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A sharp lower bound for the log canonical threshold

by

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1. Notation and main results

Here we put $d^c=(i/2\pi)(\bar\partial-\partial)$, so that $dd^c=(i/\pi)\partial\bar\partial$. The normalization of the d^c operator is chosen so that we have precisely $(dd^c\log|z|)^n=\delta_0$ for the Monge–Ampère operator in $\mathbb{C}^n$. The Monge–Ampère operator is defined on locally bounded plurisubharmonic functions according to the definition of Bedford–Taylor [BT1], [BT2]; it can also be extended to plurisubharmonic functions with isolated or compactly supported poles by [D3]. If Ω is an open subset of $\mathbb{C}^n$, we let $\mathrm{PSH}(\Omega)$ (resp. $\mathrm{PSH}^-(\Omega)$) be the set of plurisubharmonic (resp. psh$\leqslant 0$) functions on Ω.

Definition 1.1. Let Ω be a bounded hyperconvex domain (i.e. a domain possessing a negative psh exhaustion). Following Cegrell [Ce], we introduce certain classes of psh functions on Ω, in relation with the definition of the Monge–Ampère operator:

$$\mathcal{E}_0(\Omega)=\left\{\varphi\in\mathrm{PSH}^-(\Omega): \lim_{z\to\partial\Omega}\varphi(z)=0 \text{ and } \int_\Omega (dd^c\varphi)^n<\infty\right\}, \tag{a}$$

$$\mathcal{F}(\Omega)=\left\{\varphi\in\mathrm{PSH}^-(\Omega): \text{there is } \mathcal{E}_0(\Omega)\ni\varphi_p\searrow\varphi \text{ such that } \sup_{p\geqslant 1}\int_\Omega (dd^c\varphi_p)^n<\infty\right\}, \tag{b}$$

$$\mathcal{E}(\Omega)=\{\varphi\in\mathrm{PSH}^-(\Omega): \text{there is } \varphi_K\in\mathcal{F}(\Omega) \text{ such that } \varphi_K=\varphi \text{ on } K \text{ for all } K\Subset\Omega\}. \tag{c}$$

It is proved in [Ce] that the class $\mathcal{E}(\Omega)$ is the biggest subset of $\mathrm{PSH}^-(\Omega)$ on which the Monge–Ampère operator is well defined. For a general complex manifold X, after removing the negativity assumption of the functions involved, one can in fact extend the Monge–Ampère operator to the class

$$\tilde{\mathcal{E}}(X)\subset\mathrm{PSH}(X) \tag{1.1}$$

of psh functions which, on a neighborhood $\Omega \ni x_0$ of an arbitrary point $x_0 \in X$, are equal to a sum $u+v$ with $u\in\mathcal{E}(\Omega)$ and $v\in C^\infty(\Omega)$; again, this is the biggest subclass of functions of $\mathrm{PSH}(X)$ on which the Monge–Ampère operator is locally well defined. It is easy to see that $\tilde{\mathcal{E}}(X)$ contains the class of psh functions which are locally bounded outside isolated singularities.

For $\varphi\in\mathrm{PSH}(\Omega)$ and $0\in\Omega$, we introduce the *log canonical threshold* at 0,

$$c(\varphi)=\sup\{c>0 : e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } 0\}, \tag{1.2}$$

and for $\varphi\in\tilde{\mathcal{E}}(\Omega)$ we introduce the *intersection numbers*

$$e_j(\varphi)=\int_{\{0\}} (dd^c\varphi)^j \wedge (dd^c \log\|z\|)^{n-j}, \tag{1.3}$$

which can be seen also as the Lelong numbers of $(dd^c\varphi)^j$ at 0. Our main result is the following sharp estimate. It is a generalization and a sharpening of similar inequalities discussed in [Co1], [Co2], [FEM1] and [FEM2]; such inequalities have fundamental applications to birational geometry (see [IM], [P1], [P2], [I] and [Ch]).

THEOREM 1.2. *Let $\varphi\in\tilde{\mathcal{E}}(\Omega)$ and $0\in\Omega$. Then $c(\varphi)=\infty$ if $e_1(\varphi)=0$ and, otherwise,*

$$c(\varphi)\geqslant \sum_{j=0}^{n-1} \frac{e_j(\varphi)}{e_{j+1}(\varphi)}.$$

Remark 1.3. By Lemma 2.1 below, we have $(e_1(\varphi),...,e_n(\varphi))\in D$, where

$$D=\{t=(t_1,...,t_n)\in[0,\infty)^n : t_1^2\leqslant t_2 \text{ and } t_j^2\leqslant t_{j-1}t_{j+1} \text{ for } j=2,...,n-1\},$$

i.e. $\log e_j(\varphi)$ is a convex sequence. In particular, we have $e_j(\varphi)\geqslant e_1(\varphi)^j$, and the denominators do not vanish in Theorem 1.2 if $e_1(\varphi)>0$. On the other hand, a well-known inequality due to Skoda [S] tells us that

$$\frac{1}{e_1(\varphi)}\leqslant c(\varphi)\leqslant \frac{n}{e_1(\varphi)},$$

and hence $c(\varphi)<\infty$ if and only if $e_1(\varphi)>0$. To see that Theorem 1.2 is optimal, let us choose

$$\varphi(z)=\max\{a_1\log|z_1|,...,a_n\log|z_n|\},$$

with $0<a_1\leqslant a_2\leqslant ...\leqslant a_n$. Then $e_j(\varphi)=a_1a_2\dots a_j$, and a change of variable $z_j=\zeta_j^{1/a_j}$ on $\mathbb{C}\setminus\mathbb{R}_-$ easily shows that

$$c(\varphi)=\sum_{j=1}^{n}\frac{1}{a_j}.$$

Assume that we have a function $f\colon D\to[0,\infty)$ such that $c(\varphi)\geqslant f(e_1(\varphi),...,e_n(\varphi))$ for all $\varphi\in\tilde{\mathcal{E}}(\Omega)$. Then, by the above example, we must have

$$f(a_1,a_1a_2,...,a_1\,...\,a_n)\leqslant\sum_{j=1}^{n}\frac{1}{a_j}$$

for all a_j as above. By taking $a_j=t_j/t_{j-1}$, with $t_0=1$, this implies that

$$f(t_1,...,t_n)\leqslant\frac{1}{t_1}+\frac{t_1}{t_2}+...+\frac{t_{n-1}}{t_n}\quad\text{for all } t\in D,$$

whence the optimality of our inequality.

Remark 1.4. Theorem 1.2 is of course stronger than Skoda's lower bound

$$c(\varphi)\geqslant\frac{1}{e_1(\varphi)}.$$

By the inequality between the arithmetic and geometric means, we infer the main inequality of [FEM1], [FEM2] and [D4]:

$$c(\varphi)\geqslant\frac{n}{e_n(\varphi)^{1/n}}. \tag{1.4}$$

By applying the arithmetic-geometric inequality for the indices $1\leqslant j\leqslant n-1$ in our summation $\sum_{j=0}^{n-1}e_j(\varphi)/e_{j+1}(\varphi)$, we also infer the stronger inequality

$$c(\varphi)\geqslant\frac{1}{e_1(\varphi)}+(n-1)\left(\frac{e_1(\varphi)}{e_n(\varphi)}\right)^{1/(n-1)}. \tag{1.5}$$

2. Log convexity of the multiplicity sequence

The log convexity of the multiplicity sequence can be derived from very elementary integration by parts and the Cauchy–Schwarz inequality, using an argument from [Ce].

LEMMA 2.1. *Let $\varphi\in\tilde{\mathcal{E}}(\Omega)$ and $0\in\Omega$. We have*

$$e_j(\varphi)^2\leqslant e_{j-1}(\varphi)e_{j+1}(\varphi)\quad\textit{for all } j=1,...,n-1.$$

Proof. Without loss generality, by replacing φ with a sequence of local approximations $\varphi_p(z)=\max\{\varphi(z)-C,p\log|z|\}$ of $\varphi(z)-C$, $C\gg1$, we may assume that Ω is the unit ball and $\varphi\in\mathcal{E}_0(\Omega)$. Take also $h,\psi\in\mathcal{E}_0(\Omega)$. Then integration by parts and the Cauchy–Schwarz inequality yield

$$\begin{aligned}
&\biggl(\int_\Omega -h(dd^c\varphi)^j\wedge(dd^c\psi)^{n-j}\biggr)^2\\
&\qquad=\biggl(\int_\Omega d\varphi\wedge d^c\psi\wedge(dd^c\varphi)^{j-1}\wedge(dd^c\psi)^{n-j-1}\wedge dd^c h\biggr)^2\\
&\qquad\leqslant\int_\Omega d\psi\wedge d^c\psi\wedge(dd^c\varphi)^{j-1}\wedge(dd^c\psi)^{n-j-1}\wedge dd^c h\\
&\qquad\qquad\times\int_\Omega d\varphi\wedge d^c\varphi\wedge(dd^c\varphi)^{j-1}\wedge(dd^c\psi)^{n-j-1}\wedge dd^c h\\
&\qquad=\int_\Omega -h(dd^c\varphi)^{j-1}\wedge(dd^c\psi)^{n-j+1}\int_\Omega -h(dd^c\varphi)^{j+1}\wedge(dd^c\psi)^{n-j-1}.
\end{aligned}$$

Now, as $p\to\infty$, take

$$h(z)=h_p(z)=\max\biggl\{-1,\frac{1}{p}\log\|z\|\biggr\}\nearrow\begin{cases}0, & \text{if } z\in\Omega\setminus\{0\},\\ -1, & \text{if } z=0.\end{cases}$$

By the monotone convergence theorem, we get in the limit

$$\biggl(\int_{\{0\}}(dd^c\varphi)^j\wedge(dd^c\psi)^{n-j}\biggr)^2\leqslant\int_{\{0\}}(dd^c\varphi)^{j-1}\wedge(dd^c\psi)^{n-j+1}\int_{\{0\}}(dd^c\varphi)^{j+1}\wedge(dd^c\psi)^{n-j-1}.$$

For $\psi(z)=\log\|z\|$, this is the desired estimate. □

COROLLARY 2.2. *Let $\varphi\in\tilde{\mathcal{E}}(\Omega)$ and $0\in\Omega$. We have the inequalities*

$$\begin{aligned}
&e_j(\varphi)\geqslant e_1(\varphi)^j && \text{for } 0\leqslant j\leqslant n,\\
&e_k(\varphi)\leqslant e_j(\varphi)^{(l-k)/(l-j)}e_l(\varphi)^{(k-j)/(l-j)} && \text{for } 0\leqslant j<k<l\leqslant n.
\end{aligned}$$

In particular $e_1(\varphi)=0$ implies that $e_k(\varphi)=0$ for $k=2,...,n-1$ if $n\geqslant3$.

Proof. If $e_j(\varphi)>0$ for all j, Lemma 2.1 implies that $j\mapsto e_j(\varphi)/e_{j-1}(\varphi)$ is increasing, at least equal to $e_1(\varphi)/e_0(\varphi)=e_1(\varphi)$, and the inequalities follow from the log convexity. The general case can be proved by considering $\varphi_\varepsilon(z)=\varphi(z)+\varepsilon\log\|z\|$, since $0<\varepsilon^j\leqslant e_j(\varphi_\varepsilon)\to e_j(\varphi)$ as $\varepsilon\to0$. The last statement is obtained by taking $j=1$ and $l=n$. □

3. Proof of the main theorem

We start with a monotonicity statement.

LEMMA 3.1. *Let $\varphi,\psi\in\tilde{\mathcal{E}}(\Omega)$ be such that $\varphi\leqslant\psi$ (i.e. φ is "more singular" than ψ). Then*

$$\sum_{j=0}^{n-1}\frac{e_j(\varphi)}{e_{j+1}(\varphi)}\leqslant\sum_{j=0}^{n-1}\frac{e_j(\psi)}{e_{j+1}(\psi)}.$$

Proof. As in Remark 1.3, we set

$$D=\{t=(t_1,...,t_n)\in[0,\infty)^n:t_1^2\leqslant t_2 \text{ and } t_j^2\leqslant t_{j-1}t_{j+1} \text{ for } j=2,...,n-1\}.$$

Then D is a convex set in $\mathbb{R}^n$, as can be checked by a straightforward application of the Cauchy–Schwarz inequality. We consider the function $f:\operatorname{int}D\to[0,\infty)$ given by

$$f(t_1,...,t_n)=\frac{1}{t_1}+\frac{t_1}{t_2}+...+\frac{t_{n-1}}{t_n}. \tag{3.1}$$

We have

$$\frac{\partial f}{\partial t_j}(t)=-\frac{t_{j-1}}{t_j^2}+\frac{1}{t_{j+1}}\leqslant 0 \quad \text{for all } t\in D.$$

For $a,b\in\operatorname{int}D$ such that $a_j\geqslant b_j$ for all $j=1,...,n$, $[0,1]\ni\lambda\mapsto f(b+\lambda(a-b))$ is thus a decreasing function. This implies that $f(a)\leqslant f(b)$ for $a,b\in\operatorname{int}D$, with $a_j\geqslant b_j$ for $j=1,...,n$. On the other hand, the hypothesis $\varphi\leqslant\psi$ implies that $e_j(\varphi)\geqslant e_j(\psi)$ for $j=1,...,n$, by the comparison principle (see e.g. [D1]). Therefore

$$f(e_1(\varphi),...,e_n(\varphi))\leqslant f(e_1(\psi),...,e_n(\psi)).$$ □

3.1. Proof of the main theorem in the "toric case"

It will be convenient here to introduce Kiselman's refined Lelong numbers (cf. [K1] and [K2]).

Definition 3.2. Let $\varphi\in\mathrm{PSH}(\Omega)$. Then the function

$$\nu_\varphi(x)=\lim_{t\to-\infty}\frac{\max\{\varphi(z):|z_1|=e^{x_1t},...,|z_n|=e^{x_nt}\}}{t}$$

is called the *refined Lelong number* of φ at 0. This function is increasing in each variable x_j and concave on $\mathbb{R}_+^n$.

By "toric case", we mean that $\varphi(z_1,...,z_n)=\varphi(|z_1|,...,|z_n|)$ depends only on $|z_j|$ for all j; then φ is psh if and only if $(t_1,...,t_n)\mapsto\varphi(e^{t_1},...,e^{t_n})$ is increasing in each t_j and convex. By replacing φ with $\varphi(\lambda z)-\varphi(\lambda,...,\lambda)$, $0<\lambda\ll1$, we may assume that $\Omega=\Delta^n$ is the unit polydisk, $\varphi(1,...,1)=0$ (so that $\varphi\leqslant0$ on Ω), and we have

$$e_1(\varphi)=n\nu_\varphi\left(\frac{1}{n},...,\frac{1}{n}\right).$$

By convexity, the slope

$$\frac{\max\{\varphi(z):|z_j|=e^{x_j t}\}}{t}$$

is increasing in t for $t<0$. Therefore, by taking $t=-1$, we get

$$\nu_\varphi(-\log|z_1|,...,-\log|z_n|)\leqslant -\varphi(z_1,...,z_n).$$

Notice also that $\nu_\varphi(x)$ satisfies the 1-homogeneity property $\nu_\varphi(\lambda x)=\lambda\nu_\varphi(x)$ for $\lambda\in\mathbb{R}_+$. As a consequence, ν_φ is entirely characterized by its restriction to the set

$$\Sigma=\left\{x=(x_1,...,x_n)\in\mathbb{R}_+^n:\sum_{j=1}^n x_j=1\right\}.$$

We choose $x^0=(x_1^0,...,x_n^0)\in\Sigma$ such that

$$\nu_\varphi(x^0)=\max\{\nu_\varphi(x):x\in\Sigma\}\in\left[\frac{e_1(\varphi)}{n},e_1(\varphi)\right].$$

By [K2, Theorem 5.8] (see also [H] for similar results in an algebraic context) we have the formula

$$c(\varphi)=\frac{1}{\nu_\varphi(x^0)}.$$

Set

$$\zeta(x)=\nu_\varphi(x^0)\min\left\{\frac{x_1}{x_1^0},...,\frac{x_n}{x_n^0}\right\}\quad\text{for } x\in\mathbb{R}_+^n.$$

Then ζ is the smallest non-negative concave 1-homogeneous function on $\mathbb{R}_+^n$ that is increasing in each variable x_j and such that $\zeta(x^0)=\nu_\varphi(x^0)$. Therefore we have $\zeta\leqslant\nu_\varphi$, and hence

$$\begin{aligned}\varphi(z_1,...,z_n)&\leqslant-\nu_\varphi(-\log|z_1|,...,-\log|z_n|)\leqslant-\zeta(-\log|z_1|,...,-\log|z_n|)\\&\leqslant\nu_\varphi(x^0)\max\left\{\frac{\log|z_1|}{x_1^0},...,\frac{\log|z_n|}{x_n^0}\right\}=:\psi(z_1,...,z_n).\end{aligned}$$

By Lemma 3.1 and Remark 1.3 we get

$$f(e_1(\varphi),...,e_n(\varphi))\leqslant f(e_1(\psi),...,e_n(\psi))=c(\psi)=\frac{1}{\nu_\varphi(x^0)}=c(\varphi).$$

3.2. Reduction to the case of psh functions with analytic singularities

In the second step, we reduce the proof to the case $\varphi=\log(|f_1|^2+...+|f_N|^2)$, where $f_1,...,f_N$ are germs of holomorphic functions at 0. Following the technique introduced in [D2], we let $\mathcal{H}_{m\varphi}(\Omega)$ be the Hilbert space of holomorphic functions f on Ω such that

$$\int_\Omega |f|^2e^{-2m\varphi}\,dV<\infty,$$

and let

$$\psi_m = \frac{1}{2m}\log\sum_{k=1}^{\infty}|g_{m,k}|^2,$$

where $\{g_{m,k}\}_{k\geqslant 1}$ is an orthonormal basis of $\mathcal{H}_{m\varphi}(\Omega)$. Due to [DK, Theorem 4.2], mainly based on the Ohsawa–Takegoshi L^2 extension theorem [OT] (see also [D2]), there are constants $C_1, C_2>0$ independent of m such that

$$\varphi(z)-\frac{C_1}{m}\leqslant\psi_m(z)\leqslant\sup_{|\zeta-z|<r}\varphi(\zeta)+\frac{1}{m}\log\frac{C_2}{r^n}$$

for every $z\in\Omega$ and $r<d(z,\partial\Omega)$,

$$\nu(\varphi)-\frac{n}{m}\leqslant\nu(\psi_m)\leqslant\nu(\varphi)\quad\text{and}\quad\frac{1}{c(\varphi)}-\frac{1}{m}\leqslant\frac{1}{c(\psi_m)}\leqslant\frac{1}{c(\varphi)}.$$

By Lemma 3.1, we get that

$$f(e_1(\varphi),...,e_n(\varphi))\leqslant f(e_1(\psi_m),...,e_n(\psi_m))\quad\text{for all } m\geqslant 1.$$

The above inequalities show that in order to prove the lower bound of $c(\varphi)$ in Theorem 1.2, we only need to prove it for $c(\psi_m)$ and let m tend to infinity. Also notice that since the Lelong numbers of a function $\varphi\in\tilde{\mathcal{E}}(\Omega)$ occur only on a discrete set, the same is true for the functions ψ_m.

3.3. Reduction of the main theorem to the case of monomial ideals

The final step consists of proving the theorem for

$$\varphi=\log(|f_1|^2+...+|f_N|^2),$$

where $f_1, ..., f_N$ are germs of holomorphic functions at 0 (this is because the ideals $(g_{m,k})_{k\in\mathbb{N}}$ in the Noetherian ring $\mathcal{O}_{\mathbb{C}^n,0}$ are always finitely generated). Set $\mathcal{J}=(f_1, ..., f_N)$, $c(\mathcal{J})=c(\varphi)$ and $e_j(\mathcal{J})=e_j(\varphi)$ for all $j=0, ..., n$. By the final observation of §3.2, we may assume that $\mathcal{J}$ has an isolated zero at 0. Now, by fixing a multiplicative order on the monomials $z^\alpha=z_1^{\alpha_1}...z_n^{\alpha_n}$ (see [E, Chapter 15] and [FEM2]), it is well known that one can construct a flat family $(\mathcal{J}_s)_{s\in\mathbb{C}}$ of ideals of $\mathcal{O}_{\mathbb{C}^n,0}$ depending on a complex parameter $s\in\mathbb{C}$, such that $\mathcal{J}_0$ is a monomial ideal, $\mathcal{J}_1=\mathcal{J}$ and $\dim(\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}_s^t)=\dim(\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}^t)$ for all s and $t\in\mathbb{N}$; in fact $\mathcal{J}_0$ is just the initial ideal associated with $\mathcal{J}$ with respect to the monomial order. Moreover, we can arrange, by a generic rotation of coordinates $\mathbb{C}^p\subset\mathbb{C}^n$, so that the family of ideals $\mathcal{J}_s|_{\mathbb{C}^p}$ is also flat, and that the dimensions

$$\dim\left(\frac{\mathcal{O}_{\mathbb{C}^p,0}}{(\mathcal{J}_s|_{\mathbb{C}^p})^t}\right)=\dim\left(\frac{\mathcal{O}_{\mathbb{C}^p,0}}{(\mathcal{J}|_{\mathbb{C}^p})^t}\right)$$

compute the intermediate multiplicities

$$e_p(\mathcal{J}_s) = \lim_{t\to\infty} \frac{p!}{t^p} \dim\left(\frac{\mathcal{O}_{\mathbb{C}^p,0}}{(\mathcal{J}_s|_{\mathbb{C}^p})^t}\right) = e_p(\mathcal{J})$$

(notice, in the analytic setting, that the Lelong number of the (p,p)-current $(dd^c\varphi)^p$ at 0 is the Lelong number of its slice on a generic $\mathbb{C}^p\subset\mathbb{C}^n$); in particular $e_p(\mathcal{J}_0)=e_p(\mathcal{J})$ for all p. The semicontinuity property of the log canonical threshold (see for example [DK]) now implies that $c(\mathcal{J}_0)\leqslant c(\mathcal{J}_s)$ for s small. As $c(\mathcal{J}_s)=c(\mathcal{J})$ for $s\neq 0$ ($\mathcal{J}_s$ being a pull-back of $\mathcal{J}$ by a biholomorphism, in other words $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}_s\simeq\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}$ as rings; see again [E, Chapter 15]), the lower bound is valid for $c(\mathcal{J})$ if it is valid for $c(\mathcal{J}_0)$.

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