norm of μ in $\mathcal{M}(X)$ is given by $|\mu|$. $\mathcal{M}_0(X)$ is defined as the linear subspace of $\mathcal{M}(X)$ consisting of all measures of total mass 0. For x in X, we denote by δ_x the point measure at x.

Now consider a kernel function k in $C(X\times X)$. For μ and ν in $\mathcal{M}(X)$ we set

$$I_k(\mu,\nu)=\int_X\int_X k(x,y)\,d\mu(x)\,d\nu(y)$$

and

$$I_k(\mu)=I_k(\mu,\mu).$$

Further we define the linear mapping

$$T_k\colon \mathcal{M}(X)\longrightarrow C(X),$$
$$T_k(\mu)=k^\mu,\quad \mu \text{ in } \mathcal{M}(X),$$

where k^μ (the *potential* of μ with respect to k) is given by

$$k^\mu(x)=\int_X k(x,y)\,d\mu(y),\quad x \text{ in } X.$$

Recall that the kernel function k is called *symmetric*, if

$$k(x,y)=k(y,x) \quad \text{for all } x \text{ and } y \text{ in } X.$$

Further the kernel function k is of (*strictly*) *positive type* for a linear subspace L of $\mathcal{M}(X)$, if

$$I_k(\mu)\geq 0 \quad \text{for all } \mu \text{ in } L,$$
$$I_k(\mu)>0 \quad \text{for all } \mu\neq 0 \text{ in } L,$$

respectively.

It is straightforward and well known, that a symmetric kernel function k in $C(X\times X)$, which is of positive type for a linear subspace L of $\mathcal{M}(X)$, defines a semi-inner product space, we call it $E_k(L)$, given by

$$\begin{aligned} E_k(L) &= (L, \|\cdot\|), \\ (\mu\,|\,\nu) &= I_k(\mu,\nu), \qquad \mu,\nu \text{ in } L, \\ \|\mu\|^2 &= (\mu\,|\,\mu), \qquad \mu \text{ in } L. \end{aligned}$$

Note that $E_k(L)$ is an inner product space if and only if the kernel function k is of strictly positive type for L.

This paper deals with the question of completeness of the (semi)-inner product space $E_k(L)$ for certain subspaces L of $\mathcal{M}(X)$.

Questions of that type naturally appear in classical potential theory in a slightly different setting: for example, recall that for the *Riesz kernels* $k_\alpha\colon \mathbb{R}^n\to(-\infty,\infty]$ on the euclidean space $(\mathbb{R}^n,\|\cdot\|)$ defined by

$$k_\alpha(x,y)=\|x-y\|^{\alpha-n}, \quad x,y \text{ in } \mathbb{R}^n,\ 1<\alpha<n,$$

the inner product space E_α of all signed Borel measures μ (finite on compact subsets of $\mathbb{R}^n$) with finite energy is not complete. The inner product of the measures μ and ν in E_α is given by

$$(\mu\,|\,\nu)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k_\alpha(x,y)\,d\mu(x)\,d\nu(y)$$

and the energy of μ is given by $(\mu|\mu)$. For a proof of this result see Theorem 1.19 in [3].

Furthermore the above defined spaces $E_k(L)$ play a key role in the analysis of certain compact metric spaces: Let (X,d) be a compact metric space. Recall that the space (X,d) is called (*strictly*) *quasihypermetric* (or of (*strictly*) *negative type*) if the kernel function $k=-d$ is of (strictly) positive type for $\mathcal{M}_0(X)$. For examples and discussion of such spaces see e.g. [2], [4], [6], [7], [5] and [8].

In [4] (Theorem 6.1) it is shown, that the semi-inner product space $E_{-d}(\mathcal{M}_0(X))$ is complete if and only if X is finite under the additional assumption, that

$$\sup_{\substack{\mu\in\mathcal{M}(X)\\ \mu(X)=1}} \int_X\int_X d(x,y)\,d\mu(x)\,d\mu(y) < \infty.$$

Theorem 2.8 of this paper will prove this result without the above given additional assumption.

2. The results

We first need the following simple result.

Lemma 2.1. *Let X be a compact Hausdorff space and k be a function in $C(X\times X)$. Then*

(1) *the linear mapping*

$$T_k\colon \mathcal{M}(X) \longrightarrow C(X),$$
$$T_k(\mu) = k^{\mu}, \quad \mu \text{ in } \mathcal{M}(X),$$

is bounded, where $\mathcal{M}(X)$ is equipped with the usual measure norm;

(2) *for $x, x_1, x_2, \ldots, x_n$ in X set*

$$\alpha = \max_{\substack{i\geq 1\\ j\leq n}} |k(x_i,x_j)-k(x,x)|.$$

Then for all $\alpha_1,\alpha_2,\ldots,\alpha_n$ in $\mathbb{R}$ we have

$$\sum_{i,j=1}^{n} \alpha_i\alpha_j k(x_i,x_j) \leq \alpha\left(\sum_{i=1}^{n}|\alpha_i|\right)^2 + k(x,x)\left(\sum_{i=1}^{n}\alpha_i\right)^2.$$

Proof. (1) Just use the boundedness of k and the Hahn–Jordan decomposition of measures μ in $\mathcal{M}(X)$.

(2) This is straightforward by the triangle inequality. $\square$

Theorem 2.2. *Let X be a compact Hausdorff space and f be a function in $C(X)$. Let k be a symmetric kernel function in $C(X\times X)$, which is of strictly positive type for the linear subspace L of $\mathcal{M}(X)$ given by $L=\{\mu\in\mathcal{M}(X)\,|\,\mu(f)=0\}$. Then*

$$E_k(L) \text{ is complete} \quad \text{if and only if} \quad X \text{ is finite.}$$

Proof. If X is finite, then of course $E_k(L)$ is complete. So let us assume that there exists some infinite compact Hausdorff space X with the property that $E_k(L)$ is complete.

By part (1) of Lemma 2.1, we can find some $A>0$, such that

$$\|k^\mu\|_\infty \le A|\mu| \quad \text{for all } \mu \text{ in } \mathcal{M}(X).$$

Hence for all μ in L we get

$$\|\mu\|^2 = I_k(\mu) = \mu(k^\mu) \le |\mu|\,\|k^\mu\|_\infty \le A|\mu|^2,$$

and so

$$\|\mu\| \le A^{1/2}|\mu| \quad \text{for all } \mu \text{ in } L.$$

By assumption $E_k(L)=(L,\|\cdot\|)$ is a Banach space and since $(L,|\cdot|)$ is a closed linear subspace of $(\mathcal{M}(X),|\cdot|)$ and hence a Banach space as well, an application of the open mapping theorem implies the existence of some constant $B>0$, such that

$$|\mu| \le B\|\mu\| \quad \text{for all } \mu \text{ in } L.$$

Since X is compact and infinite we can find some non-isolated point x in X. Choose some $\alpha>0$, such that

$$\max\Big(\|f\|_\infty, \sup_{z\in X}|k(z,z)|\Big) \le \alpha$$

and consider the following two cases:

(1) $f(x)\neq 0$.

Choose some $\varepsilon>0$, such that

$$4\alpha^2\varepsilon+\alpha\varepsilon^2 < \frac{|f(x)|^2}{B^2}.$$

Since k and f are continuous functions we can find some neighbourhood $U(x)$ of x such that $|k(x,x)-k(u,v)|<\varepsilon$ and $|f(x)-f(u)|<\varepsilon$ for all u and v in $U(x)$. Since x is non-isolated, there is some $y\neq x$ in $U(x)$. Now define the measure $\nu= f(x)\delta_y - f(y)\delta_x$. Of course $\nu(f)=0$ and hence ν is in L. Moreover

$$\|\nu\|^2 = I_k(\nu) \le \varepsilon\,(|f(x)|+|f(y)|)^2 + \alpha\,(f(x)-f(y))^2$$

by part (2) of Lemma 2.1, and so

$$\|\nu\|^2 \le 4\alpha^2\varepsilon+\alpha\varepsilon^2.$$

But

$$|f(x)|^2 \le (|f(x)|+|f(y)|)^2 = |\nu|^2 \le B^2\|\nu\|^2 \le B^2(4\alpha^2\varepsilon+\alpha\varepsilon^2) < |f(x)|^2,$$

a contradiction.

(2) $f(x)=0$.

Choose some $\varepsilon>0$, such that

$$\max(4\varepsilon, \varepsilon(2+\varepsilon)^2+\alpha\varepsilon^2) < \frac{4}{B^2}.$$

Since k is continuous we can find some neighbourhood $V(x)$ of x, such that

$$|k(x,x)-k(u,v)| < \varepsilon \quad \text{for all } u \text{ and } v \text{ in } V(x).$$

Assume first that there exists some

$$y \neq x \text{ in } V(x) \quad \text{such that} \quad f(y)=0.$$

For $\nu=\delta_x-\delta_y$ we get that ν is in L and by part (2) of Lemma 2.1, $\|\nu\|^2\leq 4\varepsilon$ and hence

$$4=|\nu|^2 \leq B^2\|\nu\|^2 \leq 4\varepsilon B^2 < 4,$$

a contradiction.

Second we may assume, that

$$f(u) \neq 0 \quad \text{for all } u \neq x \text{ in } V(x).$$

Since x is non-isolated we can find some $y\neq x$ in $V(x)$ (note that $f(y)\neq 0$). By the continuity of f and the Hausdorff property, there exists some neighbourhood $W(x)$ of x such that

$$W(x) \subseteq V(x), \quad y \notin W(x)$$

and

$$|f(u)| = |f(x)-f(u)| < \varepsilon|f(y)| \quad \text{for all } u \text{ in } W(x).$$

Again since x is non-isolated we can find some $z\neq x$ in $W(x)$. Now define

$$\nu = \delta_x - \delta_z + \frac{f(z)}{f(y)}\delta_y$$

and note that ν is in L and

$$\|\nu\|^2 \leq \varepsilon\left(2+\frac{|f(z)|}{|f(y)|}\right)^2 + \alpha\left(\frac{f(z)}{f(y)}\right)^2,$$

by part (2) of Lemma 2.1. So $\|\nu\|^2\leq\varepsilon(2+\varepsilon)^2+\alpha\varepsilon^2$ and therefore

$$4 \leq \left(2+\frac{|f(z)|}{|f(y)|}\right)^2 = |\nu|^2 \leq B^2\|\nu\|^2 \leq B^2(\varepsilon(2+\varepsilon)^2+\alpha\varepsilon^2) < 4,$$

a contradiction. □

Letting f be a constant function on X we obtain the following result.

Corollary 2.3. *Let X be a compact Hausdorff space and k be a symmetric kernel function in $C(X\times X)$, which is of strictly positive type for $\mathcal{M}_0(X)$ resp. $\mathcal{M}(X)$. Then $E_k(\mathcal{M}_0(X))$ resp. $E_k(\mathcal{M}(X))$ is complete if and only if X is finite.*

Before discussing the case when the kernel function is of positive type, but not necessarily of strictly positive type for certain subspaces of $\mathcal{M}(X)$, we recall the following basic remark.

Remark 2.4. Let X be a compact Hausdorff space and k be a symmetric kernel function in $C(X\times X)$ of positive type for some linear subspace L of $\mathcal{M}(X)$. Further let $F=\{\mu\in L\,|\,\|\mu\|=0\}$. Then

(1) F is a linear subspace of L;

(2) $F=\{\mu\in L\,|\,(\mu|\nu)=0$ for all ν in $L\}$;

(3) $E_k(L)|_F$ is an inner product space, where the inner product is given by

$$(\mu+F\,|\,\nu+F)=(\mu\,|\,\nu)\quad \text{for } \mu \text{ and } \nu \text{ in } L;$$

(4) the (semi)-inner product space $E_k(L)$ is complete if and only if the inner product space $E_k(L)|_F$ is complete;

(5) for $L=\mathcal{M}_0(X)$ we have

$$F=\{\mu\in\mathcal{M}_0(X)\,|\,k^\mu \text{ is constant on } X\}.$$

Proof. Parts (1)–(4) are well known. To show part (5) consider some μ in F and let x and y be in X. By part (2) we get

$$k^\mu(x)-k^\mu(y)=(\mu\,|\,\delta_x-\delta_y)=0$$

and so k^μ is constant on X. Conversely if k^μ is constant on X for some μ in $\mathcal{M}_0(X)$ we obtain $\|\mu\|^2=I_k(\mu)=\mu(k^\mu)=0$ and hence μ is in F. □

Now we note, that the assertion of Theorem 2.2 is not true in general, if we consider kernel functions of positive type, but not of strictly positive type.

Remark 2.5. For $n\geq 2$, let S^{n-1} denote the euclidean unit sphere in $\mathbb{R}^n$, let X be a compact subset of S^{n-1} and let $k(x,y)=-\|x-y\|^2$ for x and y in X, where $\|\cdot\|$ denotes the euclidean norm on $\mathbb{R}^n$. Routine calculations show, that for a given measure $\mu\in\mathcal{M}_0(X)$ we have

$$I_k(\mu)=2\sum_{i=1}^{n}\left(\int_X (x\,|\,e_i)\,d\mu(x)\right)^2,$$

where $e_1, e_2, ..., e_n$ are the canonical unit vectors in $\mathbb{R}^n$. This implies, that k is of positive type for $\mathcal{M}_0(X)$. As shown in Remark 2.3 in [4] one can easily see, that

$$\dim T_k(\mathcal{M}(X)) \le n+1 < \infty$$

(to be precise, the operator under consideration in Remark 2.3 in [4] is $-T_k$ instead of T_k). If T_k^0 denotes the restriction of T_k to $\mathcal{M}_0(X)$, we get

$$\dim(\mathcal{M}_0(X)|_F) \le \dim\big(\mathcal{M}_0(X)|_{\operatorname{kern} T_k^0}\big) = \dim T_k^0(\mathcal{M}_0(X)) \le \dim T_k(\mathcal{M}(X)) < \infty,$$

by part (5) of Remark 2.4.

Hence the inner product space $E_k(\mathcal{M}_0(X))|_F$ is of finite dimension and therefore complete. Summing up (recall part (4) of Remark 2.4) we have that $E_k(\mathcal{M}_0(X))$ is complete for every compact subset X of S^{n-1}, no matter if X is finite or infinite.

Note that the reason for the completeness of $E_k(\mathcal{M}_0(X))$ in the above given remark was the fact that $T_k(\mathcal{M}(X))$ is of finite dimension even for infinite X. This cannot happen for kernels defined by a metric because of the following result given in [4] (Theorem 2.1).

Theorem 2.6. *Let (X, d) be a compact metric space. Let*

$$T : \mathcal{M}(X) \longrightarrow C(X)$$
$$T(\mu)(x) = \int_X d(x,y)\, d\mu(y), \quad x \text{ in } X,\ \mu \text{ in } \mathcal{M}(X).$$

($T = T_d$ in our notation.) Then $T(\mathcal{M}(X))$ is of finite dimension if and only if X is finite.

Remark 2.7. Compare Theorem 2.6 to the comments given in Remark 2.5: it can happen that $-k$ (where $k(x,y) = -\|x-y\|^2$ for x and y in $X \subseteq S^{n-1}$) defines a metric on certain subsets X of S^{n-1}, but a theorem of Danzer and Grünbaum (see [1]) tells us that this can only be the case if $|X| \le 2^n < \infty$.

In light of Theorem 2.6, Remarks 2.5 and 2.7 we now prove the following result, which was shown in [4] (Theorem 6.1) under the additional assumption that

$$\sup_{\substack{\mu \in \mathcal{M}(X) \\ \mu(X)=1}} \int_X \int_X d(x,y)\, d\mu(x)\, d\mu(y) < \infty.$$

Recall from Section 1 that a compact metric space (X, d) is quasihypermetric if the corresponding kernel function $k = -d$ is of positive type for $\mathcal{M}_0(X)$.

Theorem 2.8. *Let (X,d) be a compact quasihypermetric space. Then the (semi)-inner product space $E_{-d}(\mathcal{M}_0(X))$ is complete if and only if X is finite.*

Proof. It is enough to show that the completeness of $E_{-d}(\mathcal{M}_0(X))$ implies that X is finite. To use our usual notation, let $k{=}-d$ on $X\times X$. By part (1) of Lemma 2.1, we can find some $A>0$ such that

$$\|k^{\mu}\|_{\infty}\leq A|\mu| \quad \text{for all } \mu\in\mathcal{M}(X).$$

As in the proof of Theorem 2.2 it follows that $\|\mu\|\leq A^{1/2}|\mu|$ for all μ in $\mathcal{M}_0(X)$. Let $F{=}\{\mu{\in}\mathcal{M}_0(X)\,|\,\|\mu\|{=}0\}$. Part (5) of Remark 2.4 tells us that F is given by $F{=}\{\mu{\in}\mathcal{M}_0(X)\,|\,k^{\mu}$ is constant on $X\}$.

By part (1) of Lemma 2.1, we know that F is a closed subspace of $\mathcal{M}_0(X)$, where $\mathcal{M}_0(X)$ is equipped with the usual measure norm and hence $\mathcal{M}_0(X)|_F$ (equipped with the usual factor norm) is a Banach space. Now let μ be in $\mathcal{M}_0(X)$ and ν be in F, then $\|\mu\|{=}\|\mu{+}\nu\|\leq A^{1/2}|\mu{+}\nu|$ and therefore $\|\mu{+}F\|\leq A^{1/2}|\mu{+}F|$ for all $\mu{+}F\in\mathcal{M}_0(X)|_F$.

By assumption (recall part (4) of Remark 2.4) $E_k(\mathcal{M}_0(X))|_F$ is a Banach space and so again applying the open mapping theorem, we can find some $B>0$ such that

$$|\mu{+}F|\leq B\|\mu{+}F\| \quad \text{for all } \mu \text{ in } \mathcal{M}_0(X).$$

Now fix some x and y in $X, x{\neq}y$: Of course $|\delta_x-\delta_y+F|\leq 2$. For arbitrary ν in F we get

$$|\delta_x-\delta_y+\nu| = \sup_{\substack{f\in C(X)\\ \|f\|_{\infty}=1}} |f(x)-f(y)+\nu(f)| \geq |f_0(x)-f_0(y)+\nu(f_0)|,$$

where f_0 in $C(X)$ is given by

$$f_0(z)=\frac{d(x,z)-d(y,z)}{d(x,y)}, \quad z \text{ in } X.$$

(Note that $\|f_0\|_{\infty}{=}1$, by the triangle inequality of the metric d.) As mentioned above, ν in F implies that $k^{\nu}{=}(-d)^{\nu}$ is a constant function on X. Hence

$$\nu(f_0)=\frac{-k^{\nu}(x)+k^{\nu}(y)}{d(x,y)}=0.$$

Therefore

$$|\delta_x-\delta_y+\nu|\geq|f_0(x)-f_0(y)|=2.$$

Summing up we get

$$|\delta_x-\delta_y+F|=2.$$

On the other hand

$$\|\delta_x - \delta_y + F\|^2 = \|\delta_x - \delta_y\|^2 = 2d(x,y)$$

and hence

$$\|\delta_x - \delta_y + F\| = (2d(x,y))^{1/2}.$$

Now it follows, that

$$2 = |\delta_x - \delta_y + F| \le B\|\delta_x - \delta_y + F\| = B(2d(x,y))^{1/2}$$

and hence $d(x,y) \ge 2/B^2$. Since $x \ne y$ were chosen arbitrarily, the compactness of X implies that X is finite. □

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Received January 29, 2009
published online December 9, 2009