Acta Math., 206 (2011), 55–92
DOI: 10.1007/s11511-011-0059-x

Radial Fourier multipliers in high dimensions

by

Yaryong Heo
University of Wisconsin, Madison
Madison, WI, U.S.A.

Fëdor Nazarov
University of Wisconsin, Madison
Madison, WI, U.S.A.

Andreas Seeger
University of Wisconsin, Madison
Madison, WI, U.S.A.

In memory of Brent Smith.

Introduction

In this paper we study convolution operators with radial kernels acting on functions defined in $\mathbb{R}^d$. These can also be described as Fourier multiplier transformations T_m defined by

$$\widehat{T_m f} = m\hat{f},$$

with radial m. The main question we will be interested in is when the operator T_m is bounded on $L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$. By duality, the boundedness of T_m on L^p is equivalent to its boundedness on $L^{p'}$, where $1/p + 1/p' = 1$, so we may restrict ourselves to the range $1 \leqslant p \leqslant 2$.

A simple characterization of convolution operators bounded on L^p (whether radial or not) is known only in two cases: $p=1$ and $p=2$; namely, boundedness on L^1 holds if and only if the convolution kernel is a finite Borel measure, and boundedness on L^2 holds if and only if the multiplier is an essentially bounded function (see [12]). It is currently widely believed that for $1<p<2$, a full characterization of all $\mathcal{F}L^p$ multipliers in reasonable terms is impossible. For the class of *radial* multipliers we deal with in this paper, numerous sufficient conditions for boundedness on L^p have been obtained in the literature. Many of them are in some or another sense close to being necessary

Y. H. was supported by grants from the Korea Research Foundation and the National Research Foundation of Korea. F. N. and A. S. were supported in part by grants from the National Science Foundation.

(cf. [1], [2], [3], [14], [17], [29], and references in those papers) but no nice necessary and sufficient conditions have been known. However, recently, Garrigós and the third author [8] obtained a perhaps surprising characterization of the radial multiplier transformations that are bounded on the invariant subspace L^p_{rad} of radial L^p functions in the range $1<p<2d/(d+1)$ (which is optimal for their result). This raised the question whether the necessary and sufficient conditions in [8] actually give a characterization of the radial multiplier transformations bounded on the entire space $L^p(\mathbb{R}^d)$. The main result of the present paper is to show that this is indeed the case if the dimension is sufficiently large, namely if $d>(2+p)/(2-p)$, $1<p<2$.

1. Statement of results

Theorem 1.1. *Let* $d\geqslant 4$, $1<p<p_d:=(2d-2)/(d+1)$, *and let* m *be radial. Fix an arbitrary Schwartz function* η *that is not identically* 0. *Then*

$$\|T_m\|_{L^p\to L^p} \asymp \sup_{t>0} t^{d/p} \|T_m[\eta(t\,\cdot)]\|_p. \tag{1.1}$$

The finiteness of the right-hand side is, obviously, necessary for the L^p boundedness, and the main result here is that it is also sufficient. The constants implicit in this characterization depend (of course) on the choice of η. The condition in (1.1) is equivalent to $\sup_{t>0} \|\mathcal{F}^{-1}[m(t\,\cdot)\hat{\eta}]\|_p<\infty$. If one chooses η to be radial and such that $\hat{\eta}$ is compactly supported away from the origin, then one recovers one of the characterizations for L^p_{rad} boundedness in [8]. Consequently, in the given range L^p boundedness is equivalent to L^p_{rad} boundedness. We refer the reader to [8] for other equivalent formulations.

One special situation is worth mentioning here. Namely, if m is compactly supported away from the origin and $1<p<p_d$, then the convolution operator is bounded on $L^p(\mathbb{R}^d)$ if and only if the (radial) convolution kernel $\widehat{m}$ belongs to $L^p(\mathbb{R}^d)$.

We have no reason to believe that the range for p in Theorem 1.1 is even close to the optimal one. It is conceivable that the characterization holds in low dimensions or even in the optimal range $p<2d/(d+1)$, but proving that will certainly require new ideas. We also emphasize that the theorem gives no improvements for the Bochner–Riesz multiplier problem that is by now understood in the range $p<(2d+4)/(d+4)$, $d\geqslant 2$ (see [3] and [14]). Our result just goes in a different direction: it applies to all, however irregular, radial kernels and it is to be expected that, using some additional structural or regularity conditions, one may get some better range of p for each particular case. Nevertheless, our technique does yield some improvements upon the existing results in the so-called local smoothing problem for the wave equation in high dimensions. This concerns inequalities of the form

$$\left(\int_I \big\|e^{it\sqrt{-\Delta}}f\big\|_q^q\,dt\right)^{1/q} \leqslant C_I\|f\|_{L^q_\alpha}, \tag{1.2}$$

for $q>2$; here I is a compact interval and $L^q_\alpha(\mathbb{R}^d)$ denotes the usual Sobolev (or potential) space where q is the Lebesgue exponent and α is the number of derivatives. Sharp L^q-Sobolev inequalities for fixed time were obtained by Miyachi [15] and Peral [20]; they showed that the operator $e^{it\sqrt{-\Delta}}$ maps $L^q_\beta(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$ provided that

$$\beta \geqslant (d-1)\left|\frac{1}{2}-\frac{1}{q}\right|, \quad 1<q<\infty.$$

In [23] Sogge raised the question whether the averaged inequality (1.2) could hold with a gain of almost $1/q$ derivatives compared to the fixed time estimate, i.e. with

$$\alpha>\alpha(q)=d\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2},$$

in the best possible range $q>2d/(d-1)$ for such an estimate. This conjecture is at the top of a tree of other conjectures in harmonic analysis (including the cone multiplier, Bochner–Riesz, Fourier-restriction and Kakeya conjectures) and the relation between the different questions is discussed, for example, in [26]. The current techniques seem to be insufficient to settle this problem, as well as many of its consequences, in the full range of q's. Some evidence for the smoothing conjecture can be found in [17] where the analogous question for the $L^q_{\mathrm{rad}}(L^2_{\mathrm{sph}})$ scale of spaces is settled. For the L^q spaces even partial results proved to be rather hard and the first result was obtained by Wolff [29]; he established, in a deep and fundamental paper, the validity of Sogge's conjecture in two dimensions for the range $q>74$. Versions of this result for the higher-dimensional cases were obtained by Łaba and Wolff [13] and further improvements on the range of q's are in [9] and [10]; it is now known that Wolff's main $\ell^q(L^q)\to L^q$ inequality for plate decompositions of cone multipliers, which implies (1.2) for $\alpha>\alpha(q)$, holds with

$$q>\begin{cases} 20, & \text{if } d=2,\\ 2+\dfrac{8}{d-2}\,\dfrac{2d+1}{2d+2}, & \text{if } d\geqslant 3\end{cases}$$

(cf. [10]).

We improve the current results on the smoothing problem in two ways. First, we widen the range in dimensions $d\geqslant 5$. Secondly we strengthen Sogge's conjecture to obtain an endpoint result in (1.2) in dimensions $d\geqslant 4$.

THEOREM 1.2. *Suppose that* $d\geqslant 4$ *and* $q>q_d:=2+4/(d-3)$. *Then there is a constant* $C_{q,d}$ *such that for all* $L>0$,

$$\frac{1}{2L}\int_{-L}^{L}\|e^{it\sqrt{-\Delta}}f\|_q^q\,dt\leqslant C_{q,d}^q\|(I-L^2\Delta)^{\alpha/2}f\|_q^q \tag{1.3}$$

holds for

$$\alpha=\alpha(q)=d\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2}.$$

We remark that this result can be strengthened further by using suitable Triebel–Lizorkin spaces, see §10. A similar phenomenon occurs for solutions of Schrödinger type equations, see [21].

A downside of our method is, of course, that it currently does not yield L^p results in two and three dimensions. However, when it does apply, it is somewhat simpler than the induction on scales methods introduced by Wolff. We also remark that we do not improve on the current range of the above-mentioned Wolff inequality for plate decompositions, which has other applications and is interesting in its own right.

Structure of the paper. In §2 we explain the basic idea of the paper, which is that weak orthogonality properties may be combined with support size estimates to prove satisfactory L^p bounds. Here we also state a basic interpolation lemma which is related to the Marcinkiewicz theorem and will be used throughout the paper. The main section is §3 where we outline the proof of a discretized version of Theorem 1.1 for a fixed scale. A crucial L^2 estimate needed for this proof is done in §4. The characterization of L^p boundedness for radial multipliers that are compactly supported away from the origin is proved in §5. In §6 we give an important refinement of the earlier estimates, which is crucial for putting scales together. This is completed in §7 where the relevant atomic decomposition techniques are introduced and applied. The proof of Theorem 1.1 is concluded in §8. In §9 we state an extension to H^p spaces, $p\leqslant 1$, which holds for dimensions $d\geqslant 2$; moreover we obtain Lorentz space bounds (including weak type (p,p) inequalities). The last section §10 contains the proof of (a somewhat strengthened version of) Theorem 1.2.

Notation. For two quantities A and B, we shall write $A\lesssim B$ if $A\leqslant CB$ for some positive constant C, depending on the dimension and possibly other parameters apparent from the context, for instance Lebesgue exponents. We write $A\asymp B$ if $A\lesssim B$ and $B\lesssim A$. The cardinality of a finite set $\mathcal{E}$ is denoted by $\#\mathcal{E}$. The d-dimensional Lebesgue measure of a set $E\subset\mathbb{R}^d$ will be denoted by $\mathrm{meas}(E)$ or by $|E|$.

Remark. This paper is a descendant of the unpublished manuscript [18] with the same title in which Theorems 1.1 and 1.2 were proved in dimensions $d \geqslant 5$ for slightly smaller ranges of p and q. The approach in the present paper simplifies the one in [18] and was inspired in part by an idea in [11]. The authors would like to thank Gustavo Garrigós and Keith Rogers for their comments on various preliminary versions of [18].

2. L^2 bounds versus support: A simple model case

Since we do not know how to exploit cancellations in L^p directly, we use the strategy of controlling the L^2 norm and the size of the support simultaneously to get our L^p bounds. We start with describing a simple model case for which we have some limited orthogonality, but not enough to prove a favorable L^2 bound.

LEMMA 2.1. *Suppose we are given a finite number of complex-valued L^2-functions $\{f_z\}$ indexed by z in a subset of $\mathbb{Z}^d$, such that each function f_z is supported in a cube Q_z of sidelength* 1*. Suppose also that the family $\{f_z\}$ satisfies*

$$|\langle f_z, f_{z'}\rangle| \leqslant (1+|z-z'|)^{-\beta}, \tag{2.1}$$

for some $\beta \in (0,d)$. Then for $p<2d/(2d-\beta)$,

$$\Big\|\sum_z a_z f_z\Big\|_p \lesssim \Big(\sum_z |a_z|^p\Big)^{1/p}. \tag{2.2}$$

The implicit constant in (2.2) depends on d, β and p. Note that (2.2) is trivial for $p \leqslant 1$. We remark that if (2.1) were assumed for some $\beta > d$, then inequality (2.2) would also be true for $p=2$ and thereby for $1<p<2$ by interpolation. The assumption (2.1) for $\beta<d$ is too weak to yield the $\ell^2 \to L^2$ bound. Instead we have to use some improved support properties when several of the cubes Q_z overlap.

Proof. We shall first prove a weaker (so-called *restricted strong type*) inequality that includes the endpoint; namely for $1 \leqslant p \leqslant 2d/(2d-\beta)$,

$$\Big\|\sum_{z\in E} a_z f_z\Big\|_p \lesssim (\#E)^{1/p} \sup_z |a_z|. \tag{2.3}$$

We may assume that $\sup_z |a_z|=1$. Let $x_z \in \mathbb{R}^d$ be the center of the cube Q_z of sidelength 1 supporting f_z. Split $\mathbb{R}^d$ into non-overlapping cubes J of sidelength 1, put

$$E_J = \{z \in E : x_z \in J\},$$

and define $u_J = \# E_J$ so that $\# E = \sum_J u_J$. We have to bound the L^p norm of $\sum_J F_J$, where $F_J = \sum_{z \in E_J} a_z f_z$.

Now observe that at each point $x \in \mathbb{R}$, at most 3^d of the functions F_J can be non-zero simultaneously. Therefore,

$$\Big\| \sum_J F_J \Big\|_p^p \leqslant 3^{dp} \sum_J \|F_J\|_p^p.$$

Now, according to our weak orthogonality assumption about the functions f_z, we have

$$\|F_J\|_2^2 \leqslant \sum_{z \in E_J} \sum_{z' \in E_J} (1+|z-z'|)^{-\beta} \leqslant \sum_{z \in E_J} \sum_{z' : |z-z'| \leqslant \sqrt{d} u_J^{1/d}} (1+|z-z'|)^{-\beta} \lesssim u_J^{2-\beta/d}.$$

The measure of the support of F_J is at most 2^d and therefore, by Hölder's inequality, $\|F_J\|_p \lesssim \|F_J\|_2$. Hence,

$$\Big\| \sum_J F_J \Big\|_p \lesssim \Big(\sum_J \|F_J\|_2^p \Big)^{1/p} \lesssim \Big(\sum_J u_J^{(2-\beta/d)p/2} \Big)^{1/p}$$

and if $(2-\beta/d)p/2 \leqslant 1$, then the last expression is bounded by $\big(\sum_J u_J\big)^{1/p} \leqslant (\# E)^{1/p}$. This yields (2.3).

The improved bound (2.2) can be deduced by using interpolation theorems for Lorentz spaces (see [24, Chapter V]). Consider the operator on sequences $\mathfrak{a} = \{a_z\}_{z \in \mathbb{Z}^d}$, given by $T[\mathfrak{a}] = \sum_z a_z f_z$. Then (2.3) states that T maps the Lorentz space $\ell^{p,1}$ to L^p, for $p \leqslant 2d/(2d-\beta)$ and, by interpolation, one deduces the inequality (2.2) in the open range $p < 2d/(2d-\beta)$. □

We wish to give a direct proof of the last interpolation result based on a dyadic interpolation lemma, which will be frequently used in this paper. For closely related considerations see also the expository note [27] by Tao.

LEMMA 2.2. *Let $0 < p_0 < p_1 < \infty$. Let $\{F_j\}_{j \in \mathbb{Z}}$ be a sequence of measurable functions on a measure space $\{\Omega, \mu\}$, and let $\{s_j\}_{j \in \mathbb{Z}}$ be a sequence of non-negative numbers. Assume that, for all j, the inequality*

$$\|F_j\|_{p_\nu}^{p_\nu} \leqslant 2^{j p_\nu} M^{p_\nu} s_j \tag{2.4}$$

holds for $\nu = 0$ and $\nu = 1$. Then for all $p \in (p_0, p_1)$, there is a constant $C = C(p_0, p_1, p)$ such that

$$\Big\| \sum_{j \in \mathbb{Z}} F_j \Big\|_p^p \leqslant C^p M^p \sum_{j \in \mathbb{Z}} 2^{jp} s_j. \tag{2.5}$$

There is an analogous statement for the case $p_0 = 0$, where the assumption (2.4) for $\nu = 0$ is replaced by $\operatorname{meas}(\{x : F_j(x) \neq 0\}) \leqslant s_j$, and the conclusion (2.5) holds for $0 < p < p_1$.

To see how this is used to derive (2.2) from (2.3), we consider the sets of indices

$$E_j = \{z \in \mathbb{Z}^d : 2^{j-1} < |a_z| \leqslant 2^j\}$$

and define $F_j = \sum_{z \in E_j} a_z f_z$. Then $\|F_j\|_p^p \lesssim 2^{jp} \# E_j$ for all $p \in (0, 2d/(2d-\beta)]$ by (2.3). Thus Lemma 2.2 immediately yields

$$\Big\| \sum_{z \in \mathbb{Z}_d} a_z f_z \Big\|_p^p = \Big\| \sum_{j \in \mathbb{Z}} F_j \Big\|_p^p \lesssim \sum_{j \in \mathbb{Z}} 2^{pj} \# E_j \lesssim \sum_{z \in \mathbb{Z}_d} |a_z|^p$$

for all $p < 2d/(2d-\beta)$.

Proof. First, replacing F_j by $M^{-1} F_j$, we can reduce the statement to the case $M=1$. Now, for $n \in \mathbb{Z}$, denote by $E_{j,n}$ the set where $2^{j+n} \leqslant |F_j| < 2^{j+n+1}$ and put $F_{j,n} = \chi_{E_{j,n}} F_j$. Then $F_j = \sum_{n \in \mathbb{Z}} F_{j,n}$. Observe that if b_j is any numerical sequence such that for all j the absolute value of b_j either is 0 or belongs to $[2^j, 2^{j+1})$, then $\big|\sum_{j \in \mathbb{Z}} b_j\big|^p \lesssim \sum_{j \in \mathbb{Z}} |b_j|^p$. Applying this observation to $2^{-n} \sum_{j \in \mathbb{Z}} F_{j,n}$, we see that for fixed n and x,

$$\Big| \sum_{j \in \mathbb{Z}} F_{j,n}(x) \Big| \lesssim \Big(\sum_{j \in \mathbb{Z}} |F_{j,n}(x)|^p \Big)^{1/p},$$

and therefore

$$\Big\| \sum_{j \in \mathbb{Z}} F_{j,n} \Big\|_p^p \lesssim \sum_{j \in \mathbb{Z}} \|F_{j,n}\|_p^p \lesssim \sum_{j \in \mathbb{Z}} 2^{(j+n)p} \operatorname{meas}(\{x : |F_j| \geqslant 2^{j+n}\}).$$

By Chebyshev's inequality,

$$\operatorname{meas}(\{x : |F_j| \geqslant 2^{j+n}\}) \leqslant \min\{2^{-p_0 n}, 2^{-p_1 n}\} s_j.$$

Thus,

$$\Big\| \sum_{j \in \mathbb{Z}} F_{j,n} \Big\|_p \lesssim 2^{-\sigma |n|/p} \Big(\sum_{j \in \mathbb{Z}} 2^{jp} s_j \Big)^{1/p},$$

where $\sigma = \min\{p_1 - p, p - p_0\}$. We sum over n to get the statement of the lemma for the case $p_0 > 0$. The case $p_0 = 0$ is very similar and is left to the reader. □

3. The main inequality

In this section we shall prove the main inequality of this paper, which turns out to be the key estimate for the case when our multiplier has compact support away from the origin; this application is discussed at the end of the section.

In what follows, we denote by σ_r the surface measure on the $(d-1)$-dimensional sphere of radius r centered at the origin. We shall denote by $\psi_\circ$ a fixed radial C^∞ function that is compactly supported in a ball of radius $\frac{1}{10}$ centered at the origin, and whose Fourier transform $\hat{\psi}_\circ$ vanishes to high order (say, $20d$) at the origin. We set $\psi=\psi_\circ*\psi_\circ$.

Consider a 1-separated set $\mathcal{Y}$ of points in $\mathbb{R}^d$ and a 1-separated set $\mathcal{R}$ of radii $\geqslant 1$. Also set

$$\mathcal{R}_k=\mathcal{R}\cap[2^k,2^{k+1}),\quad k\geqslant 0.$$

For $y\in\mathcal{Y}$ and $r\in\mathcal{R}$, define

$$F_{y,r}=\sigma_r*\psi(\cdot-y). \tag{3.1}$$

Proposition 3.1. *Let $\mathcal{E}$ be a finite subset of $\mathcal{Y}\times\mathcal{R}$ and let $\mathcal{E}_k=\mathcal{E}\cap(\mathcal{Y}\times\mathcal{R}_k)$. Let $c:\mathcal{E}\to\mathbb{C}$ satisfy $|c(y,r)|\leqslant 1$ for all $(y,r)\in\mathcal{E}$. Then, for $p<p_d=(2d-2)/(d+1)$,*

$$\bigg\|\sum_{(y,r)\in\mathcal{E}}c(y,r)F_{y,r}\bigg\|_p^p\lesssim\sum_{k=0}^\infty 2^{k(d-1)}\#\mathcal{E}_k. \tag{3.2}$$

Here the implicit constant depends only on p, d and ψ.

Proposition 3.1 implies stronger estimates, namely the following.

Corollary 3.2. *For $F_{y,r}$ as in* (3.1) *and $p<p_d=(2d-2)/(d+1)$,*

$$\bigg\|\sum_{(y,r)\in\mathcal{Y}\times\mathcal{R}}\gamma(y,r)F_{y,r}\bigg\|_p\lesssim\bigg(\sum_{(y,r)\in\mathcal{Y}\times\mathcal{R}}|\gamma(y,r)|^p r^{d-1}\bigg)^{1/p}. \tag{3.3}$$

Also,

$$\bigg\|\int_{\mathbb{R}^d}\int_1^\infty h(y,r)F_{y,r}\,dr\,dy\bigg\|_p\lesssim\bigg(\int_{\mathbb{R}^d}\int_1^\infty|h(y,r)|^p r^{d-1}\,dr\,dy\bigg)^{1/p}. \tag{3.4}$$

Proof. Denote by $\mathcal{E}^j$, $j\in\mathbb{Z}$, the set of all $(y,r)\in\mathcal{Y}\times\mathcal{R}$ for which $2^{j-1}<|\gamma(y,r)|\leqslant 2^j$. By Proposition 3.1 we see that

$$\bigg\|\sum_{(y,r)\in\mathcal{E}^j}\gamma(y,r)F_{y,r}\bigg\|_p^p$$

is dominated by

$$C_p^p 2^{jp}\sum_{(y,r)\in\mathcal{E}^j}r^{d-1}$$

for all $p<p_d$, and (3.3) follows by the dyadic interpolation Lemma 2.2.

To prove (3.4), we write $y=z+w$, where $z\in\mathbb{Z}^d$, $w\in Q_\circ:=[0,1)^d$ and $r=n+\tau$, with $n\in\mathbb{N}$ and $0\leqslant\tau<1$. Then, by Minkowski's inequality, the left-hand side of (3.4) is dominated by

$$\iint_{Q_\circ\times[0,1)}\biggl\|\sum_{z\in\mathbb{Z}^d}\sum_{n=1}^{\infty}h(z+w,n+\tau)F_{z+w,n+\tau}\biggr\|_p\,dw\,d\tau$$
$$\lesssim\iint_{Q_\circ\times[0,1)}\biggl(\sum_{z\in\mathbb{Z}^d}\sum_{n=1}^{\infty}|h(z+w,n+\tau)|^p(n+\tau)^{d-1}\biggr)^{1/p}\,dw\,d\tau.$$

Now (3.4) follows by Hölder's inequality. □

If h has a tensor product structure, namely, $h(y,r)=g(y)\beta(r)$, then the expression $\iint h(y,r)F_{y,r}\,dy\,dr$ can be interpreted as a convolution of a radial kernel with g. In §5 we shall see how this model case implies the version of our theorem for radial multipliers that are compactly supported away from the origin.

We shall present the proof of Proposition 3.1 (leaving one part to the next section).

Estimates for scalar products

We aim at a good L^2 estimate for $\sum_{y,r}c_{y,r}F_{y,r}$ and make use of some (albeit weak) orthogonality property of the summands. This property is expressed by the following lemma.

LEMMA 3.3. *For any choice of* $r,r'>1$ *and* $y,y'\in\mathbb{R}^d$,

$$|\langle F_{y,r},F_{y',r'}\rangle|\lesssim\frac{(rr')^{(d-1)/2}}{(1+|y-y'|+|r-r'|)^{(d-1)/2}}.\tag{3.5}$$

Proof. Note that $\sigma_r=r^{-1}\sigma_1(r^{-1}\cdot)$ in the sense of measures and $\hat\sigma_r(\xi)=r^{d-1}\hat\sigma_1(r\xi)$. Next, we have $\hat\sigma_1(\xi)=B_d(|\xi|)$, where $B_d(s)=c_ds^{-(d-2)/2}J_{(d-2)/2}(s)$ (and $J.$ denotes the usual Bessel functions). Thus $|B_d(s)|\lesssim(1+|s|)^{-(d-1)/2}$ (see [24, Chapter IV]). Now $\hat\psi$ is radial and we can write $\hat\psi(\xi)=a(|\xi|)$, where a is rapidly decaying and vanishes to high order at the origin. By Plancherel's theorem, the scalar product $\langle F_{y,r},F_{y',r'}\rangle$ is equal to a constant times

$$\int_{\mathbb{R}^d}\hat\sigma_r(\xi)\hat\sigma_{r'}(\xi)|\hat\psi(\xi)|^2e^{i\langle y'-y,\xi\rangle}\,d\xi$$
$$=c(rr')^{d-1}\int_0^\infty B_d(r\varrho)B_d(r'\varrho)B_d(|y-y'|\varrho)|a(\varrho)|^2\varrho^{d-1}\,d\varrho.$$

The decay properties of B_d and the behavior of a imply that

$$|\langle F_{y,r},F_{y',r'}\rangle|\lesssim\frac{(rr')^{(d-1)/2}}{(1+|y-y'|)^{(d-1)/2}},$$

which gives the claimed bound for the range $|r-r'|\leqslant C(1+|y-y'|)$. On the other hand, if $|r-r'|\gg 1+|y-y'|$, then $F_{y,r}$ and $F_{y',r'}$ have disjoint supports. Therefore in this case $\langle F_{y,r}, F_{y',r'}\rangle=0$. The lemma is proved. □

Remark 3.4. Taking into account the oscillation of the Bessel functions, one can obtain the improved bound

$$|\langle F_{y,r}, F_{y',r'}\rangle| \leqslant C_N (rr')^{(d-1)/2}(1+|y-y'|)^{-(d-1)/2} \sum_{\pm,\pm} \big(1+\big|r\pm r'\pm|y-y'|\big|\big)^{-N}.$$

We shall not use this in our proof.

The exponent $\frac{1}{2}(d-1)$ in the denominator in (3.5) is too small to use orthogonality in a straightforward way; this is analogous to the weak orthogonality assumption in Lemma 2.1. However if we impose a suitable density assumption on the sets $\mathcal{E}_k$, then we can prove a satisfactory L^2 bound. To quantify this, we give a definition.

Definition 3.5. Fix $R\geqslant 1$ and $u\geqslant 1$. Assume that $\mathcal{E}$ is a finite 1-separated subset of $\mathbb{R}^d\times[R,2R)$. We say that $\mathcal{E}$ is of *density type* (u,R) if

$$\#(B\cap\mathcal{E})\leqslant u\,\mathrm{diam}(B)$$

for any ball $B\subset\mathbb{R}^{d+1}$ of diameter $\leqslant R$.

If we drop the restriction on the diameter, then for any ball B and any set $\mathcal{E}$ of density type (u,R),

$$\#(B\cap\mathcal{E})\leqslant C_d\bigg(1+\frac{\mathrm{diam}(B)}{R}\bigg)^d u\,\mathrm{diam}(B). \tag{3.6}$$

This is immediate from the definition.

We shall prove in §4 the following L^2 inequality based on Lemma 3.3.

LEMMA 3.6. *Let $u\geqslant 1$ and, for each $k\geqslant 0$, let $\mathcal{E}_k\subset\mathcal{Y}\times\mathcal{R}_k$ be a set of density type $(u,2^k)$. Assume that $|c(y,r)|\leqslant 1$ for $(y,r)\in\mathcal{Y}\times\mathcal{R}$. Then*

$$\bigg\|\sum_{k=0}^{\infty}\sum_{(y,r)\in\mathcal{E}_k} c(y,r)F_{y,r}\bigg\|_2^2 \lesssim u^{2/(d-1)}\log(2+u)\sum_{k=0}^{\infty}2^{k(d-1)}\#\mathcal{E}_k. \tag{3.7}$$

Density decompositions of sets

Assume that $\mathcal{E}\subset\mathcal{Y}\times\mathcal{R}$ is a finite 1-separated set. Let $\mathcal{E}_k=\mathcal{E}\cap(\mathcal{Y}\times\mathcal{R}_k)$ (i.e. only radii in $[2^k,2^{k+1})$ are involved). We consider $u\in\mathcal{U}=\{2^\nu:\nu=0,1,2,\dots\}$ and decompose the sets $\mathcal{E}_k$ into subsets of density type $(u,2^k)$.

Let $\widehat{\mathcal{E}}_k(u)$ be the set of all points $(y,r)\in\mathcal{E}_k$ that are contained in some ball B of radius $\operatorname{rad}(B)\leqslant 2^k$ such that

$$\#(\mathcal{E}_k\cap B)\geqslant u\operatorname{rad}(B). \tag{3.8}$$

Also set

$$\mathcal{E}_k(u)=\widehat{\mathcal{E}}_k(u)\setminus\bigcup_{\substack{u'\in\mathcal{U}\\u'>u}}\widehat{\mathcal{E}}_k(u').$$

Finally set $\mathcal{E}(u)=\bigcup_{k=0}^{\infty}\mathcal{E}_k(u)$.

Lemma 3.7. *The sets $\mathcal{E}(u)$ have the following properties*:

(i) $\mathcal{E}=\bigcup_{u\in\mathcal{U}}\mathcal{E}(u)=\bigcup_{u\in\mathcal{U}}\bigcup_{k=0}^{\infty}\mathcal{E}_k(u)$ *and the unions are disjoint*;

(ii) *if B is any ball of radius $\leqslant 2^k$ containing at least $u\operatorname{rad}(B)$ points of $\mathcal{E}_k$, then*

$$B\cap\mathcal{E}_k\subset\widehat{\mathcal{E}}_k(u)\equiv\bigcup_{\substack{u'\in\mathcal{U}\\u'\geqslant u}}\mathcal{E}_k(u');$$

(iii) *there are finitely many disjoint balls $B_1,...,B_N$ (depending on u and k), of radii $\leqslant 2^k$ such that*

$$\sum_{i=1}^{N}\operatorname{rad}(B_i)\leqslant\frac{\#\mathcal{E}_k}{u} \tag{3.9}$$

and

$$\widehat{\mathcal{E}}_k(u)\subset\bigcup_{i=1}^{N}B_i^*, \tag{3.10}$$

where B_i^* denotes the ball with $\operatorname{rad}(B_i^*)=5\operatorname{rad}(B_i)$ and the same center as B_i;

(iv) *$\mathcal{E}_k(u)$ is a set of density type $(u,2^k)$.*

Proof. In order to prove (i), it suffices to observe that $\widehat{\mathcal{E}}_k(2^0)=\mathcal{E}_k$ and $\widehat{\mathcal{E}}_k(u)=\varnothing$ when u is sufficiently large. Property (ii) follows immediately from the definition of the sets $\widehat{\mathcal{E}}_k(u)$ and $\mathcal{E}_k(u)$.

To prove (iii), cover the set $\widehat{\mathcal{E}}_k(u)$ by a finite number of balls satisfying (3.8). We apply the Vitali covering lemma to this family of balls and select *disjoint* balls B_i, $i=1,...,N(k,u,\mathcal{E})$, so that the five times dilated balls B_i^* cover $\widehat{\mathcal{E}}_k(u)$. This yields (3.10). The inequality (3.9) follows from the disjointness of the selected balls and condition (3.8).

To prove (iv), let $(y,r)\in\mathcal{E}_k(u)$. By definition $(y,r)\notin\widehat{\mathcal{E}}_k(2u)$ and thus, for any ball B of radius $\operatorname{rad}(B)\leqslant 2^k$, the number of points in $\mathcal{E}_k$ contained in B is less than $2u\operatorname{rad}(B)=u\operatorname{diam}(B)$. Thus $\mathcal{E}_k(u)$ is of density type $(u,2^k)$. □

We now set

$$G_{u,k}=\sum_{(y,r)\in\mathcal{E}_k(u)}c(y,r)F_{y,r}\quad\text{and}\quad G_u=\sum_{k=0}^{\infty}G_{u,k}. \tag{3.11}$$

From the support properties of $\sigma_r * \psi$ it follows immediately that $G_{u,k}$ is supported in a set of measure $\lesssim 2^{k(d-1)} \# \mathcal{E}_k(u)$, and hence of measure $\lesssim 2^{k(d-1)} \# \mathcal{E}_k$. By the properties of $\mathcal{E}_k(u)$ we get the following improved bound.

LEMMA 3.8. *For all $u \in \mathcal{U}$, the Lebesgue measure of the support of $G_{u,k}$ is*

$$\lesssim \frac{2^{k(d-1)} \# \mathcal{E}_k}{u}.$$

Proof. We use (3.10). Let (y_i, r_i) be the center of B_i^*. Then, for every pair (y, r) contained in B_i^*, the support of $c(y,r)\sigma_r * \psi(\cdot - y)$ is contained in an annulus of width not exceeding $4\,\mathrm{rad}(B_i^*)+1$ built on the sphere centered at y_i of radius r_i. Also, note that the estimate for the width of the annulus does not exceed the estimate for the radius of the sphere it is built upon, so we can conclude that the volume of this annulus is $\lesssim 2^{k(d-1)}\,\mathrm{rad}(B_i^*)$. Consequently the measure of the support of $G_{u,k}$ does not exceed $C_d 2^{k(d-1)} \sum_{i=1}^N \mathrm{rad}(B_i^*)$, and hence, by (3.9), it does not exceed $5C_d 2^{k(d-1)} u^{-1} \# \mathcal{E}_k$. $\square$

We now combine the L^2 bound of Lemma 3.6 and the support bound of Lemma 3.8 to get an L^p bound; for later reference in §6 this is formally stated as follows.

LEMMA 3.9. *Suppose that $d \geqslant 4$. Let G_u be as in* (3.11), *where the sets $\mathcal{E}_k(u)$ are defined using the density decomposition of $\mathcal{E}_k$. Then, for $p \leqslant 2$,*

$$\|G_u\|_p \lesssim u^{-(1/p-1/p_d)} \sqrt{\log(2+u)} \left(\sum_{k=0}^{\infty} 2^{k(d-1)} \# \mathcal{E}_k \right)^{1/p}.$$

Proof. By Lemma 3.6, $\|G_u\|_2^2 \lesssim \log(2+u) u^{2/(d-1)} \sum_{k=0}^{\infty} 2^{k(d-1)} \# \mathcal{E}_k$. Combining this with the support bound of Lemma 3.8, we obtain

$$\|G_u\|_p^p \leqslant \mathrm{meas}(\mathrm{supp}(G_u))^{1-p/2} \|G_u\|_2^p \leqslant \left(\sum_{k=0}^{\infty} \mathrm{meas}(\mathrm{supp}(G_{u,k})) \right)^{1-p/2} \|G_u\|_2^p,$$

which is

$$\lesssim u^{-(1-p/2)} (\log(2+u) u^{2/(d-1)})^{p/2} \sum_{k=0}^{\infty} 2^{k(d-1)} \# \mathcal{E}_k.$$

We finally note that $-1+p/2+p/(d-1) = (1/p_d - 1/p)p$, and the lemma is proved. $\square$

The proof of Proposition 3.1 is now complete since for $p < p_d$, we can sum the bounds for $\|G_u\|_p$ over $u \in \mathcal{U}$.

4. Proof of Lemma 3.6

We are working with sets $\mathcal{E}_k \subset \mathcal{Y} \times \mathcal{R}_k$, which have the property that every ball of radius $\varrho \leqslant 2^k$ contains $\lesssim u\varrho$ points in $\mathcal{E}_k$. Let

$$G_k = \sum_{(y,r)\in\mathcal{E}_k} c(y,r) F_{y,r}$$

with $\|c\|_\infty \leqslant 1$. Our task is to estimate the L^2 norm of $\sum_{k=0}^{\infty} G_k$. We may break up this sum into ten separate sums, each with the property that k ranges over a 10-separated set of natural numbers. We shall assume this separation property in all sums involving a k-summation.

It will be convenient to avoid scalar products of expressions of G_k involving

$$k \lesssim \log(2+u).$$

Let $N(u)$ be the smallest integer larger than $10\log_2(2+u)$. Split the sum as

$$\sum_{k\leqslant N(u)} G_k + \sum_{k>N(u)} G_k$$

and then apply the Cauchy–Schwarz inequality. We thus obtain

$$\begin{aligned} \Big\| \sum_k G_k \Big\|_2^2 &\lesssim \log(2+u) \Big(\sum_{k\leqslant N(u)} \|G_k\|_2^2 + \Big\| \sum_{k>N(u)} G_k \Big\|_2^2 \Big) \\ &\lesssim \log(2+u) \Big(\sum_k \|G_k\|_2^2 + 2 \sum_{k'>k>N(u)} |\langle G_{k'}, G_k\rangle| \Big). \end{aligned} \tag{4.1}$$

We begin with estimating the double sum $\sum_{k'>k>N(u)} |\langle G_{k'}, G_k\rangle|$. In this sum we have various scalar products of $F_{y,r}$ with $F_{Y,R}$, where $r \leqslant 2^{-5}R$. Let us fix the pair (Y,R) and examine the sum of the absolute values of such scalar products when (y,r) runs over $\mathcal{E}_k$ with $2^k < \frac{1}{4}R$. The scalar product $\langle F_{y,r}, F_{Y,R}\rangle$ can be different from 0 only if y lies in the annulus of width $2^{k+1}+2$ built upon the sphere of radius R centered at Y. Moreover $2^k \leqslant r < 2^{k+1}$. The set of all pairs $(y,r) \in \mathcal{Y} \times \mathcal{R}$ satisfying these conditions can be covered by $\lesssim R^{d-1} 2^{-k(d-1)}$ balls (in $\mathbb{R}^{d+1}$) of radius 2^k. Each such ball can contain only $u2^{k+1}$ pairs $(y,r) \in \mathcal{E}_k$, by our assumption on $\mathcal{E}_k$. For each such (y,r), the scalar product $\langle F_{y,r}, F_{Y,R}\rangle$ is $O(2^{k(d-1)/2})$ by Lemma 3.3. Consequently, for fixed (Y,R),

$$\sum_{(y,r)\in\mathcal{E}_k} |\langle F_{y,r}, F_{Y,R}\rangle| \lesssim R^{d-1} 2^{-k(d-1)/2} u 2^k,$$

and therefore, as $N(u)=10\log_2(2+u)$,

$$\sum_{k:2^{N(u)}<2^k<R/4}\ \sum_{(y,r)\in\mathcal{E}_k}|\langle F_{y,r},F_{Y,R}\rangle|\lesssim R^{d-1}\sum_{k>N(u)}2^{-k(d-1)/2}(u2^k)\lesssim R^{d-1};$$

here we used the fact that $d>3$ and summed a decaying geometric progression whose maximal term corresponds to $k=N(u)+10$. Since $(d-1)/2>1$, we see that the geometric decay cancels the large factor u in the last displayed formula. It remains to sum these estimates over pairs (Y,R) to get the bound $\sum_{(Y,R)\in\mathcal{E}}R^{d-1}\lesssim\sum_k 2^{k(d-1)}\#\mathcal{E}_k$ for the sum of scalar products in (4.1).

Now that we have dealt with the interaction of incomparable radii, we can concentrate on estimating $\|G_k\|_2^2$ for each k separately. It is convenient to arrange the radii in intervals of length u^a for some $a>0$, and then apply the estimates of Lemma 3.3 to scalar products arising from different intervals; we shall see later that the choice of $a=2/(d-1)$ is optimal.

Let $I_{k,\mu}=[2^k+(\mu-1)u^a, 2^k+\mu u^a)$ for $\mu=1,2,...$, and let $\mathcal{E}_{k,\mu}$ be the set of all $(y,r)\in\mathcal{Y}\times I_{k,\mu}$ that belong to $\mathcal{E}_k$. Set

$$G_{k,\mu}=\sum_{(y,r)\in\mathcal{E}_{k,\mu}}c(y,r)F_{y,r}.$$

We need to estimate the L^2 norm of $\sum_{\mu=1}^{\infty}G_{k,\mu}$. By splitting the μ sum into ten different sums, we may assume that μ ranges over a 10-separated set and bound

$$\Big\|\sum_\mu G_{k,\mu}\Big\|_2^2\lesssim\sum_\mu\|G_{k,\mu}\|_2^2+2\sum_{\mu'>\mu+10}|\langle G_{k,\mu'},G_{k,\mu}\rangle|.$$

Again, we shall first estimate the sum of the various scalar products, using the assumption that the sets $\mathcal{E}_k$ are of density type $(u,2^k)$. We claim that

$$\sum_{\mu'>\mu+10}|\langle G_{k,\mu'},G_{k,\mu}\rangle|\lesssim u^{1-a(d-3)/2}2^{k(d-1)}\#\mathcal{E}_k. \tag{4.2}$$

To see this, we pick again some pair $(Y,R)\in\mathcal{E}_{k,\mu'}$ and examine how it interacts with pairs in $\mathcal{E}_{k,\mu}$, where $\mu\leqslant\mu'-10$. Note that if (y,r) is such a pair for which the scalar product is non-zero, then we must have $|y-Y|\leqslant 2^{k+3}$ and, since $|r-R|\leqslant 2^{k+1}$, we conclude that $|(y,r)-(Y,R)|\leqslant 2^{k+4}$ in $\mathbb{R}^{d+1}$. Moreover, $|r-R|\geqslant u^a$ and thus the sum of the scalar products in which the pair (Y,R) participates is

$$\lesssim 2^{k(d-1)}\sum_{\substack{(y,r)\in\mathcal{E}_k\\ u^a\leqslant|(y,r)-(Y,R)|\leqslant 2^{k+5}}}|(y,r)-(Y,R)|^{-(d-1)/2}.$$

Now we use the assumption that $\mathcal{E}_k$ is of density type $(u, 2^k)$ (cf. (3.6)) and estimate the displayed sum by

$$C_d 2^{k(d-1)} \sum_{2^\ell \geqslant u^a} (u 2^\ell) 2^{-\ell(d-1)/2} \lesssim 2^{k(d-1)} u^{1-a(d-3)2};$$

here we have used again that $d>3$. We sum over all $(Y, R) \in \mathcal{E}_{k,\mu'}$ and then over all μ'. The left-hand side of (4.2) is then $\lesssim u^{1-a(d-3)/2} 2^{k(d-1)} \sum_\mu \#\mathcal{E}_{k,\mu}$; and (4.2) follows.

We now estimate the L^2 norm of each $G_{k,\mu}$. For each $r \in \mathcal{R}_{k,\mu} := I_{k,\mu} \cap \mathcal{R}$, let

$$G_{k,\mu,r} = \sum_{y:(y,r)\in\mathcal{E}_k} c(y,r) F_{y,r}.$$

The conclusion of Lemma 3.3 is now too weak to give satisfactory results; instead we apply the Cauchy–Schwarz inequality with respect to r and use the fact that the cardinality of $\mathcal{R}_{k,\mu}$ is $\lesssim u^a$. Thus

$$\|G_{k,\mu}\|_2^2 \lesssim u^a \sum_{r\in\mathcal{R}_{k,\mu}} \|G_{k,\mu,r}\|_2^2.$$

Now $G_{k,\mu,r}$ is the convolution of $\sum_{y:(y,r)\in\mathcal{E}_{k,\mu}} c(y,r)\psi_\circ(\cdot - y)$ with $\sigma_r * \psi_\circ$. By the standard decay estimate for the Fourier transform of the surface measure on the unit sphere, we have

$$|\hat{\sigma}_r(\xi)| \leqslant r^{d-1}(1+r|\xi|)^{-(d-1)/2}$$

and, since $\hat{\psi}_\circ$ vanishes to high order at the origin, we also have, for $r \geqslant 1$, that

$$\|\hat{\sigma}_r \hat{\psi}_\circ\|_\infty \lesssim r^{(d-1)/2}. \tag{4.3}$$

As $\mathcal{Y}$ is 1-separated and the support of ψ is contained in a ball of radius $\frac{1}{2}$, we conclude that

$$\|G_{k,\mu,r}\|_2^2 \lesssim r^{d-1} \#\{y \in \mathcal{Y} : (y,r) \in \mathcal{E}_{k,\mu}\},$$

and thus

$$\sum_\mu \|G_{k,\mu}\|_2^2 \lesssim u^a \sum_\mu \sum_{r\in\mathcal{R}_{k,\mu}} \|G_{k,\mu,r}\|_2^2 \lesssim u^a 2^{k(d-1)} \#\mathcal{E}_k.$$

Combining this bound with (4.2) yields

$$\|G_k\|_2^2 \lesssim (u^a + u^{1-a(d-3)/2}) 2^{k(d-1)} \#\mathcal{E}_k.$$

The two terms balance if $a=2/(d-1)$, and with this choice the previous bound becomes

$$\|G_k\|_2^2 \lesssim u^{2/(d-1)} 2^{k(d-1)} \#\mathcal{E}_k.$$

Finally, we use this to estimate the first term in (4.1) and combine the resulting bound with the earlier bound for the mixed terms in (4.1) to complete the proof of the lemma.

5. Application to compactly supported multipliers

Now let m be a radial Fourier multiplier supported in $\{\xi:\frac{1}{2}<|\xi|<2\}$ and let $K=\mathcal{F}^{-1}[m]$. Since K is radial, we can also write $K=\varkappa(|\cdot|)$ for some $\varkappa$. We shall prove the estimate

$$\|K*f\|_p \lesssim \|K\|_p\|f\|_p, \quad 1\leqslant p<p_d. \tag{5.1}$$

Let $\eta_\circ$ be a radial Schwartz function whose Fourier transform is supported in the set $\{\xi:\frac{1}{4}<|\xi|<4\}$ and such that $\hat{\eta}_\circ(\xi)=1$ on the support of m. Let $\psi_\circ$ be a radial C^∞ function with compact support in $\{x:|x|\leqslant\frac{1}{10}\}$ with the property that $\hat{\psi}_\circ$ and all its derivatives up to order $20d$ vanish at the origin but $\hat{\psi}_\circ(\xi)>0$ on $\{\xi:\frac{1}{4}\leqslant|\xi|\leqslant 4\}$. This is easy to achieve (take a radial function $\chi\in C_0^\infty$ such that $\widehat{\chi}(0)=1$, then define $\psi_\circ=\lambda^d\Delta^{10d}[\chi(\lambda\,\cdot)]$ for a sufficiently large λ; here Δ denotes the Laplacian in $\mathbb{R}^d$).

Let $\eta=\mathcal{F}^{-1}[\hat{\eta}_\circ(\hat{\psi}_\circ)^{-2}]$. Then

$$K*f=\psi_\circ*K*\psi_\circ*g,$$

where $g=\eta*f$ and clearly $\|g\|_p\lesssim\|f\|_p$. We split $K=K_0+K_\infty$, where $K_0=K\chi_{\{x:|x|\leqslant 1\}}$. Since $\|K_0\|_1\lesssim\|K\|_p$, the operator of convolution with K_0 is clearly bounded on all L^p, $1\leqslant p\leqslant\infty$, with operator norm $O(\|K\|_p)$. Therefore it suffices to show that the L^p norm of $\psi_\circ*K_\infty*\psi_\circ*g$ is controlled by $C\|K\|_p\|g\|_p$. We set $\psi=\psi_\circ*\psi_\circ$ and observe that

$$\psi*K_\infty*g=\int_1^\infty\int_{\mathbb{R}^d}\psi*\sigma_r(\cdot-y)\varkappa(r)g(y)\,dy\,dr. \tag{5.2}$$

By Corollary 3.2,

$$\|\psi*K_\infty*g\|_p\lesssim\biggl(\int_1^\infty|\varkappa(r)|^p r^{d-1}\,dr\biggr)^{1/p}\biggl(\int_{\mathbb{R}^d}|g(y)|^p\,dy\biggr)^{1/p}.$$

This establishes (5.1).

6. A variant of Corollary 3.2 involving large radii

The following estimate for convolution operators with radial kernels will be used in conjunction with atomic decompositions to extend the one scale situation of §5 to the general case. We consider radial kernels with cancellation that are supported in the set $\{x:|x|>2^\ell\}$. The crucial feature is an exponential gain in ℓ, which will be useful when putting different scales together. For $\nu\in\mathbb{Z}$, let $\mathcal{W}^\nu$ be the tiling of $\mathbb{R}^d$ with dyadic cubes of sidelength 2^ν, i.e. the set of cubes of the form

$$[z_1 2^\nu,(z_1+1)2^\nu)\times\ldots\times[z_d 2^\nu,(z_d+1)2^\nu), \quad z=(z_1,\ldots,z_d)\in\mathbb{Z}^d.$$

Proposition 6.1. *Let $1<p<p_d$ and $\varepsilon<(d-1)(1/p-1/p_d)$. Let $\ell\geqslant 0$. Let K be a radial convolution kernel supported in $\{x:|x|>2^\ell\}$. For $s\in\mathbb{Z}$, let $K_s=2^{sd}K(2^s\cdot)$ and $\psi_s=2^{sd}\psi(2^s\cdot)$. Then*

$$\|\psi_s*K_s*g\|_p\lesssim\|K\|_p 2^{-\ell\varepsilon}\biggl(\sum_{W\in\mathcal{W}^{\ell-s}}\operatorname{meas}(W)\|g\chi_W\|_\infty^p\biggr)^{1/p}. \tag{6.1}$$

The implicit constant in (6.1) depends on ε.

We prove a variant of Corollary 3.2, which involves only radii $r\geqslant 2^\ell$ and corresponds to the case $s=0$ of the proposition. Let $F_{y,r}$ be as in (3.1).

Lemma 6.2. *Let $1<p<p_d$ and $\varepsilon<(d-1)(1/p-1/p_d)$. Then, for $\ell\geqslant 0$,*

$$\biggl\|\int_{\mathbb{R}^d}\int_{2^\ell}^\infty h(y,r)F_{y,r}\,dr\,dy\biggr\|_p\lesssim 2^{-\ell\varepsilon}2^{\ell d/p}\biggl(\int_{2^\ell}^\infty\sum_{W\in\mathcal{W}^\ell}\sup_{y\in W}|h(y,r)|^p r^{d-1}\,dr\biggr)^{1/p}. \tag{6.2}$$

Proof. We shall base the proof on the arguments in §3 and first prove a discretized version. Let $\mathcal{Y}$ and $\mathcal{R}$ be 1-separated subsets of $\mathbb{R}^d$ and $[1,\infty)$, respectively. Inequality (6.2) follows from the following discretized version by the averaging argument employed in the proof of Corollary 3.2:

$$\biggl\|\sum_{\substack{(y,r)\in\mathcal{Y}\times\mathcal{R}\\ r\geqslant 2^\ell}}\gamma(y,r)F_{y,r}\biggr\|_p\lesssim 2^{-\ell\varepsilon}2^{\ell d/p}\biggl(\sum_{r\in\mathcal{R}}\sum_{W\in\mathcal{W}^\ell}\sup_{y\in\mathcal{Y}\cap W}|\gamma(y,r)|^p r^{d-1}\biggr)^{1/p}. \tag{6.3}$$

For $j\in\mathbb{Z}$ and $r\in\mathcal{R}$, let $\mathcal{W}^\ell(j,r)$ be the set of all $W\in\mathcal{W}^\ell$ for which

$$2^j\leqslant\sup_{x\in W}|\gamma(x,r)|<2^{j+1}.$$

For each $y\in\mathcal{Y}$, let $W(y)$ be the unique cube in $\mathcal{W}^\ell$ that contains y, and for each $j\in\mathbb{Z}$, let $\mathcal{E}_k(j)$ be the set of all $(y,r)\in\mathcal{Y}\times\mathcal{R}_k$ with the property that $W(y)\in\mathcal{W}^\ell(j,r)$. Apply the density decomposition of Lemma 3.7 to the sets $\mathcal{E}_k(j)$ and write $\mathcal{E}_k(j)=\sum_{u\in\mathcal{U}}\mathcal{E}_k(j,u)$ as in that lemma. Lemma 3.9 applied to the set $\bigcup_{k=\ell}^\infty\mathcal{E}_k(j,u)$ yields

$$\biggl\|\sum_{(y,r)\in\bigcup_{k=\ell}^\infty\mathcal{E}_k(j,u)}\gamma(y,r)F_{y,r}\biggr\|_p^p\lesssim u^{-\delta p}2^{jp}\sum_{k=\ell}^\infty\sum_{(y,r)\in\mathcal{E}_k(j,u)}r^{d-1} \tag{6.4}$$

for $\delta<1/p-1/p_d$. We now use the fact that $\mathcal{E}_k(j,u)$ is of density type $(u,2^k)$. Since $k\geqslant\ell$, this implies that for every $u\in\mathcal{U}$, every j, every $W\in\mathcal{W}^\ell$ and every $r\in[2^k,2^{k+1})$ the slice $\mathcal{E}_k(j,u,W,r):=\{y\in\mathcal{Y}\cap W:(y,r)\in\mathcal{E}_k(j,u)\}$ contains $O(u2^\ell)$ points. Also, since $\mathcal{Y}$ is

1-separated, the cardinality of each slice is $\lesssim 2^{\ell d}$. Therefore the right-hand side of (6.4) is controlled by

$$2^{jp}u^{-\delta p}\sum_{k=\ell}^{\infty}\sum_{r\in\mathcal{R}_k} r^{d-1}\sum_{W\in\mathcal{W}^\ell}\#\mathcal{E}_k(j,u,W,r)\lesssim 2^{jp}C(\ell,u)\sum_{k=\ell}^{\infty}\sum_{r\in\mathcal{R}_k} r^{d-1}\#\mathcal{W}^\ell(j,r),$$

with $C(\ell,u):=u^{-\delta p}\min\{u2^\ell,\, 2^{\ell d}\}$. By interpolation (Lemma 2.2),

$$\begin{aligned}\bigg\|\sum_{j\in\mathbb{Z}}\sum_{(y,r)\in\bigcup_{k=\ell}^{\infty}\mathcal{E}_k(j,u)}\gamma(y,r)F_{y,r}\bigg\|_p^p &\lesssim C(\ell,u)\sum_{j\in\mathbb{Z}}2^{jp}\sum_{k=\ell}^{\infty}\sum_{r\in\mathcal{R}_k} r^{d-1}\#\mathcal{W}^\ell(j,r)\\ &\lesssim C(\ell,u)\sum_{W\in\mathcal{W}^\ell}\sum_{r\in\mathcal{R}} r^{d-1}\sup_{y\in W}|\gamma(y,r)|^p.\end{aligned}$$

We sum geometric progressions to get

$$\sum_{u\in\mathcal{U}} C(\ell,u)^{1/p}\lesssim 2^{-\ell\delta(d-1)}2^{\ell d/p}.$$

Hence, with $\varepsilon=(d-1)\delta$,

$$\bigg\|\sum_{j\in\mathbb{Z}}\sum_{(y,r)\in\bigcup_{k=\ell}^{\infty}\mathcal{E}_k(j)}\gamma(y,r)F_{y,r}\bigg\|_p^p\lesssim 2^{-\ell\varepsilon p}\sum_{r\in\mathcal{R}} r^{d-1}\sum_{W\in\mathcal{W}^\ell}\operatorname{meas}(W)\sup_{y\in W}|\gamma(y,r)|^p.$$

This proves (6.3). □

Proof of Proposition 6.1. By scaling, we may assume that $s=0$. As in §5, we write

$$\psi*K*g=\int_{2^\ell}^{\infty}\int_{\mathbb{R}^d}\psi*\sigma_r(\cdot-y)\,\varkappa(r)g(y)\,dy\,dr.$$

Apply Lemma 6.2 with $h(y,r)=\varkappa(r)g(y)$ and notice that the right-hand side of (6.2) is equal to

$$2^{-\ell\varepsilon}\bigg(\int_{2^\ell}^{\infty}|\varkappa(r)|^p r^{d-1}\,dr\bigg)^{1/p}\bigg(\sum_{W\in\mathcal{W}^\ell}\operatorname{meas}(W)\|g\chi_W\|_\infty^p\bigg)^{1/p}. \qquad \square$$

7. Atomic decompositions and the proof of Theorem 1.1

The purpose of this chapter is to prove Theorem 1.1 for one particular Schwartz function η whose Fourier transform is compactly supported away from the origin (for the extension to more general η see §8). We follow the presentation in §3.1 and introduce a radial Schwartz function $\eta_\circ$ such that $\hat{\eta}_\circ$ is supported in $\{\xi:\frac{1}{2}<|\xi|<2\}$ and satisfies

$$\sum_{s\in\mathbb{Z}}[\hat{\eta}_\circ(2^{-s}\xi)]^2=1 \tag{7.1}$$

for all $\xi\neq 0$. Let $\psi_\circ$ be a C^∞ function compactly supported in $\left\{x:|x|\leqslant\frac{1}{10}\right\}$ such that $\hat{\psi}_\circ$ does not vanish in $\left\{\xi:\frac{1}{4}\leqslant|\xi|\leqslant 4\right\}$ and does vanish to order $10d$ at the origin. Let $\psi=\psi_\circ*\psi_\circ$ and

$$\eta=\mathcal{F}^{-1}\left[\frac{\hat{\eta}_\circ}{\hat{\psi}}\right]. \tag{7.2}$$

We shall use this particular η in the assumption of our theorem; in other words, we shall assume that $\sup_{t>0}\|T_m[t^{d/p}\eta(t\,\cdot)]\|_p\leqslant B_p<\infty$. For $s\in\mathbb{Z}$, let

$$H_s=\mathcal{F}^{-1}[\hat{\eta}(\cdot)m(2^s\,\cdot)].$$

By our assumption,

$$\sup_{s\in\mathbb{Z}}\|H_s\|_p\leqslant B_p. \tag{7.3}$$

Now let $K_s=2^{sd}H_s(2^s\,\cdot)$, $\psi_s=2^{sd}\psi(2^s\,\cdot)=2^{sd}(\psi_\circ*\psi_\circ)(2^s\,\cdot)$ and $\eta_s=2^{sd}\eta(2^s\,\cdot)$. By (7.1) and our definitions, we have the decomposition

$$T_mf=\sum_{s\in\mathbb{Z}}\psi_s*\psi_s*K_s*f_s,$$

where

$$f_s=\eta_s*f. \tag{7.4}$$

We may assume that f is a Schwartz function whose Fourier transform is compactly supported away from the origin; this class is dense in $L^p(\mathbb{R}^d)$, $1<p<\infty$. For those functions, the sum in s is finite.

We shall work with atomic decompositions constructed from Peetre's maximal square function (cf. [7], [19], [22] and [28]) using ideas from work by Chang and Fefferman [4]. The non-tangential version of Peetre's expression is

$$Sf(x)=\left(\sum_{s\in\mathbb{Z}}\sup_{|y|\leqslant 10d2^{-s}}|f_s(x+y)|^2\right)^{1/2}.$$

Then the L^p norm of Sf is controlled by $\|f\|_p$ in case $1<p<\infty$, and by the Hardy space (quasi-)norm $\|f\|_{H^p}$ if $p\leqslant 1$. These statements follow, for example, from the Fefferman–Stein inequalities for the vector-valued Hardy–Littlewood maximal operator ([6]).

Put $\Psi_s=\psi_s*\psi_s$. The proof of the L^p boundedness of T_m reduces to the inequality

$$\left\|\sum_{s\in\mathbb{Z}}\Psi_s*K_s*f_s\right\|_p\lesssim B_p\|Sf\|_p,\quad 1<p<p_d; \tag{7.5}$$

here we now assume that the sum in s is over a finite set of integers. In what follows, we will make several decompositions of the Schwartz functions f_s (involving even rough cutoffs) and the a-priori convergence of various sums can be justified by using the rapid decay of the functions.

The cancellation of the functions ψ_s is crucial for the estimation of the left-hand side in (7.5) and various similar expressions. A simple tool is the inequality

$$\Big\|\sum_{s\in\mathbb{Z}}\psi_s*h_s\Big\|_\tau\leqslant C\bigg(\sum_{s\in\mathbb{Z}}\|h_s\|_\tau^\tau\bigg)^{1/\tau},\quad 1\leqslant\tau\leqslant 2, \tag{7.6}$$

with a constant C depending only on ψ. This is immediate from Plancherel's theorem for $\tau=2$, trivial for $\tau=1$ and true by interpolation for $1<\tau<2$. Inequality (7.6) is not enough to put the estimates for the various scales together, and in addition we have to use an "atomic decomposition" of each f_s, which we now describe.

For fixed s, we tile $\mathbb{R}^d$ by the dyadic cubes of sidelength 2^{-s}; and we shall write $L(Q)=-s$ to indicate that the sidelength of a dyadic cube is 2^{-s}. For each integer j, we introduce the set $\Omega_j=\{x:Sf(x)>2^j\}$. Let $\mathcal{Q}_j^s$ be the set of all dyadic cubes for which $L(Q)=-s$ and which have the property that $|Q\cap\Omega_j|\geqslant\frac{1}{2}|Q|$ but $|Q\cap\Omega_{j+1}|<\frac{1}{2}|Q|$. We also set

$$\Omega_j^*=\{x:M\chi_{\Omega_j}(x)>100^{-d}\},$$

where M is the Hardy–Littlewood maximal operator. Note that Ω_j^* is an open set containing Ω_j and $|\Omega_j^*|\lesssim|\Omega_j|$. We work with a Whitney decomposition $\mathcal{W}_j$ of Ω_j^* into dyadic cubes W. Specifically, $\mathcal{W}_j$ is the set of all dyadic cubes W such that the 20-fold dilate of W is contained in Ω_j^* and W is maximal with respect to this property. We note that each $Q\in\mathcal{Q}_j^s$ is contained in a unique $W\in\mathcal{W}_j$. This is verified by showing that the 20-fold dilate Q^* of Q belongs to Ω_j^*. Indeed,

$$\frac{|Q^*\cap\Omega_j|}{|Q^*|}\geqslant 20^{-d}\frac{|Q\cap\Omega_j|}{|Q|}\geqslant 40^{-d};$$

and hence $Q^*\subset\Omega_j^*$. We shall also need that the quadruple dilates W^* of W, $W\in\mathcal{W}_j$, have bounded overlap (uniformly in j).

We now define some building blocks that are analogous to the usual atoms; however they are not normalized and, since we are mainly interested in L^p bounds for $p>1$, we do not insist on cancellation. For each $W\in\mathcal{W}_j$, set

$$A_{s,W,j}=\sum_{\substack{Q\in\mathcal{Q}_j^s\\ Q\subset W}}f_s\chi_Q;$$

note that only terms with $L(W)+s\geqslant 0$ occur. We also need to consider "cumulative atoms", as any dyadic cube W can be a Whitney cube for several Ω_j^*. We set

$$A_{s,W}=\sum_{j:W\in\mathcal{W}_j}A_{s,W,j}.$$

Note that

$$f_s = \sum_{W\in\bigcup_{j\in\mathbb{Z}}\mathcal{W}_j} A_{s,W} = \sum_{j\in\mathbb{Z}}\sum_{W\in\mathcal{W}_j} A_{s,W,j}.$$

The following observations about atomic decompositions are standard (see, e.g., [4]), but included here for completeness.

LEMMA 7.1. *For each $j\in\mathbb{Z}$, the following inequalities hold*:

(i)

$$\sum_{W\in\mathcal{W}_j}\sum_{s\in\mathbb{Z}} \|A_{s,W,j}\|_2^2 \lesssim 2^{2j}\operatorname{meas}(\Omega_j);$$

(ii) *there is a constant C_d such that for every assignment $W\mapsto s(W)$ defined on $\mathcal{W}_j$, and every $0\leqslant p\leqslant 2$,*

$$\sum_{W\in\mathcal{W}_j}\operatorname{meas}(W)\|A_{s(W),W,j}\|_\infty^p \leqslant C_d 2^{pj}\operatorname{meas}(\Omega_j).$$

Proof. Using the definitions of the atoms, part (i) follows from the inequality

$$\sum_{s\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_j^s}\|f_s\chi_Q\|_2^2 \lesssim 2^{2j}\operatorname{meas}(\Omega_j).$$

To see this, observe that $\operatorname{meas}(Q\setminus\Omega_{j+1})\geqslant\frac{1}{2}\operatorname{meas}(Q)$ for each $Q\in\mathcal{Q}_j^s$, and we also have $Q\subset\Omega_j^*$. We use this together with Fubini's theorem and see that the left-hand side of (i) is bounded by

$$\begin{aligned}\sum_{s\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_j^s}\operatorname{meas}(Q)\|f_s\chi_Q\|_\infty^2 &\leqslant \sum_{s\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_j^s} 2\operatorname{meas}(Q\setminus\Omega_{j+1})\|f_s\chi_Q\|_\infty^2\\ &\leqslant 2\int_{\Omega_j^*\setminus\Omega_{j+1}}\biggl(\sum_{s\in\mathbb{Z}}\sup_{|y|\leqslant 2^{-s}\sqrt{d}}|f_s(x+y)|^2\biggr)dx\\ &\leqslant 2\cdot 2^{2(j+1)}\operatorname{meas}(\Omega_j^*),\end{aligned}$$

which is $\lesssim 2^{2j}\operatorname{meas}(\Omega_j)$.

Part (ii) of the lemma follows since

$$\|A_{s,W,j}\|_\infty \lesssim \sup_{\substack{Q\in\mathcal{Q}_j^s\\ Q\subset W}}|f_s\chi_Q| \leqslant \sup_{x\in\Omega_j^*\setminus\Omega_{j+1}}|Sf(x)| \leqslant 2^{j+1}$$

and $\sum_{W\in\mathcal{W}_j}|W|\leqslant|\Omega_j^*|\lesssim|\Omega_j|$. $\square$

To establish (7.5), we need to verify the inequality

$$\Big\| \sum_{s,j\in\mathbb{Z}} \sum_{\ell=0}^{\infty} \sum_{\substack{W\in\mathcal{W}_j\\ L(W)=-s+\ell}} \Psi_s * K_s * A_{s,W,j} \Big\|_p \lesssim B_p \|Sf\|_p. \tag{7.7}$$

For each integer ℓ in this sum, we split the convolution operator K_s into short range and long range pieces, $K^{\rm sh}_{s,\ell}$ and $K^{\rm lg}_{s,\ell}$. To define them, we first look at the rescaled kernels H_s and set

$$H^{\rm sh}_{s,\ell}(x) = \begin{cases} H_s(x), & \text{if } |x|\leqslant 2^\ell,\\ 0, & \text{if } |x|>2^\ell. \end{cases}$$

Also $H^{\rm lg}_{s,\ell}(x)=H_s(x)-H^{\rm sh}_{s,\ell}$. Now set $K^{\rm sh}_{s,\ell}=2^{sd}H^{\rm sh}_{s,\ell}(2^s\,\cdot)$ and $K^{\rm lg}_{s,\ell}=2^{sd}H^{\rm lg}_{s,\ell}(2^s\,\cdot)$. Finally, we split the sum in (7.7) into two parts, replacing K_s by $K^{\rm sh}_{s,\ell}$ and $K^{\rm lg}_{s,\ell}$, respectively.

Now consider W with $L(W)=-s+\ell$ and note that the short range convolution $\psi_s * K^{\rm sh}_{s,\ell} * A_{s,W,j}$ is supported in the quadruple dilate W^* of W; thus, for fixed j, all these terms are supported in Ω_j^*. We prove the short range inequality

$$\Big\| \sum_{s,j\in\mathbb{Z}} \sum_{\ell=0}^{\infty} \sum_{\substack{W\in\mathcal{W}_j\\ L(W)=-s+\ell}} \Psi_s * K^{\rm sh}_{s,\ell} * A_{s,W,j} \Big\|_\tau \lesssim B_p \|Sf\|_\tau \tag{7.8}$$

for $p<2d/(d+1)$ and $\tau<2$. The choice $\tau=p$ is, of course, permitted for the p-range of Theorem 1.1. To prove (7.8), it suffices to show that for fixed j, and for $p<2d/(d+1)$ and $\tau\leqslant 2$,

$$\Big\| \sum_{s\in\mathbb{Z}} \sum_{\ell=0}^{\mathbb{Z}} \sum_{\substack{W\in\mathcal{W}_j\\ L(W)=-s+\ell}} \Psi_s * K^{\rm sh}_{s,\ell} * A_{s,W,j} \Big\|_\tau^\tau \lesssim B_p^\tau 2^{j\tau} \operatorname{meas}(\Omega_j). \tag{7.9}$$

Indeed, by Lemma 2.2, inequality (7.9) implies that the left-hand side of (7.8) is controlled by $B_p^\tau \sum_j 2^{j\tau}\operatorname{meas}(\Omega_j) \lesssim B_p^\tau \|Sf\|_\tau^\tau$ for $\tau<2$.

Inequality (7.9) for $\tau<2$ follows from (7.9), and for $\tau=2$ by Hölder's inequality. Here we use the fact that the relevant expressions are supported in Ω_j^* and $|\Omega_j^*|\lesssim|\Omega_j|$. To prove (7.9) for $\tau=2$, we use a standard estimate for the Fourier transform of radial kernels $K=\int_0^\infty \varkappa(r)\sigma_r\,dr$, namely,

$$\|\widehat{K}\hat{\psi}\|_\infty \leqslant C_p\|K\|_p = c\bigg(\int_0^\infty |\varkappa(r)|^p r^{d-1}\,dr\bigg)^{1/p}, \quad p<\frac{2d}{d+1}. \tag{7.10}$$

Indeed using Bessel functions as in the proof of Lemma 3.3, one can use Hölder's inequality to estimate

$$\begin{aligned} |\widehat{K}(\xi)| &= c'\int_0^\infty \varkappa(r) r^{d-1} B_d(r|\xi|)\,dr \\ &\lesssim \bigg(\int_0^\infty |\varkappa(r)|^p r^{d-1}\,dr\bigg)^{1/p} \bigg(\int_0^\infty r^{d-1}(1+r|\xi|)^{-(d-1)p'/2}\,dr\bigg)^{1/p'}. \end{aligned}$$

It is easy to see that the last $L^{p'}$ norm is $O(|\xi|^{-d/p'})$, provided that $p<2d/(d+1)$. The bound (7.10) follows since $\hat{\psi}$ is a Schwartz function that vanishes to high order at 0.

We return to (7.9) for $\tau=2$. As $\Psi_s * K^{\mathrm{sh}}_{s,\ell} * A_{s,W,j}$ is supported in W^* and the W^* have bounded overlap, we can dominate the left-hand side of the inequality by

$$\begin{aligned} &\Big\| \sum_{W\in\mathcal{W}_j} \sum_s \psi_s*\psi_s*K^{\mathrm{sh}}_{s,L(W)+s}*A_{s,W,j}\Big\|_2^2 \\ &\qquad\lesssim \sum_{W\in\mathcal{W}_j} \Big\| \sum_s \psi_s*\psi_s*K^{\mathrm{sh}}_{s,L(W)+s}*A_{s,W,j}\Big\|_2^2 \\ &\qquad\lesssim \sum_{W\in\mathcal{W}_j} \sum_s \|\psi_s*K^{\mathrm{sh}}_{s,L(W)+s}*A_{s,W,j}\|_2^2 \\ &\qquad\lesssim \sup_{s,\nu} \|\hat{\psi}_s \widehat{K^{\mathrm{sh}}_{s,\nu}}\|_\infty^2 \sum_{W\in\mathcal{W}_j} \sum_s \|A_{s,W,j}\|_2^2. \end{aligned} \tag{7.11}$$

Here we used the L^2 case of (7.6). Now, by (7.10), the Fourier transform of $\psi_s*K^{\mathrm{sh}}_{s,\nu}$ has L^∞ norm $\lesssim\|H^{\mathrm{sh}}_{s,\nu}\|_p\lesssim\|H_s\|_p\leqslant B_p$. Thus, by Lemma 7.1 (i), the last displayed quantity is $\lesssim B_p^2 2^{2j}|\Omega_j|$. This finishes the proof of (7.9).

We now turn to the long range estimate, that is,

$$\Big\| \sum_{s,j\in\mathbb{Z}} \sum_{\ell=0}^{\infty} \sum_{\substack{W\in\mathcal{W}_j \\ L(W)=-s+\ell}} \psi_s*\psi_s*K^{\mathrm{lg}}_{s,\ell}*A_{s,W,j}\Big\|_p \lesssim B_p\|Sf\|_p. \tag{7.12}$$

We use the j-sum to combine the atoms into the cumulative atoms $A_{s,W}$, take out the ℓ-sum by Minkowski's inequality, and use (7.6). Thus the left-hand side of (7.12) is dominated by a constant times

$$\sum_{\ell=0}^{\infty}\bigg(\sum_{s\in\mathbb{Z}}\Big\|\psi_s*K^{\mathrm{lg}}_{s,\ell}*\sum_{W:L(W)=-s+\ell}A_{s,W}\Big\|_p^p\bigg)^{1/p}. \tag{7.13}$$

Now $\|H^{\mathrm{lg}}_{s,\ell}\|_p\leqslant\|H_s\|_p\leqslant B_p$ and therefore Proposition 6.1 implies that, for fixed ℓ,

$$\Big\|\psi_s*K^{\mathrm{lg}}_{s,\ell}*\sum_{W:L(W)=-s+\ell}A_{s,W}\Big\|_p \lesssim 2^{-\ell\varepsilon}B_p\bigg(\sum_{W:L(W)=-s+\ell}\mathrm{meas}(W)\|A_{s,W}\|_\infty^p\bigg)^{1/p} \tag{7.14}$$

for $p<p_d$, with some $\varepsilon=\varepsilon(p)>0$. Note that for fixed s and W, the functions $A_{s,W,j}$ live on disjoint sets (since the dyadic cubes of sidelength 2^{-s} are disjoint and each is in exactly one family $\mathcal{Q}^s_j$). Thus, clearly,

$$\|A_{s,W}\|_\infty^p \lesssim \sum_{j\in\mathbb{Z}}\|A_{s,W,j}\|_\infty^p.$$

It follows that the expression (7.13) is

$$\begin{aligned}&\lesssim B_p \sum_{\ell=0}^{\infty} 2^{-\ell\varepsilon} \biggl(\sum_{j\in\mathbb{Z}} \sum_{W\in\mathcal{W}_j} \operatorname{meas}(W) \|A_{\ell-L(W),W,j}\|_{L^\infty(W)}^p \biggr)^{1/p} \\ &\lesssim B_p \sum_{\ell=0}^{\infty} 2^{-\ell\varepsilon} \biggl(\sum_{j\in\mathbb{Z}} \operatorname{meas}(\Omega_j) 2^{jp} \biggr)^{1/p} \lesssim B_p \|Sf\|_p,\end{aligned}$$

by part (ii) of Lemma 7.1. This yields (7.12). Finally, (7.7) follows from (7.8) and (7.12). This concludes the proof of the L^p boundedness of T_m under the assumption (7.3).

8. Conclusion of the proof

We still have to prove (1.1) for an arbitrary choice of η. To this end, we fix the radial multiplier m and consider the family Θ of all C^∞ functions φ compactly supported away from the origin such that the condition

$$\|\mathcal{F}^{-1}[\varphi m(t\,\cdot)]\,\|_p < \infty \tag{8.1}$$

holds. Note that if $\varphi\in\Theta$, then $\varphi(\lambda\,\cdot)\in\Theta$ for every $\lambda>0$, moreover $\varphi\circ R\in\Theta$ for every rotation R of $\mathbb{R}^d$ (here we use the fact that m is radial). Also, if χ is any compactly supported C^∞ function, then $\chi\varphi\in\Theta$, simply because χ is an $\mathcal{F}L^p$ multiplier. Finally, if $\varphi_1,\varphi_2\in\Theta$, then $\varphi_1+\varphi_2\in\Theta$.

Now assume that there exists at least one non-identically-zero function $\varphi_\circ\in\Theta$. Let V be a non-empty open subset of $\mathbb{R}^{d+1}$ such that $|\varphi_\circ|>0$ on V. Let φ be any other C^∞ function compactly supported away from the origin. For every $\xi\in\mathbb{R}^d\setminus\{0\}$, one can find a rotation R_ξ and a number $\lambda_\xi>0$ such that $\lambda_\xi R_\xi \xi\in V$ or, equivalently, $\xi\in\lambda_\xi^{-1}R_\xi^{-1}V$. Then the open sets $\lambda_\xi^{-1}R_\xi^{-1}V$, $\xi\in\operatorname{supp}\varphi$, form a cover of $\operatorname{supp}\varphi$. Choose a finite subcover $\lambda_{\xi_j}^{-1}R_{\xi_j}^{-1}V$, $j=1,\dots,n$, and put

$$\zeta = \sum_{j=1}^{n} \overline{\varphi_\circ(\lambda_{\xi_j}R_{\xi_j}\,\cdot)}\,\varphi_\circ(\lambda_{\xi_j}R_{\xi_j}\,\cdot).$$

Note that $\zeta\in\Theta$ and $\zeta>0$ on $\bigcup_{j=1}^n \lambda_{\xi_j}^{-1}R_{\xi_j}^{-1}V\supset\operatorname{supp}\varphi$. Hence, the function χ defined as φ/ζ on $\operatorname{supp}\varphi$ and 0 on $\mathbb{R}^d\setminus\operatorname{supp}\varphi$ is a C^∞ function with compact support, so $\varphi=\chi\zeta\in\Theta$.

Proof of Theorem 1.1 (concluded). Let g be an arbitrary Schwartz function, then the condition

$$\sup_{t>0} \|T_m[t^{d/p}g(\cdot)]\,\|_p < \infty$$

is clearly necessary for L^p boundedness. Conversely, suppose that this condition is satisfied; it is equivalent to

$$\sup_{t>0} \|\mathcal{F}^{-1}[m(t\cdot)\hat{g}]\|_p < \infty.$$

We may pick $\chi\in C^\infty$ with compact support in $\mathbb{R}^d\setminus\{0\}$ so that $\chi\hat{g}$ is not identically 0. Since χ is a Fourier multiplier, we see that $\chi\hat{g}\in\Theta$. By the above considerations, we also have $\hat{\eta}\in\Theta$, where η is as in (7.2). But for this η, the characterization is already proved and the L^p boundedness of T_m follows. □

9. Variants and extensions

Hardy space estimates

We now give an extension of Theorem 1.1 to the range $p\leqslant 1$. We prove, in dimension $d\geqslant 2$, a full characterization of the convolution operators with radial kernels mapping the Hardy space H^p to L^p.

THEOREM 9.1. *Suppose that $d\geqslant 2$ and $0<p\leqslant 1$. Let m be radial and let η be a Schwartz function whose Fourier transform is compactly supported away from the origin and is not identically* 0. *Then*

$$\|T_m\|_{H^p\to L^p} \asymp \sup_{t>0} t^{d/p} \|T_m[\eta(t\cdot)]\|_{L^p}.$$

Remarks. (i) The $H^p\to L^p$ boundedness is equivalent to the $H^p\to H^p$ boundedness, by [16, Theorem 3.4].

(ii) The proof is substantially simpler than the L^p result for $p>1$; in particular, the crucial orthogonality Lemma 3.3 plays no role, and is replaced by the L^∞ multiplier bound (4.3). This allows us to include dimensions 2 and 3.

Sketch of proof of Theorem 9.1. We first note the chain of inequalities

$$\|\widehat{H}\|_\infty \leqslant \|H\|_1 \leqslant \sum_{z\in\mathbb{Z}^d} \sup_{y\in[0,1]^d} |H(z+y)| \leqslant \biggl(\sum_{z\in\mathbb{Z}^d} \sup_{y\in[0,1]^d} |H(z+y)|^p\biggr)^{1/p},$$

since $p\leqslant 1$. Now note that if $\widehat{H}$ is supported in $\{\xi:|\xi|\leqslant 2\}$, then the last expression is $O(\|K\|_p)$, by a Plancherel–Pólya-type estimate (cf. [28, §1.3.3]).

Now the proof of the short range estimate (7.8) for $\tau\leqslant 1$ is rather similar to the argument in §7. Note that $\Psi_s * K^{\mathrm{sh}}_{s,\ell} * A_{s,W,j}$ is supported in $W^*\subset\Omega_j^*$. Thus we can bound the left-hand side of (7.8), for $\tau\leqslant 1$, by

$$\biggl(\sum_{j\in\mathbb{Z}} |\Omega_j|^{1-\tau/2} \biggl\| \sum_{W\in\mathcal{W}_j} \sum_{s\in\mathbb{Z}} \sum_{\ell=0}^{\infty} \Psi_s * K^{\mathrm{sh}}_{s,\ell} * A_{s,W,j} \biggr\|_2^\tau \biggr)^{1/\tau}.$$

By (7.11) and Lemma 7.1 (i), this is dominated by

$$\sup_{s,\nu} \|H^{\mathrm{sh}}_{s,\nu}\|_p \biggl(\sum_{j\in\mathbb{Z}} |\Omega_j|^{1-\tau/2}(2^{2j}|\Omega_j|)^{\tau/2}\biggr)^{1/\tau},$$

which is $\lesssim \sup_s \|H_s\|_p \|Sf\|_\tau$. Of course, we may choose $\tau=p$.

We prove the analogue of the long range estimate (7.12). As $p\leqslant 1$, we can apply the triangle inequality for the pth power of the L^p (quasi)-norm for the sums in s, ℓ, j and W. After rescaling to the case $s=0$, matters are reduced to the estimation of the convolution with a radial kernel

$$\int_{2^\ell}^\infty \varkappa(r)\sigma_r*\psi_0\,dr,$$

where $\varkappa(|\cdot|)$ is the Fourier transform of a function supported in $\{\xi:\frac12<|\xi|<2\}$. The relevant estimate is then

$$\biggl\|\int_{2^\ell}^\infty \varkappa(r)\sigma_r*\psi_0*A_{0,W,j}\,dr\biggr\|_p \lesssim 2^{-\ell\varepsilon(p)}\biggl(\int_{2^\ell}^\infty |\varkappa(r)|^p r^{d-1} dr\biggr)^{1/p} |W|^{1/p}\|A_{0,W,j}\|_\infty, \quad (9.1)$$

where $|W|=2^{\ell d}$. Now let $\varkappa_n^*=\sup_{n\leqslant r\leqslant n+1}|\varkappa(r)|$. We shall establish that

$$\biggl\|\int_{2^\ell}^\infty \varkappa(r)\sigma_r*\psi_0*A_{0,W,j}\,dr\biggr\|_p \lesssim 2^{-\ell\varepsilon(p)}\biggl(\sum_{n=2^\ell}^\infty |\varkappa_n^*|^p n^{d-1}\biggr)^{1/p} |W|^{1/p}\|A_{0,W,j}\|_\infty, \quad (9.2)$$

and (9.1) will follow by the Plancherel–Pólya-type estimate

$$\biggl(\sum_{n=2^\ell}^\infty |\varkappa_n^*|^p n^{d-1}\biggr)^{1/p} \lesssim_p \biggl(\int_{2^\ell}^\infty |\varkappa(r)|^p r^{d-1}\,dr\biggr)^{1/p}.$$

We now prove (9.2), with $\varepsilon(p)=(d-1)(1/p-1/2)$. Since $p\leqslant 1$, the left-hand side is dominated by

$$\biggl(\sum_{n=2^\ell}^\infty \biggl\|\int_n^{n+1} \varkappa(r)\sigma_r*\psi_0*A_{0,W,j}\,dr\biggr\|_p^p\biggr)^{1/p}.$$

As $n\geqslant 2^\ell$, the term $\sigma_r*\psi_0*A_{s,W,j}$, for $n\leqslant r\leqslant n+1$, is supported in an annulus with width $c2^\ell$ and inner and outer radii comparable to n, and hence of measure $\lesssim n^{d-1}2^\ell$. By (4.3),

$$\sup_\xi \biggl|\int_n^{n+1} \varkappa(r)\hat\sigma_r(\xi)\hat\psi_0(\xi)\,dr\biggr| \lesssim |\varkappa_n^*| n^{(d-1)/2}.$$

We use Hölder's inequality and estimate

$$\biggl\|\int_n^{n+1} \varkappa(r)\sigma_r*\psi_0*A_{0,W,j}\,dr\biggr\|_p$$

by

$$(n^{d-1}2^{\ell})^{1/p-1/2}\bigg\|\int_n^{n+1} \varkappa(r)\sigma_r*\psi_0*A_{0,W,j}\,dr\bigg\|_2 \lesssim \varkappa_n^* n^{(d-1)/p}2^{\ell(1/p-1/2)}\|A_{0,W,j}\|_2.$$

But $\|A_{0,W,j}\|_2\lesssim 2^{\ell d/2}\|A_{0,W,j}\|_\infty$, and therefore the last displayed expression is controlled by

$$\varkappa_n^* n^{(d-1)/p}2^{\ell((d-1)/2+1/p)}\|A_{0,W,j}\|_\infty \lesssim \varkappa_n^* n^{(d-1)/p}2^{-\ell(d-1)(1/p-1/2)}|W|^{1/p}\|A_{0,W,j}\|_\infty.$$

Finally, we remark that the arguments in §8 carry over to the H^p case, $p\leqslant 1$. □

Lorentz space estimates

Weak type (p,p) (i.e. $L^p\to L^{p,\infty}$) estimates for convolutions with radial kernels, in particular for Bochner–Riesz means, have been considered in [25] and the references therein. We shall indicate here how to prove $L^p\to L^{p,\nu}$ estimates by combining our previous arguments with interpolation by the real method (the general Marcinkiewicz theorem). We will use the following simple fact about Lorentz spaces.

LEMMA 9.2. *Let $(\mathcal{X}_1,\mu_1)$ and $(\mathcal{X}_2,\mu_2)$ be σ-finite measure spaces, and let $\mu=\mu_1\times\mu_2$ be the product measure on $\mathcal{X}_1\times\mathcal{X}_2$. Then, for $1\leqslant p<\infty$, $p\leqslant\nu\leqslant\infty$, and any μ-measurable function G,*

$$\|G\|_{L^{p,\nu}(\mathcal{X}_1\times\mathcal{X}_2,\mu)} \leqslant C_{p,\nu}\bigg(\int_{\mathcal{X}_1}\|G(x_1,\cdot\,)\|^p_{L^{p,\nu}(\mathcal{X}_2,\mu_2)}\,d\mu_1\bigg)^{1/p}. \tag{9.3}$$

Proof. Let $p\leqslant\nu<\infty$. By Fubini's theorem, the Lorentz space norm $\|G\|_{L^{p,\nu}(\mathcal{X}_1\times\mathcal{X}_2)}$ is controlled by

$$\bigg(\int_0^\infty \alpha^{\nu-1}\bigg(\int_{\mathcal{X}_1}\mu_2(\{x_2\in\mathcal{X}_2:|G(x_1,x_2)|>\alpha\})\,d\mu_1\bigg)^{\nu/p}d\alpha\bigg)^{1/\nu}.$$

By Minkowski's inequality, this is bounded by

$$\bigg(\int_{\mathcal{X}_1}\bigg(\int_0^\infty \alpha^{\nu-1}\mu_2(\{x_2\in\mathcal{X}_2:|G(x_1,x_2)|>\alpha\})^{\nu/p}\,d\alpha\bigg)^{p/\nu}d\mu_1\bigg)^{1/p},$$

which is comparable to the right-hand side of (9.3). The case $\nu=\infty$ is similar. □

We state a result only for multipliers that are compactly supported away from the origin.

THEOREM 9.3. *Let $d\geqslant 4$, $1<p<p_d=(2d-2)/(d+1)$ and $p\leqslant\nu\leqslant\infty$. Let further m be radial and supported in $\{\xi:\frac{1}{2}\leqslant|\xi|\leqslant 2\}$. Then*

$$\|T_m\|_{L^p\to L^{p,\nu}} \asymp \|\widehat{m}\|_{L^{p,\nu}}, \tag{9.4}$$

$$\|T_m\|_{L^{p,\nu}\to L^{p,\nu}} \asymp \|\widehat{m}\|_{L^p}. \tag{9.5}$$

Proof. The lower bound for the operator norm in (9.4) follows in the usual way, by testing on suitable Schwartz functions. By Colzani's theorem ([5]) for convolution operators, the $L^p\to L^p$ operator norm is controlled by the $L^{p,\nu}\to L^{p,\nu}$ operator norm and this implies the lower bound for (9.5).

For the upper bounds, we apply real interpolation to the second inequality in Corollary 3.2 and obtain

$$\left\|\int_{\mathbb{R}^d}\int_0^\infty h(y,r)F_{y,r}\,dr\,dy\right\|_{L^{p,\nu}(\mathbb{R}^d)} \lesssim \|h\|_{L^{p,\nu}(\mathbb{R}^d\times[1,\infty);dy\,r^{d-1}\,dr)}. \tag{9.6}$$

Now let $K=\widehat{m}$. We argue as in §5. Split $K=K_0+K_\infty$. Then $\|K_0\|_1\lesssim\|K_0\|_{L^{p,\nu}}$ and therefore $\|K_0*f\|_{L^{p,\nu}}\lesssim\|K\|_{L^{p,\nu}}\|f\|_{L^{p,\nu}}$. To estimate the main term $K_\infty*f=\psi*K_\infty*g$, we express it as in (5.2) and then apply (9.6). Using Lemma 9.2, we can estimate $\|\psi*K_\infty*g\|_{L^{p,\nu}}$ by either

$$\|\varkappa\|_{L^{p,\nu}(\mathbb{R}^+,r^{d-1}\,dr)}\|g\|_{L^p(\mathbb{R}^d)} = C\|K\|_{L^{p,\nu}(\mathbb{R}^d)}\|g\|_{L^p(\mathbb{R}^d)},$$

or by

$$\|\varkappa\|_{L^p(\mathbb{R}^+,r^{d-1}\,dr)}\|g\|_{L^{p,\nu}(\mathbb{R}^d)} = C\|K\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{p,\nu}(\mathbb{R}^d)}.$$

□

Remark. One can also obtain $L^p\to L^{p,\nu}$ estimates for multipliers that are not necessarily compactly supported. However the proper generalization of the $L^{p,\nu}\to L^{p,\nu}$ bound in (9.5) presents some difficulties at the current stage. We hope to consider these and related matters later.

10. The regularity result for the wave equation

In this section we shall prove Theorem 1.2. We first note that by a standard scaling argument, it suffices to prove the inequality

$$\left(\int_1^2 \left\|e^{it\sqrt{-\Delta}}f\right\|_q^q\,dt\right)^{1/q} \lesssim \|(I-\Delta)^{\alpha/2}f\|_q. \tag{10.1}$$

Indeed, let us first show how (1.3) follows assuming (10.1) (here $q<\infty$). We may assume by symmetry that in (1.3) we integrate over $[0,L]$. We then write

$$\begin{aligned}\biggl(\frac{1}{L}\int_0^L \|e^{it\sqrt{-\Delta}}f\|_q^q\,dt\biggr)^{1/q} &\leqslant \sum_{n=1}^\infty \biggl(\frac{1}{L}\int_{2^{-n}L}^{2^{-n+1}L}\|e^{it\sqrt{-\Delta}}f\|_q^q\,dt\biggr)^{1/q}\\ &= \sum_{n=1}^\infty 2^{-n/q}\biggl(\int_1^2\|e^{iL2^{-n}s\sqrt{-\Delta}}f\|_q^q\,ds\biggr)^{1/q} = \sum_{n=1}^\infty 2^{-n/q}(*)_n,\end{aligned}$$

where

$$(*)_n = \biggl(\int_1^2\int_{\mathbb{R}^d}\bigl|\bigl(e^{is\sqrt{-\Delta}}f_{L,n}\bigr)(L^{-1}2^nx)\bigr|^q\,dx\,ds\biggr)^{1/q} \quad\text{and}\quad f_{L,n}(y)=f(L2^{-n}y).$$

We change variables in x, apply (10.1), and then change variables again to see that

$$(*)_n \lesssim (L2^{-n})^{d/q}\|(I-\Delta)^{\alpha/2}f_{L,n}\|_q = \|(I-2^{-2n}L^2\Delta)^{\alpha/2}f\|_q.$$

Now we have for $\alpha\geqslant 0$ and $n\geqslant 0$,

$$\|(I-2^{-2n}L^2\Delta)^{\alpha/2}f\|_q \leqslant C_q\|(I-L^2\Delta)^{\alpha/2}f\|_q,$$

where C does not depend on L and n; for $1<q<\infty$, this follows, for example, from the Mikhlin–Hörmander multiplier theorem. Thus $(*)_n$ is bounded by the right-hand side of (1.3) uniformly in $n\geqslant 1$, and, for $q<\infty$, the sum $\sum_{n=1}^\infty 2^{-n/q}(*)_n$ is essentially dominated by the same quantity.

We shall actually obtain an improvement of (10.1), which is formulated using dyadic decompositions. Let $\eta_\circ$ be as in (7.1). Define P_k by $\widehat{P_kf}=(\hat\eta_\circ(2^{-k}\xi))^2\hat f$ for $k\geqslant 1$ and $P_0=I-\sum_{k=1}^\infty P_k$. We have chosen k as our index for the dyadic frequency pieces instead of s, firstly to distinguish it from the homogeneous expression ($s\in\mathbb{Z}$) used earlier and, secondly, to match it with the notation in §3; the term for large frequencies $\approx 2^k$ will correspond, after an appropriate rescaling, to the situation of Corollary 3.2 when the radii are taken in $[2^k,2^{k+1}]$.

Theorem 10.1. *Let $d\geqslant 4$, $(2d-2)/(d-3)<q<\infty$ and $\alpha=d(1/2-1/q)-1/2$. Then*

$$\biggl(\int_1^2\biggl\|\sum_{k=0}^\infty|P_ke^{it\sqrt{-\Delta}}f|\biggr\|_q^q\,dt\biggr)^{1/q} \lesssim \biggl(\sum_{k=0}^\infty 2^{k\alpha q}\|P_kf\|_q^q\biggr)^{1/q}. \tag{10.2}$$

The slightly weaker inequality for Sobolev spaces follows if we replace the ℓ^1 norm in k on the left-hand side of (10.2) and the ℓ^q norm on the right-hand side (with $q>2$) by the ℓ^2 norms. Inequality (10.2) can be restated using Triebel–Lizorkin spaces, namely,

$$\biggl(\int_1^2\|e^{it\sqrt{-\Delta}}f\|_{F^q_{0,1}}^q\,dt\biggr)^{1/q} \lesssim \|f\|_{F^q_{\alpha,q}}.$$

It will be convenient to dispose of the terms corresponding to $k=0,1$. Let χ_0 be a radial $C_0^\infty(\mathbb{R}^d)$ function such that $\chi_0(\xi)=1$ for $|\xi|\leqslant 1$ and $\chi_0(\xi)=0$ for $|\xi|\geqslant \frac{3}{2}$. One easily checks that $\chi_0(\xi/\lambda)e^{i|\xi|}$ is the Fourier transform of an L^1 function for any λ (with L^1 norm growing in λ as $\lambda\to\infty$). Indeed, the contribution of the multiplier near the origin is handled by considering $m_\varkappa(\xi)=(\chi_0(2^\varkappa\xi)-\chi_0(2^{\varkappa+1}\xi))(e^{i|\xi|}-1)$. One bounds the derivatives of $m_\varkappa(2^{-\varkappa}\xi)$ for $\varkappa>0$ to see that the L^1 norm of $\mathcal{F}^{-1}[m_\varkappa]$ is $O(2^{-\varkappa})$.

Next, we describe a further reduction to an inequality involving spherical means (cf. (10.7) and (10.9) below). This can be done in various ways. One way is to apply the method of stationary phase in conjunction with multiplier theorems. We will give a more direct approach based on the principle that every radial function can be written as an average of spherical measures. As before, we let σ_ϱ denote the surface measure on the sphere of radius ϱ.

Let ϑ be a C^∞-function on the real line supported in $\left(\frac{1}{8},8\right)$ such that $\vartheta(s)=1$ on $\left(\frac{1}{4},4\right)$. For $k\geqslant 1$, define the convolution kernel K_k by

$$\widehat{K}_k(\xi)=e^{i|\xi|}\vartheta(2^{-k}|\xi|).$$

LEMMA 10.2. *Let $d\geqslant 2$. Then, for $k\geqslant 1$,*

$$K_k=2^{k(d-1)/2}\int_{1/2}^{2} w_k(\varrho)\sigma_\varrho\,d\varrho+E_k, \tag{10.3}$$

where

$$\sup_k \int_{1/2}^{2} |w_k(\varrho)|\,d\varrho<\infty, \tag{10.4}$$

and, for any M,

$$\|E_k\|_1\leqslant C_{M,d}2^{-kM}. \tag{10.5}$$

Proof. We use polar coordinates for the Fourier integral defining K_k and then write an integral over the sphere S^{d-1} in terms of integrals over $(d-2)$-dimensional spheres perpendicular to x. We get

$$\begin{aligned}(2\pi)^d K_k(x)&=\int_{\mathbb{R}^d}\vartheta(2^{-k}|\xi|)e^{i|\xi|}e^{i\langle\xi,x\rangle}\,d\xi\\ &=2^{k(d-1)}\int_0^\infty \vartheta(2^{-k}s)(2^{-k}s)^{d-1}e^{is}\int_{S^{d-1}} e^{is|x|\langle x/|x|,\theta\rangle}\,d\sigma(\theta)\,ds\\ &=c_{d-2}2^{k(d-1)}\int_{-1}^{1} 2^k\Theta(2^k(1+\tau|x|))(1-\tau^2)^{(d-3)/2}\,d\tau,\end{aligned}$$

where c_{d-2} is the surface measure of the unit sphere S^{d-2} and

$$\Theta(\sigma)=\int_0^\infty \vartheta(s)s^{d-1}e^{is\sigma}\,ds.$$

Clearly $\Theta\in\mathcal{S}(\mathbb{R})$.

From the above formula it is clear that (10.3) holds with

$$w_k(\varrho)=c_{d-2}(2\pi)^{-d}2^{k(d-1)/2}\int_{-1}^{1}2^k\Theta(2^k(1+\tau\varrho))(1-\tau^2)^{(d-3)/2}\,d\tau$$

and $E_k(x)=2^{k(d-1)/2}w_k(|x|)[1-\chi_{[1/2,2]}(|x|)]$.

Let $\gamma>-1$ be fixed and let Θ be any Schwartz function on $\mathbb{R}$ whose Fourier transform is supported in $\left(\frac{1}{8},8\right)$. We prove that for $\beta\geqslant1$ and $\varrho>0$,

$$\int_{-1}^{1}\Theta(\beta(1+\tau\varrho))(1-\tau^2)^{\gamma}\,d\tau\leqslant C\beta^{-\gamma-1}(1+\beta|1-\varrho|)^{-N}\tag{10.6}$$

for any $N>1$. Here $C\geqslant0$ depends on Θ, γ and N but not on β or ϱ. Clearly, the $L_1((0,\infty))$ norm of the right-hand side of (10.6) is $O(\beta^{-\gamma-2})$. Thus (10.6) applied with $\gamma=\frac{1}{2}(d-3)$ and $\beta=2^k$ yields the bounds (10.4) and (10.5).

The bound (10.6) is straighforward; one examines separately the three cases $0<\varrho<\frac{1}{2}$, $\frac{1}{2}\leqslant\varrho\leqslant1$, and $\varrho>1$. We may assume that $N\geqslant1$.

Let $C_0=\sup_{x\in\mathbb{R}}|\Theta(x)|(1+2|x|)^{N+\gamma+2}$. Then, for $0<\varrho<\frac{1}{2}$, the integral can be estimated by

$$C_0\int_{-1}^{1}(1-\tau^2)^{\gamma}(1+\beta)^{-N-\gamma-2}\,d\tau,$$

which is better than the claimed bound.

If $\frac{1}{2}\leqslant\varrho\leqslant1$, we split the integral over $[-1,1]$ as $\int_{-1}^{0}+\int_{0}^{1}$. For the latter, we may argue as in the previous case and bound it by the last displayed expression. For the integral over $[-1,0]$, we make the change of variable $\tau=-1+t$, set $C_1=\sup_{x\in\mathbb{R}}|\Theta(x)|(1+|x|)^{N+\gamma+2}$ and bound $\left|\int_{-1}^{0}\Theta(\beta(1+\tau\varrho))(1-\tau^2)^{\gamma}\,d\tau\right|$ by

$$C_1\int_{0}^{1}\frac{t^{\gamma}(2-t)^{\gamma}}{(1+\beta(1-\varrho)+\beta\varrho t)^{N+\gamma+2}}\,dt\leqslant C_2(\beta\varrho)^{-\gamma-1}(1+\beta(1-\varrho))^{-N-1},$$

where $C_2=C_1\max\{1,2^{\gamma}\}\int_0^{\infty}t^{\gamma}(1+t)^{-N-\gamma-2}\,dt$.

Finally, we consider the last case, $\varrho\geqslant1$. Here we use the fact that the Fourier transform of Θ is supported in $\left(\frac{1}{8},8\right)$ and thus Θ extends to an entire function satisfying $|\Theta(x+iy)|\leqslant Ce^{-y/8}$ for $y\geqslant0$. Now set $g_\gamma(z)=(1-z^2)^{\gamma}$ so that g_γ is analytic in the upper half-plane and $g_\gamma(x)$ is non-negative for $x\in[-1,1]$. By Cauchy's theorem and limiting arguments, the integral over the real line of Θg_γ vanishes, and therefore

$$\left|\int_{-1}^{1}\Theta(x)g_\gamma(x)\,dx\right|=\left|\int_{\mathbb{R}\setminus[-1,1]}\Theta(x)g_\gamma(x)\,dx\right|.$$

The latter integral is bounded by

$$C_3\int_1^\infty \frac{(\tau^2-1)^\gamma}{(1+\beta(\tau\varrho-1))^{N+2\gamma+2}}\,d\tau = C_3\int_0^\infty \frac{(t(2+t))^\gamma}{(1+\beta(\varrho-1)+\beta\varrho t)^{N+2\gamma+3}}\,dt,$$

where $C_3=2\sup_{x\in\mathbb{R}}|\Theta(x)|(1+|x|)^{N+2\gamma+3}$. One separately considers the cases $\gamma\geqslant 0$ and $-1<\gamma<0$. It is not hard to see that in both cases the last displayed expression can be estimated by

$$C_3\max\{1,4^\gamma\}\int_0^\infty t^\gamma(1+\beta(\varrho-1)+\beta\varrho t)^{-N-\gamma-2}\,dt,$$

which in turn is equal to

$$C_4(\beta\varrho)^{-\gamma-1}(1+\beta(\varrho-1))^{-N-2}$$

with

$$C_4=C_3\max\{1,4^\gamma\}\int_0^\infty \frac{t^\gamma}{(1+t)^{N+\gamma+2}}\,dt. \qquad \square$$

We continue with the proof of (10.2). Let $K_{k,t}=t^{-d}K_k(t^{-1}\,\cdot)$ with K_k as in the lemma and observe that

$$P_k\big[e^{it\sqrt{-\Delta}}f\big]=P_k[K_{k,t}*f],\quad \tfrac{1}{2}\leqslant t\leqslant 2.$$

We first dispose of the error terms E_k. Let $E_{k,t}=t^{-d}E_k(t^{-1}\,\cdot)$. Then for any fixed $t\in[1,2]$,

$$\bigg\|\sum_{k=0}^\infty |E_{k,t}*P_kf|\bigg\|_q \lesssim \sum_{k=0}^\infty 2^{-kM}\|P_kf\|_q,$$

which, by Hölder's inequality, is controlled by the right-hand side of (10.2).

Now define

$$\mu_{k,t}=\int_{1/2}^2 w_k(\varrho)\sigma_{\varrho t}\,d\varrho, \tag{10.7}$$

with w_k satisfying (10.4). In view of Lemma 10.2, it suffices to prove that, for $q>q_d$, the estimate

$$\bigg(\int_1^2\bigg\|\sum_{k=2}^\infty 2^{k(d-1)/2}|\mu_{k,t}*\psi_k*f_k|\bigg\|_q^q\,dt\bigg)^{1/q} \lesssim \bigg(\sum_{k=2}^\infty \|f_k\|_q^q 2^{kq(d(1/2-1/q)-1/2)}\bigg)^{1/q} \tag{10.8}$$

holds for all $\{f_k\}_{k=2}^\infty$ with $\hat{f}_k$ supported in $\mathcal{A}_k:=\{\xi:2^{k-1}<|\xi|<2^{k+1}\}$. Here ψ_k are suitably chosen so that $\psi_k=2^{kd}\psi(2^k\,\cdot)$, where $\psi=\psi_\circ*\psi_\circ$ and $\psi_\circ$ is supported in $\big\{x:|x|\leqslant\frac{1}{10}\big\}$ with $10d$ vanishing moments (see the discussion leading to (7.2)). In addition we assume that $\hat{\psi}_\circ(\xi)\neq 0$ for $\frac{1}{2}\leqslant|\xi|\leqslant 4$. To see how (10.8) implies (10.2) we choose $f_k=2^{k(d-1)/2}L_kf$

with $\widehat{L_k f}(\xi)=\eta_\circ^2(2^{-k}\xi)[\hat\psi(2^{-k}\xi)]^{-1}\hat f(\xi)$ and use the fact that $\zeta/\hat\psi$ is the Fourier transform of a Schwartz function for every ζ that is smooth and compactly supported in $\{\xi:\frac13\leqslant|\xi|\leqslant 3\}$.

It suffices to prove (10.8) for families $\{f_k\}_{k=2}^\infty$ for which all but finitely many of the f_k's are zero, with constant independent of the number of summands. By duality the desired bound then follows from the inequality

$$\bigg(\sum_{k=2}^\infty 2^{kdp/p'}\bigg\|\int_1^2 \mu_{k,t}*\psi_k*g_k(\,\cdot\,,t)\,dt\bigg\|_p^p\bigg)^{1/p} \lesssim \bigg(\int_1^2 \Big\|\sup_k |g_k(\,\cdot\,,t)|\Big\|_p^p\,dt\bigg)^{1/p} \tag{10.9}$$

with $p<p_d$, for all $\{g_k\}_{k=2}^\infty$ with the property that the (spatial) Fourier transform of $g_k(\,\cdot\,,t)$ is supported in $\mathcal{A}_k$.

To prove (10.9), we need the following inequality for fixed k (which will be a straightforward consequence of Lemma 6.2). Let $\mathcal{W}^{\ell-k}$ denote the set of dyadic cubes of side-length $2^{\ell-k}$.

Proposition 10.3. *Let $1\leqslant p<p_d$ and $\varepsilon<(d-1)(1/p-1/p_d)$. Then, for $0\leqslant\ell\leqslant k$,*

$$\bigg\|\int_1^2 \psi_k*\mu_{k,t}*g(\,\cdot\,,t)\,dt\bigg\|_p \lesssim_\varepsilon 2^{-kd/p'}2^{-\ell\varepsilon}\bigg(\sum_{W\in\mathcal{W}^{\ell-k}}|W|\int_1^2 \sup_{y\in W}|g(y,t)|^p\,dt\bigg)^{1/p}. \tag{10.10}$$

Proof. We first prove the inequality

$$\bigg\|\int_1^2 \psi_k*\sigma_t*g(\,\cdot\,,t)\,dt\bigg\|_p \lesssim_\varepsilon 2^{-kd/p'}2^{-\ell\varepsilon}\bigg(\sum_{W\in\mathcal{W}^{\ell-k}}|W|\int_1^2 \sup_{y\in W}|g(y,t)|^p\,dt\bigg)^{1/p}. \tag{10.11}$$

We apply a rescaling and averaging argument to deduce it from Lemma 6.2. Define $H_{k,t}$ by $\widehat{H}_{k,t}(\xi)=\hat\psi(\xi)\hat\sigma_1(2^kt\xi)$. The expression on the left-hand side of (10.11) can be written as

$$\begin{aligned}\bigg\|\int_1^2 2^{kd}H_{k,t}(2^k\,\cdot\,)*g(\,\cdot\,,t)t^{d-1}\,dt\bigg\|_p &= 2^{-kd/p}\bigg\|\int_1^2 H_{k,t}*g(2^{-k}\,\cdot\,,t)t^{d-1}\,dt\bigg\|_p\\ &= 2^{-kd/p}\bigg\|\int_{2^k}^{2^{k+1}} \psi*\sigma_r*2^{-kd}g(2^{-k}\,\cdot\,,2^{-k}r)\,dr\bigg\|_p.\end{aligned}$$

By Lemma 6.2, the last expression is

$$\lesssim 2^{-kd/p}2^{-\ell\varepsilon}\bigg(\int_{2^k}^{2^{k+1}}\sum_{W'\in\mathcal{W}^\ell}|W'|\sup_{y'\in W'}|2^{-kd}g(2^{-k}y',2^{-k}r)|^p r^{d-1}\,dr\bigg)^{1/p},$$

which is dominated by a constant times

$$2^{-\ell\varepsilon}\biggl(\int_1^2 \sum_{W\in\mathcal{W}^{\ell-k}} 2^{-kd(p-1)}|W|\sup_{y\in W}|g(y,t)|^p\,dt\biggr)^{1/p}$$
$$\lesssim 2^{-\ell\varepsilon}2^{-kd/p'}\biggl(\sum_{W\in\mathcal{W}^{\ell-k}}|W|\int_1^2\sup_{y\in W}|g(y,t)|^p\,dt\biggr)^{1/p}.$$

It remains to show how (10.11) implies the assertion of the proposition. Since

$$\int_{1/2}^2|w_k(\varrho)|\,d\varrho$$

is uniformly bounded, it suffices, by averaging, to show the uniform bound

$$\biggl\|\int_1^2\psi_k*\sigma_{\varrho t}*g(\,\cdot\,,t)\,dt\biggr\|_p\lesssim_\varepsilon 2^{-kd/p'}2^{-\ell\varepsilon}\biggl(\sum_{W\in\mathcal{W}^{\ell-k}}|W|\int_1^2\sup_{y\in W}|g(y,t)|^p\,dt\biggr)^{1/p}\tag{10.12}$$

for $\frac{1}{2}\leqslant\varrho\leqslant 2$. This is a consequence of (10.11) by scaling. For the details, assume $\varrho\in(1,2]$. After a change of variables we have to estimate the L^p norm of

$$\biggl(\int_\varrho^2+\int_2^{2\varrho}\biggr)[\psi_k*\sigma_1*g(\,\cdot\,,\varrho^{-1}t)](x)\frac{dt}{\varrho}.$$

We apply (10.11) with the function $g(\,\cdot\,,\varrho^{-1}t)\chi_{[\varrho,1]}(t)$ to bound the first integral. The second integral is equal to

$$\frac{2}{\varrho}\int_1^\varrho\biggl[\psi_k*\sigma_{2s}*g\biggl(\,\cdot\,,\frac{2s}{\varrho}\biggr)\biggr](x)\,ds=\frac{2^d}{\varrho}\int_1^\varrho\biggl[\psi_{k+1}*\sigma_s*g\biggl(2\,\cdot\,,\frac{2s}{\varrho}\biggr)\biggr]\frac{x}{2}\,ds$$

and, after conjugation with a dilation operator, we may apply (10.11) (with ψ_k replaced by ψ_{k+1}). Note that replacing $\mathcal{W}^{\ell-k}$ by $\mathcal{W}^{\ell-k-1}$ on the right-hand side of (10.12) yields an equivalent norm. The argument for $\varrho\in\bigl[\frac{1}{2},1\bigr)$ is similar. □

We now use the arguments of §7 based on "atomic" decompositions for the functions $g_k(\,\cdot\,,t)$, *for any fixed* $t\in[1,2]$. We work with the ℓ^∞ variant of Peetre's operator, namely,

$$\mathcal{M}G(x,t)=\sup_{k>0}\sup_{|y|\leqslant 10d2^{-k}}|g_k(x+y,t)|,$$

where it will always be understood that $G=\{g_k\}_{k=1}^\infty$ and $g_k(\,\cdot\,,t)$ has spectrum in the annulus $\mathcal{A}_k$. Then, with this specification, Peetre's inequality says that

$$\|\mathcal{M}G(\,\cdot\,,t)\|_{L^p(\mathbb{R}^d)}\lesssim\Bigl\|\sup_k|g_k(\,\cdot\,,t)|\Bigr\|_{L^p(\mathbb{R}^d)},\quad 0<p\leqslant\infty.\tag{10.13}$$

For each $t\in[1,2]$, let

$$\Omega_j(t)=\{x\in\mathbb{R}^d:\mathcal{M}G(x,t)>2^j\}.$$

Let $\mathcal{Q}_j(t)$ be the set of all dyadic cubes which are contained in $\Omega_j(t)$ but not in $\Omega_{j+1}(t)$.

For each dyadic cube of sidelength less than 1 we define an expanded cube $W(Q,t)$ as follows. We first let $j(Q)$ be the unique j such that $Q\in\mathcal{Q}_j(t)$. If the unique dyadic cube of sidelength 1 containing Q is contained in $\Omega_{j(Q)}(t)$, then we let $W(Q,t)$ be this cube. If not then we let $W(Q,t)$ be the maximal dyadic cube that contains Q and that is contained in $\Omega_{j(Q)}(t)$.

We let $\mathcal{Q}_j^k(t)$ be the family of cubes in $\mathcal{Q}_j(t)$ which are of sidelength 2^{-k}. Notice that if Q has sidelength 2^{-k}, then the sidelength of $W(Q,t)$ is $2^{\ell-k}$ for some non-negative integer $\ell\leqslant k$. As before, we denote by $\mathcal{W}^{\ell-k}$ the collection of dyadic cubes of sidelength $2^{\ell-k}$. We also let $\mathcal{W}_j(t)$ be the set of dyadic cubes contained in $\Omega_j(t)$ which are either of sidelength 1, or of sidelength less than 1 and maximal in $\Omega_j(t)$. Notice that the cubes in $\mathcal{W}_j(t)$ have disjoint interiors. With this notation, we note that if $Q\in\mathcal{Q}_j^k(t)$ and $W(Q,t)$ has sidelength $2^{\ell-k}$, then $W(Q,t)$ is a cube in $\mathcal{W}_j(t)\cap\mathcal{W}^{\ell-k}$.

For each $\ell=0,...,k$, define

$$A_{k,\ell,j}(x,t)=\sum_{\substack{Q\in\mathcal{Q}_j^k(t)\\ W(Q,t)\in\mathcal{W}^{\ell-k}}} g_k(x,t)\chi_Q(x).$$

We can now decompose

$$g_k=\sum_{\ell=0}^{\infty}\sum_{j\in\mathbb{Z}}A_{k,\ell,j}.$$

Using this decomposition and Minkowski's inequality, we estimate the left-hand side of (10.9) by

$$\sum_{\ell=0}^{\infty}\biggl(\sum_{k=2}^{\infty}2^{kdp/p'}\biggl\|\int_1^2\mu_{k,t}*\psi_k*\sum_{j\in\mathbb{Z}}A_{k,\ell,j}(\cdot\,,t)\,dt\biggr\|_p^p\biggr)^{1/p}$$

and, by Proposition 10.3, the term corresponding to a fixed ℓ is

$$\begin{aligned}&\lesssim 2^{-\ell\varepsilon}\biggl(\sum_{k=2}^{\infty}\sum_{W\in\mathcal{W}^{\ell-k}}\operatorname{meas}(W)\int_1^2\sup_{y\in W}\biggl|\sum_{j\in\mathbb{Z}}A_{k,\ell,j}(y,t)\biggr|^p\,dt\biggr)^{1/p}\\&\lesssim 2^{-\ell\varepsilon}\biggl(\sum_{k=2}^{\infty}\sum_{W\in\mathcal{W}^{\ell-k}}\operatorname{meas}(W)\int_1^2\sup_{y\in W}\sum_{j\in\mathbb{Z}}|A_{k,\ell,j}(y,t)|^p\,dt\biggr)^{1/p},\end{aligned}\tag{10.14}$$

where for the last estimate we have used that for each fixed k, ℓ and t the functions $y\mapsto A_{k,\ell,j}(y,t)$, $j\in\mathbb{Z}$, live on (essentially) disjoint sets.

To estimate (10.14) we set, for $W\in\mathcal{W}_j(t)$,

$$A^W_{k,j}(\cdot,t)=\sum_{\substack{Q\in\mathcal{Q}^k_j(t)\\ W(Q,t)=W}} g_k(x,t)\chi_Q(x),$$

so that

$$A_{k,\ell,j}=\sum_{W\in\mathcal{W}^{\ell-k}} A^W_{k,j}.$$

By the definitions of $\mathcal{M}$ and Ω_j, we have $\|A^W_{k,j}(\cdot,t)\|^p_\infty\leqslant 2^{(j+1)p}$ for any $W\in\mathcal{W}_j(t)$. Therefore we get, for any fixed ℓ,

$$\begin{aligned}\sum_{k=2}^{\infty}\sum_{W\in\mathcal{W}^{\ell-k}}\operatorname{meas}(W)\|A_{k,\ell,j}(\cdot,t)\chi_W\|^p_\infty &=\sum_{k=2}^{\infty}\sum_{W\in\mathcal{W}^{\ell-k}\cap\mathcal{W}_j(t)}\operatorname{meas}(W)\|A^W_{k,j}(\cdot,t)\|^p_\infty\\ &\leqslant 2^{(j+1)p}\sum_{W\in\mathcal{W}_j(t)}\operatorname{meas}(W)\\ &\leqslant 2^{(j+1)p}\operatorname{meas}(\Omega_j(t)).\end{aligned}$$

The expression (10.14) is now $\lesssim 2^{-\ell\varepsilon}(*)_\ell$, where

$$\begin{aligned}(*)_\ell &:=\bigg(\sum_{j\in\mathbb{Z}}\int_1^2\sum_{k=2}^{\infty}\sum_{W\in\mathcal{W}^{\ell-k}}\operatorname{meas}(W)\|A_{k,l,j}(\cdot,t)\chi_W\|^p_\infty\,dt\bigg)^{1/p}\\ &\lesssim\bigg(\int_1^2\sum_{j\in\mathbb{Z}}2^{jp}\operatorname{meas}(\Omega_j(t))\,dt\bigg)^{1/p}\lesssim\bigg(\int_1^2\|\mathcal{M}G(\cdot,t)\|^p_p\,dt\bigg)^{1/p}.\end{aligned}$$

We sum in ℓ and use (10.13) to conclude the proof of (10.9).

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Yaryong Heo
Department of Mathematics
University of Wisconsin, Madison
480 Lincoln Drive
Madison, WI 53706
U.S.A.
yaryong@korea.ac.kr

Current address:
Department of Mathematics
Korea University
Seoul 136-701
Republic of Korea

Fëdor Nazarov
Department of Mathematics
University of Wisconsin, Madison
480 Lincoln Drive
Madison, WI 53706
U.S.A.
nazarov@math.wisc.edu

Andreas Seeger
Department of Mathematics
University of Wisconsin, Madison
480 Lincoln Drive
Madison, WI 53706
U.S.A.
seeger@math.wisc.edu

Received February 9, 2009