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Resolution of singularities of real-analytic vector fields in dimension three

by

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1. Introduction

1.1. Main result

Let χ be an analytic vector field defined on a real-analytic manifold M. We shall say that χ is *elementary* at a point $p\in M$ if one of the following conditions holds:

(i) (nonsingular case) $\chi(p)\neq 0$, or

(ii) (singular case) $\chi(p)=0$, and the *Jacobian map*

$$D\chi(p)\colon \mathbf{m}_p/\mathbf{m}_p^2 \longrightarrow \mathbf{m}_p/\mathbf{m}_p^2,$$
$$[g]\longmapsto[\chi(g)],$$

($\mathbf{m}_p \subset \mathcal{O}_{M,p}$ is the maximal ideal) has at least one nonzero eigenvalue.

Here $\chi(\cdot)$ denotes the action of χ as a derivation in $\mathcal{O}_p$.

If we fix a local coordinate system $(x_1, ..., x_n)$ for M at p and write

$$\chi = a_1 \frac{\partial}{\partial x_1} + ... + a_n \frac{\partial}{\partial x_n},$$

then the Jacobian map is given by the real matrix

$$D\chi(p) = \left(\frac{\partial a_i}{\partial x_j}(0) \right)_{i,j=1}^{n}.$$

We say that χ is *reduced* if $\gcd(a_1, ..., a_n) = 1$ at each point $p \in M$ (this implies that the set $\mathrm{Ze}(\chi) = \{q \in M : \chi(q) = 0\}$ has codimension strictly greater than 1).

Let us state our main result. We briefly define the necessary concepts and postpone the details to the next section.

A singularly foliated manifold is a 4-tuple $\mathbf{M} = (M, \Upsilon, \mathfrak{D}, L)$, where

(i) M is a real-analytic 3-dimensional manifold with corners;

(ii) $\Upsilon \in \mathbf{L}$ is an ordered list of natural numbers;

(iii) $\mathfrak{D} = \mathfrak{D}_\Upsilon$ is a Υ-tagged divisor on M with normal crossings;

(iv) L is a singular orientable analytic line field on $(M, \mathfrak{D})$ which is $\mathfrak{D}$-preserving.

At each point $p \in M$, the line field L is locally generated by an analytic vector field χ_p which is tangent to the divisor $\mathfrak{D}$. This local generator is uniquely defined up to multiplication by a strictly positive analytic function.

We say that the singularly foliated manifold $\mathbf{M}$ is *elementary* at a point $p \in M$ if the local generator χ_p is an elementary vector field at p. The complement of the set of elementary points in M will be denoted by $\mathrm{NElem}(\mathbf{M})$.

A singularly foliated manifold $\mathbf{M}$ is said to be *elementary* if $\mathrm{NElem}(\mathbf{M}) = \varnothing$.

THEOREM 1.1. (Main theorem) *Let χ be a reduced analytic vector field defined in a real-analytic 3-dimensional manifold M without boundary. Then, for each relatively compact set $U \subset M$, there exists a finite sequence of weighted blowing-ups*

$$(U, \varnothing, \varnothing, L_\chi|_U) =: \mathbf{M}_0 \xleftarrow{\Phi_1} \mathbf{M}_1 \xleftarrow{\Phi_2} ... \xleftarrow{\Phi_n} \mathbf{M}_n \tag{1}$$

such that the resulting singularly foliated manifold $\mathbf{M}_n$ is elementary. Moreover, the center Y_i of the blowing-up Φ_i is a smooth analytic subset of $\mathrm{NElem}(\mathbf{M}_i)$ for each $i = 0, ..., n-1$.

In the above statement, L_χ denotes the singular orientable line field which is associated with the vector field χ.

1.2. Previous works

The theorem of resolution of singularities for vector fields in dimension 2 was present in the work of Bendixson [Be]. The first complete proof of this result was given by Seidenberg in [S].

In [Pe], Pelletier gives an alternative proof of this result through the use of the weighted blowing-ups.

In the book [C1], Cano proves a result of local reduction of singularities in the formal context for complex 3-dimensional vector fields.

The paper [Sa] studies generic equireduction of singularities for vector fields in arbitrary dimension.

In a recent paper [CMR], the authors prove a local uniformization theorem for analytic vector fields in dimension 3. Their proof is based on the analysis of valuations defined by nonoscillating subanalytic integral curves.

The literature on the Newton polyhedron and its applications is extensive. For some results related to the use of the Newton polyhedron in resolution of singularities, we refer the reader to [H2], [H3] and [Y].

In the book [B], Bruno uses the Newton polyhedron and normal form theory to describe many explicit algorithms for studying the asymptotic behavior of integral curves of vector fields near elementary and nonelementary singular points.

1.3. Overview of the paper

The proof of Theorem 1.1 consists of two parts: the description of the local strategy for resolution of singularities, given in §4, and the proof that this local strategy can be *globalized*, which will be explained in §5.

Let us briefly describe the ingredients used in the central result of the paper: the theorem on local resolution of singularities.

For definiteness, we assume here that $M=(\mathbf{R}^3_{(x,y,z)},0)$, and that the origin is contained in a divisor with normal crossings $\mathfrak{D}$ which is given either by $\{(x,y,z)\in\mathbf{R}^3:x=0\}$ (briefly, $\{x=0\}$) or by $\{xy=0\}$. We further assume that the reduced vector field χ defined in M is tangent to the divisor $\mathfrak{D}$, and that the origin is a nonelementary singular point. Finally, we assume that the vertical axis $\{x=y=0\}$ is not entirely contained in the set NElem of nonelementary points.

Using the logarithmic basis $\{x\partial/\partial x, y\partial/\partial y, z\partial/\partial z\}$, we can write

$$\chi = fx\frac{\partial}{\partial x}+gy\frac{\partial}{\partial y}+hz\frac{\partial}{\partial z},$$

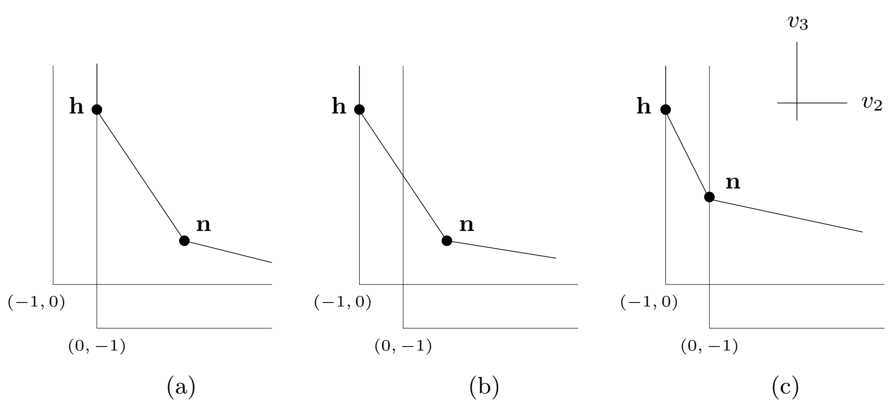


Figure 1. Regular and nilpotent configurations.

where fx, gy and hz are germs in $\mathbf{R}\{x,y,z\}$. The *Newton polyhedron* of χ (with respect to the coordinates (x,y,z)) is the convex polyhedron

$$\mathcal{N}=\operatorname{conv}(\operatorname{supp}(f,g,h))+\mathbf{R}^3_{\geqslant 0},$$

where $\operatorname{conv}(\cdot)$ denotes the operation of convex closure, $\operatorname{supp}(f,g,h)\subset\mathbf{Z}^3$ is the set of integer points $\mathbf{v}=(v_1,v_2,v_3)\in\mathbf{Z}^3$ such that the monomial $x^{v_1}y^{v_2}z^{v_3}$ has a nonzero coefficient in the Laurent expansion of either f, g or h; and the "+" operator is the usual Minkowski sum of convex polyhedrons.

The *higher vertex* of $\mathcal{N}$ is the vertex $\mathbf{h}\in\mathcal{N}$ which is minimal with respect to the lexicographical ordering in $\mathbf{R}^3$. By the hypothesis, it follows that this vertex has the form $\mathbf{h}=(0,h_2,h_3)$ for some integers $h_2,h_3\in\mathbf{Z}_{\geqslant -1}$. Moreover, the intersection of $\mathcal{N}$ with the plane $\{\mathbf{v}\in\mathbf{R}^3:v_1=0\}$ is in one of the situations shown in Figure 1.

Referring to Figure 1, the configurations (a) and (b) are called *regular* and the configuration (c) is called *nilpotent*. As indicated in the figure, we define the *main vertex* $\mathbf{m}=(m_1,m_2,m_3)$ by $\mathbf{m}=\mathbf{h}$ in cases (a) and (b), and by $\mathbf{m}=\mathbf{n}$ in case (c). Now, we consider the intersection

$$\mathcal{N}'=\mathcal{N}\cap\left\{\mathbf{v}\in\mathbf{R}^3: v_3=m_3-\tfrac{1}{2}\right\},$$

and call the polygon $\mathcal{N}'$ the *derived polygon* (see Figure 2).

The derived polygon has some similarities with the *characteristic polygon* introduced by Hironaka [H3] in his proof of the resolution of singularities for excellent surfaces. However, there are some essential differences which will be discussed in §3.3.

Let us denote by $\mathbf{m}'=\left(m_1',m_2',m_3-\frac{1}{2}\right)$ the minimal vertex of $\mathcal{N}'$ (with respect to the lexicographical ordering) and write the displacement vector $\mathbf{m}'-\mathbf{m}$ as $\frac{1}{2}(\Delta_1,\Delta_2,-1)$, for some nonzero rational vector $\Delta=(\Delta_1,\Delta_2)\in\mathbf{Q}^2$.

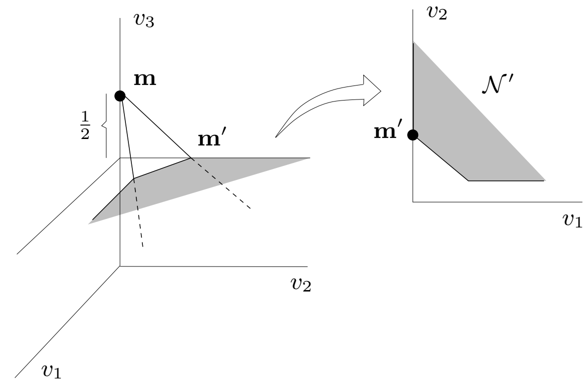


Figure 2. The derived polygon.

The *main invariant* for the vector field χ (with respect to the coordinates (x, y, z)) is given by the 6-tuple of natural numbers

$$\mathrm{inv} = (\mathfrak{h}, m_2+1, m_3, \#\iota-1, \lambda\Delta_1, \lambda \max\{0, \Delta_2\}),$$

where $\lambda=(m_3+1)!$, $\#\iota\in\{1,2\}$ is the number of local irreducible components of the divisor at the origin and the *virtual height* $\mathfrak{h}$ is the natural number defined by

$$\mathfrak{h} = \begin{cases} \lfloor m_3+1-1/\Delta_2 \rfloor, & \text{if } m_2=-1 \text{ and } \Delta_1=0, \\ m_3, & \text{if } m_2=0 \text{ or } \Delta_1>0, \end{cases}$$

where $\lfloor\alpha\rfloor:=\max\{n\in\mathbf{Z}:n\leqslant\alpha\}$ (see Figure 3 for an example).

In §4.5 we shall introduce the fundamental notion of *stable coordinates*. Roughly speaking, if we start with a system of local coordinates (x, y, z) as above, we obtain a new system of coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ by an analytic change of coordinates of the form

$$\tilde{x}=x, \quad \tilde{y}=y+G(x) \quad \text{and} \quad \tilde{z}=z+F(x,y), \tag{2}$$

in such a way that the main invariant inv, when computed with respect to the new coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$, has nice analytic properties such as being an upper semicontinuous function. Moreover, we shall see that inv is an *intrinsic* object attached to the germ of vector fields χ, up to fixing an additional geometric structure on the ambient space called an *axis* (see §2.7).

The local strategy of reduction of singularities will be read out of the Newton polyhedron $\mathcal{N}$ and the main invariant inv, *provided* that these objects are computed with respect to a stable system of coordinates.

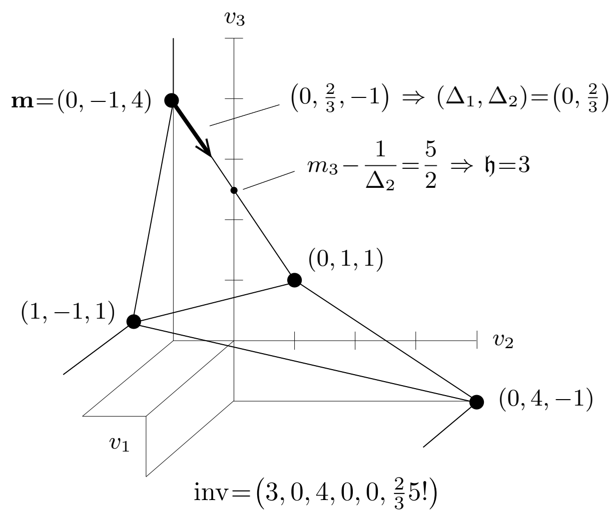


Figure 3. The Newton polyhedron for $\chi=x^2y\,\partial/\partial x+(z^4+xz)\,\partial/\partial y+y^4\,\partial/\partial z$.

The notion of stable coordinates is similar to the notions of *well-prepared* and *very well-prepared* systems of coordinates, as defined by Hironaka [H3] in the context of function germs. However, new difficulties appear in the context of vector fields, since the action of the Lie group of coordinate changes given in (2) is much harder to study in this situation. An example of new phenomena is the appearance of the so-called *resonant configurations*, described in §4.4.

Let us now briefly introduce a second ingredient of our proof: the notion of weighted blowing-up. Given a vector of nonzero natural numbers $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)\in\mathbf{N}^3_{>0}$, the $\boldsymbol{\omega}$-*weighted blowing-up* (with respect to the coordinates (x,y,z)) is the proper analytic surjective map given by

$$\begin{aligned}\Phi_{\boldsymbol{\omega}}\colon \mathbf{S}^2\times\mathbf{R}^+ &\longrightarrow \mathbf{R}^3,\\ ((\bar{x},\bar{y},\bar{z}),\tau) &\longmapsto (x,y,z)=(\tau^{\omega_1}\bar{x},\tau^{\omega_2}\bar{y},\tau^{\omega_3}\bar{z}).\end{aligned}$$

Similarly, for a weight-vector of the form $\boldsymbol{\omega}=(\omega_1,0,\omega_3)$ with ω_1 and ω_3 nonzero, we define

$$\begin{aligned}\Phi_{\boldsymbol{\omega}}\colon \mathbf{S}^1\times\mathbf{R}^+\times\mathbf{R} &\longrightarrow \mathbf{R}^3,\\ ((\bar{x},\bar{z}),\tau,y) &\longmapsto (x,y,z)=(\tau^{\omega_1}\bar{x},y,\tau^{\omega_3}\bar{z}).\end{aligned}$$

We define analogously the blowing-ups for weight-vectors $\boldsymbol{\omega}$ of the forms $(0,\omega_2,\omega_3)$ and $(\omega_1,\omega_2,0)$. Notice that the blowing-up center, in these four cases, is given respectively by $\{x=y=z=0\}$, $\{x=z=0\}$, $\{y=z=0\}$ and $\{x=y=0\}$.

Suppose now that (x,y,z) is a stable system of coordinates at the origin, and let $\mathrm{inv}\in\mathbf{N}^6$ be the corresponding main invariant. Starting from §4.8, we prove the following result on local resolution of singularities for vector fields: There exists a choice of weight-vector $\boldsymbol{\omega}$ for which the strict transform $\widetilde{\chi}$ of the vector field χ under the $\boldsymbol{\omega}$-weighted blowing-up

$$\Phi_{\boldsymbol{\omega}}\colon \widetilde{M} \longrightarrow M$$

is such that, for each nonelementary point $\tilde{p}\in\widetilde{M}\cap\Phi^{-1}(0)$, and each choice of a stable system of coordinates $(\tilde{x},\tilde{y},\tilde{z})$ with center at $\tilde{p}$, the corresponding main invariant $\widetilde{\mathrm{inv}}$ is such that

$$\widetilde{\mathrm{inv}} <_{\mathrm{lex}} \mathrm{inv},$$

where $<_{\mathrm{lex}}$ is the usual lexicographical ordering in $\mathbf{N}^6$.

The second part of the proof of Theorem 1.1 is given in §5. There we show that the local strategy for resolution of singularities described in the previous paragraph can be *globalized*.

To prove this, the main ingredient is the fact that both the main invariant inv and the choice of a weight-vector $\boldsymbol{\omega}$ are independent of the given stable system of coordinates. Moreover, we shall see in §5.2 that inv is an upper semicontinous function on the set NElem of nonelementary singular points of the vector field.

Based on these facts, the global strategy of resolution is similar to the one presented by Cano in [C2], based on the notion of generic equireducibility and bad points. The main distinction is the fact that our strategy leads to a unique choice of local center and an *a priori* absence of cycles, due to a conveniently chosen enumeration of the exceptional divisors. Moreover, we can guarantee that the blowing-up centers are always contained in the set of nonelementary points.

1.4. An example

The following example was communicated to me by F. Sanz. It justifies the use of *weighted* blowing-ups in the resolution of singularities of vector fields in dimension greater than 2.

Example 1.2. Consider the nonelementary germ of vector fields

$$\chi = x\left(x\frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y} - \beta z\frac{\partial}{\partial z}\right) + xz\frac{\partial}{\partial y} + (y-\lambda x)\frac{\partial}{\partial z}, \tag{3}$$

for some real constants $\alpha,\beta\geqslant 0$ and $\lambda>0$. We claim that this germ cannot be simplified by any sequence of *homogeneous blowing-ups* (i.e. blowing-ups with weight equal to 1) with center contained in the set of singularities.

In fact, since the blowing-up center Y is contained in the set of singularities, there are two possible blowing-up strategies:

(a) blow-up with center at the point $Y=\{x=y=z=0\}$;

(b) blow-up with center at the curve $Y=\{x=y=0\}$.

In case (a), the x-directional blowing-up is given by

$$x=\tilde{x}, \quad y=\tilde{x}\tilde{y} \quad \text{and} \quad z=\tilde{x}\tilde{z}.$$

Therefore, the blowing-up of χ will be given by the following expression (dropping the tildes to simplify the notation):

$$\chi=x\left(x\frac{\partial}{\partial x}-\alpha' y\frac{\partial}{\partial y}-\beta' z\frac{\partial}{\partial z}\right)+xz\frac{\partial}{\partial y}+(y-\lambda)\frac{\partial}{\partial z},$$

where $\alpha'=\alpha+1$ and $\beta'=\beta+1$. If we make the translations $\tilde{y}=y-\lambda$ and $\tilde{z}=z-\alpha'\lambda$, we obtain (dropping again the tildes)

$$\chi=x\left(x\frac{\partial}{\partial x}-\alpha' y\frac{\partial}{\partial y}-\beta' z\frac{\partial}{\partial z}\right)+xz\frac{\partial}{\partial y}+(y-\lambda' x)\frac{\partial}{\partial z}, \tag{4}$$

where $\lambda'=\alpha'\beta'\lambda$. Notice that the vector field (4) can be obtained from (3) simply by making the replacement of the constants

$$(\alpha,\beta,\lambda)\longmapsto(\alpha',\beta',\lambda').$$

In case (b), the x-directional blowing-up is given by

$$x=\tilde{x}, \quad y=\tilde{x}\tilde{y} \quad \text{and} \quad z=\tilde{z},$$

and we get (dropping the tildes)

$$\chi=x\left(x\frac{\partial}{\partial x}-\alpha' y\frac{\partial}{\partial y}-\beta z\frac{\partial}{\partial z}\right)+z\frac{\partial}{\partial y}+x(y-\lambda)\frac{\partial}{\partial z},$$

where $\alpha'=\alpha+1$. After the translation $\tilde{y}=y-\lambda$, we obtain

$$\chi=x\left(x\frac{\partial}{\partial x}-\alpha' y\frac{\partial}{\partial y}-\beta z\frac{\partial}{\partial z}\right)+(z-\bar{\lambda}x)\frac{\partial}{\partial y}+xy\frac{\partial}{\partial z}, \tag{5}$$

where $\bar{\lambda}=\alpha'\lambda$. Notice that the vector field (5) can be obtained from (3) simply by making the replacement of the constants

$$(\alpha,\beta,\lambda)\longmapsto(\alpha',\beta,\lambda'),$$

and interchanging the roles of the variables y and z. This proves that no improvement has been made neither by the blowing-up (a) nor by the blowing-up (b).

Remark 1.3. Up to some additional computations, we can further prove that no improvement can be made if we choose, as blowing-up centers, *arbitrary* analytic curves which are left invariant by the vector field.

1.5. Acknowledgments

I would like to thank Felipe Cano for his encouragement and suggestions. I also thank Joris van der Hoeven for useful discussions on the subject. Some parts of this work have been done at the Universidad de Valladolid, Université de Bourgogne and IMPA. I thank these institutions for their hospitality. Finally, I would like to thank the anonymous referee for the many valuable comments and suggestions, which greatly improved the presentation of this work.

1.6. Notation

- $\mathbf{N}=\{n\in\mathbf{Z}:n\geqslant 0\}$, $\mathbf{N}_{\geqslant k}=\{n\in\mathbf{Z}:n\geqslant k\}$.
- $\mathbf{R}^+=\{x\in\mathbf{R}:x\geqslant 0\}$, $\mathbf{R}^*=\{x\in\mathbf{R}:x\neq 0\}$.
- $\mathbf{R}_{>\alpha}=\{x\in\mathbf{R}:x>\alpha\}$, $\mathbf{R}_{\geqslant\alpha}=\{x\in\mathbf{R}:x\geqslant\alpha\}$.
- $\mathbf{R}^n_s=(\mathbf{R}^+)^s\times\mathbf{R}^{n-s}$.
- $\overline{\mathbf{R}}=\mathbf{R}\cup\{\infty\}$ is the extended field of real numbers, with the usual extended ordering relation (we define similarly $\overline{\mathbf{Q}}$ and $\overline{\mathbf{Z}}$).
- $\mathrm{Mat}(n,\mathbf{R})$ is the set of real $n\times n$ matrices.
- $\mathbf{R}[[\mathbf{x}]]$ is the ring of real formal series in the variables $\mathbf{x}$.
- $\mathbf{L}$ is the set of all *reverse ordered* lists of natural numbers. A typical element $\iota\in\mathbf{L}$ is written

$$\iota=[i_1,...,i_k],\quad \text{where } i_1,...,i_k\in\mathbf{N} \text{ and } i_1>i_2>...>i_k;$$

$\#\iota=k$ denotes the length of the list ι.

- For two lists $\iota,\varrho\in\mathbf{L}$, we denote by $\iota\cup\varrho$, $\iota\cap\varrho$ and $\iota\setminus\varrho$ the new lists which are obtained by the usual operations of concatenation, intersection and difference (for instance, $[3,2,1]\cup[5,3]=[5,3,2,1]$, $[5,3,2]\cap[3,1]=[3]$ and $[5,4,3,2]\setminus[5,4,1]=[3,2]$).
- For $\mathbf{u},\mathbf{v}\in\mathbf{R}^n$, the notation $\mathbf{v}<_{\mathrm{lex}}\mathbf{u}$ indicates that $\mathbf{v}$ is *lexicographically* smaller than $\mathbf{u}$, i.e. the relation

$$\bigvee_{i=0}^{n-1}[(v_1,...,v_i)=(u_1,...,u_i)\wedge v_{i+1}<u_{i+1}]$$

holds.

- For $\alpha\in\mathbf{R}$, $\lfloor\alpha\rfloor=\max\{n\in\mathbf{Z}:n\leqslant\alpha\}$ and $\lceil\alpha\rceil=\min\{n\in\mathbf{Z}:n\geqslant\alpha\}$.

2. Blowing-up and singularly foliated manifolds

2.1. Manifolds with corners

We shall work on the category of analytic manifolds with corners. Recall that an n-dimensional manifold M *with corners* is a paracompact topological space which is locally modeled by

$$\mathbf{R}^n_s = \{(x_1, ..., x_n) \in \mathbf{R}^n : x_1 \geqslant 0, ..., x_s \geqslant 0\}.$$

A *local chart* (or local coordinate system) at a point $p \in M$ is a pair (U, ϕ) such that $U \subset M$ is an open neighborhood of p and $\phi: U \to \mathbf{R}^n_s$ is a diffeomorphism with $\phi(p)=0$. Note that $\phi(U \cap \partial M)$ is mapped to $\partial \mathbf{R}^n_s := \bigcup_{i=1}^s \{x_i = 0\}$.

For notational simplicity, we shall sometimes omit the subscript s when we refer to the space $\mathbf{R}^n_s$.

The number $b(p) := s$ is the number of boundary components which meet at p (it is independent of the choice of local chart). Note that $\partial M := \{p \in M : b(p) \geqslant 1\}$.

In this work, we shall say that a subset $N \subset M$ is a *submanifold* if for each point $p \in N$ there exists a local chart (as defined above) such that

$$N = \mathbf{R}^n_s \cap \{x_{i_1} = ... = x_{i_k} = 0\},$$

for some sublist of indices $[i_1, ..., i_k] \subset [n, ..., 1]$.

The connected components of $\partial M \setminus \partial\partial M$ will play a role similar to the irreducible components of the exceptional divisors in the classical results of resolution of singularities. For this reason, we call an *irreducible divisor* (or divisor component) of M a connected submanifold of codimension 1 which is contained in ∂M. A *divisor with normal crossings* (or, shortly, a *divisor*) is a subset $\mathfrak{D} \subset \partial M$ formed by some union of irreducible divisors.

We shall denote by $\mathcal{O}_M$ the sheaf of germs of analytic functions on M. If there is no risk of ambiguity, the stalk of $\mathcal{O}_M$ at a point $p \in M$ will be simply denoted by $\mathcal{O}_p$.

We refer to [Mi] and [Me] for further details on the theory of manifolds with corners.

2.2. Singularly foliated manifolds

Let M be a real-analytic 3-dimensional manifold (with corners) and let $\Upsilon \in \mathbf{L}$ be a list of natural numbers.

Definition 2.1. A Υ*-tagged divisor* on M is a divisor with normal crossings $\mathfrak{D} \subset M$, together with a bijection

$$\Upsilon \longrightarrow \{\text{irreducible components of } \mathfrak{D}\},$$

which associates to each index $i \in \Upsilon$ an irreducible component $D_i \subset \mathfrak{D}$. We shall shortly write $\mathfrak{D} = \mathfrak{D}_\Upsilon$ to indicate that $\mathfrak{D}$ is a Υ-tagged divisor.

Let χ be an analytic vector field on M. Given a point $p \in M$ and a prime germ $g \in \mathbf{m}_p$ (where $\mathbf{m}_p \subset \mathcal{O}_p$ is the maximal ideal), consider the ideal $\mathcal{I}_\chi g \subset \mathcal{O}_p$ which is generated by the set

$$\{\chi(h) : h \in (g)\mathcal{O}_p\},$$

where $\chi(h)$ is the action of χ (seen as a derivation) on $h \in \mathcal{O}_p$.

Definition 2.2. The vector field χ will be called *nondegenerate* with respect to the divisor $\mathfrak{D}$, if for all points $p \in M$ and all primes $g \in \mathbf{m}_p$, one of the following two cases occurs:

(i) the set $\{g=0\}$ is not a local irreducible component of the divisor, and $\mathcal{I}_\chi(\mathbf{m}_p)$ is not divisible by g;

(ii) the set $\{g=0\}$ is a local irreducible component of the divisor, and $\mathcal{I}_\chi(g)$ is not divisible by g^2.

Choose some coordinate system $(x_1, ..., x_n)$ at p, and suppose that $g = x_1$. If we write

$$\chi = a_1 \frac{\partial}{\partial x_1} + ... + a_n \frac{\partial}{\partial x_n}, \quad a_1, ..., a_n \in \mathcal{O}_p,$$

then the ideal $\mathcal{I}_\chi(\mathbf{m}_p)$ is generated by $\{a_1, a_2, ..., a_n\}$, and the ideal $\mathcal{I}_\chi(g)$ is generated by $\{a_1, a_2 x_1, ..., a_n x_1\}$. Hence, conditions (i) and (ii) can be rewritten as follows:

(i) if $\{x_1=0\} \not\subset \mathfrak{D}$, then $\{a_1, ..., a_n\} \not\subset (x_1)\mathcal{O}_p$;

(ii) if $\{x_1=0\} \subset \mathfrak{D}$, then there exists no collection of germs $\{b_1, ..., b_n\} \subset \mathcal{O}_p$ such that we can write $a_1 = x_1^2 b_1$ and $a_j = x_1 b_j$ for $j \geqslant 2$.

Remark 2.3. Suppose that we consider (as in [C1]) the sheaf $\Theta_M[\log \mathfrak{D}]$ of vector fields adapted to $\mathfrak{D}$ (i.e. the dual of the sheaf of logarithmic forms with respect to $\mathfrak{D}$). Then an element $\chi_p \in \Theta_M[\log \mathfrak{D}]_p$ is nondegenerate if and only if the adapted coefficients are without a common divisor.

Note that a reduced vector field (as defined in the introduction) is automatically nondegenerate. However, a nondegenerate vector field can have a set of singularities Z of codimension 1. For instance,

$$\chi = x_1 \frac{\partial}{\partial x_1} + 0 \frac{\partial}{\partial x_2} + ... + 0 \frac{\partial}{\partial x_n}$$

is a nondegenerate vector field in $\mathbf{R}^n$ (if the divisor $\mathfrak{D}$ contains $\{x_1=0\}$). Note that each singular point on the hypersurface $Z = \{x_1=0\}$ is elementary.

Remark 2.4. Let $p \in M$ be a singular point of a nondegenerate vector field χ. Suppose that, for some neighborhood $U \subset M$ of p, we have $\mathrm{Ze}(\chi) \cap U = \{f=0\}$, for some analytic function f such that $df(p)\neq 0$. Then, writing $\chi = f\chi_1$ for some analytic vector field χ_1 defined in U, we obtain the following equivalence:

$$p \text{ is an elementary singular point of } \chi \quad \Longleftrightarrow \quad \chi_1(f)(p)\neq 0$$

(where χ_1 acts as a derivation on f).

As we shall see, even if we start with a vector field χ which is reduced, the procedure of resolution of singularities can produce new vector fields which belong to the more general class of nondegenerate vector fields. This is due to the occurrence of the so-called *dicritical* situations (see Example 2.10).

A *singular orientable analytic line field* on $(M,\mathfrak{D})$ is given by a collection of pairs $L=\{(U_\alpha,\chi_\alpha)\}_{\alpha\in A}$, where $\{U_\alpha\}_{\alpha\in A}$ is an open covering of M and

$$\chi_\alpha : U_\alpha \longrightarrow TU_\alpha$$

is an analytic vector field in U_α which is nondegenerate with respect to $U_\alpha\cap\mathfrak{D}$ (see Definition 2.2) and such that for each pair of indices $\alpha,\beta\in A$,

$$\chi_\alpha = h_{\alpha\beta}\cdot\chi_\beta$$

for some strictly positive analytic function $h_{\alpha\beta}: U_\alpha\cap U_\beta \to \mathbf{R}_{>0}$.

An analytic vector field χ defined in a neighborhood $U\subset M$ of a point p is called *local generator* of L if the collection $\{(U_\alpha,\chi_\alpha)\}_{\alpha\in A}\cup\{(U,\chi)\}$ is still a singular orientable analytic line field.

Let $Y\subset M$ be an analytic subset and L be a singular orientable analytic line field on $(M,\mathfrak{D})$. We shall say that L is *Y-preserving* if for each point $p\in M$ and each local generator χ of L at p, we have

$$\chi(g)\in\mathcal{I}(Y_p) \quad \text{for all } g\in\mathcal{I}(Y_p),$$

where $\mathcal{I}(Y_p)$ is the ideal in the local ring $\mathcal{O}_p$ which defines the germ Y_p, and $\chi(g)$ is the action of χ (seen as a derivation on $\mathcal{O}_p$) on g.

Definition 2.5. A *singularly foliated manifold* is a 4-tuple $\mathbf{M}=(M,\Upsilon,\mathfrak{D},L)$, where

(i) M is an analytic 3-dimensional manifold with corners;

(ii) $\Upsilon\in\mathbf{L}$ is a list of natural numbers;

(iii) $\mathfrak{D}=\mathfrak{D}_\Upsilon$ is a Υ-tagged divisor on M;

(iv) L is a singular orientable analytic line field on $(M,\mathfrak{D})$ which is $\mathfrak{D}$-preserving.

For each point $p \in M$, we define the *incidence list* at p as the sublist

$$\iota_p = \{i \in \Upsilon : p \in D_i\}, \tag{6}$$

where D_i is the ith irreducible component of $\mathfrak{D}$ (note that $0 \leqslant \#\iota_p \leqslant n$). We shall say that p is a *divisor point* if $\#\iota_p \geqslant 1$.

Given a singularly foliated manifold $\mathbf{M} = (M, \Upsilon, \mathfrak{D}, L)$, we consider the analytic subsets

$$\mathrm{Ze}(\mathbf{M}) = \{p \in M : \chi(p) = 0\} \quad \text{and} \quad \mathrm{Elem}(\mathbf{M}) = \{p \in M : \chi \text{ is elementary at } p\},$$

where χ is a local generator for L at p. The set $\mathrm{NElem}(\mathbf{M}) = \mathrm{Ze}(\mathbf{M}) \setminus \mathrm{Elem}(\mathbf{M})$ will be called the set of *nonelementary singular points* of L.

PROPOSITION 2.6. *The set* $\mathrm{NElem}(\mathbf{M})$ *is a closed analytic subset of* M *of codimension strictly greater than* 1.

Proof. Given a point $p \in M$, fix some local coordinates $(x_1, ..., x_n)$ and a local generator $\chi = a_1 \partial/\partial x_1 + ... + a_n \partial/\partial x_n$ for L at p. Then $\mathrm{NElem}(\mathbf{M})$ is locally defined by the analytic conditions

$$\{p \in M : \chi = 0 \text{ and the spectrum of } D\chi \text{ at } p \text{ is zero}\}.$$

Let us prove that this germ of analytic sets $\mathrm{NElem}(\mathbf{M})_p$ has codimension strictly greater than 1. If this is not the case, there exists some prime element $f \in \mathbf{m}_p$ such that $\{f = 0\}$ is contained in $\mathrm{NElem}(\mathbf{M})_p$. By the coherence of $\mathcal{O}_M$, we can suppose (possibly replacing p by some neighboring point) that $df(p) \neq 0$. Using Remark 2.4, we conclude that χ is necessarily divisible by f^2. This contradicts the hypothesis of nondegeneracy for χ. $\square$

2.3. Multiplicity and weighted blowing-up in $\mathbf{R}^n$

A *weight-vector* is a nonzero vector of natural numbers $\boldsymbol{\omega} = (\omega_1, ..., \omega_n) \in \mathbf{N}^n$.

The $\boldsymbol{\omega}$-*multiplicity* of a monomial $\mathbf{x}^{\mathbf{v}} = x_1^{v_1} ... x_n^{v_n}$ (with $\mathbf{v} \in \mathbf{Z}^n$) is the integer number

$$\mu_{\boldsymbol{\omega}}(\mathbf{x}^{\mathbf{v}}) = \langle \boldsymbol{\omega}, \mathbf{v} \rangle := \omega_1 v_1 + ... + \omega_n v_n.$$

More generally, the $\boldsymbol{\omega}$-*multiplicity* of a formal series $f \in \mathbf{R}[[\mathbf{x}]]$ is given by

$$\mu_{\boldsymbol{\omega}}(f) = \min\{d \in \mathbf{N} : f \text{ has a monomial } *\mathbf{x}^{\mathbf{v}} \text{ with } \mu_{\boldsymbol{\omega}}(\mathbf{x}^{\mathbf{v}}) = d\}$$

(where $*$ denotes some nonzero real number). We denote by $H_{\boldsymbol{\omega}}^d$ the subset of all formal series with $\boldsymbol{\omega}$-multiplicity equal to d.

Given formal series $a_1, ..., a_n \in \mathbf{R}[[\mathbf{x}]]$, the corresponding formal n-dimensional vector field

$$\chi = a_1 \frac{\partial}{\partial x_1} + ... + a_n \frac{\partial}{\partial x_n}$$

is a derivation on the ring $\mathbf{R}[[\mathbf{x}]]$. The $\boldsymbol{\omega}$-multiplicity of χ is the integer number

$$\mu_{\boldsymbol{\omega}}(\chi) = \max\{k \in \mathbf{Z} : \chi(H^d_{\boldsymbol{\omega}}) \subset H^{d+k}_{\boldsymbol{\omega}} \text{ for all } d \in \mathbf{N}\},$$

where $\chi(H^d_{\boldsymbol{\omega}})$ denotes the action of χ (seen as a derivation) on the subset $H^d_{\boldsymbol{\omega}}$.

Remark 2.7. Using the expression $\chi = a_1 \partial/\partial x_1 + ... + a_n \partial/\partial x_n$, we have

$$\mu_{\boldsymbol{\omega}}(\chi) = \min\{\mu_{\boldsymbol{\omega}}(a_1) - \omega_1, ..., \mu_{\boldsymbol{\omega}}(a_n) - \omega_n\}.$$

Given a weight-vector $\boldsymbol{\omega} \in \mathbf{N}^n_{>0}$, the *$\boldsymbol{\omega}$-weighted blowing-up* of $\mathbf{R}^n$ is the real-analytic surjective map

$$\begin{aligned} \Phi_{\boldsymbol{\omega}} : \mathbf{S}^{n-1} \times \mathbf{R}^+ &\longrightarrow \mathbf{R}^n, \\ (\bar{\mathbf{x}}, \tau) &\longmapsto \tau^{\boldsymbol{\omega}} \bar{\mathbf{x}} = (\tau^{\omega_1} \bar{x}_1, ..., \tau^{\omega_n} \bar{x}_n), \end{aligned}$$

where we put $\mathbf{S}^{n-1} = \{\bar{\mathbf{x}} \in \mathbf{R}^n : \bar{x}_1^2 + ... + \bar{x}_n^2 = 1\}$.

More generally, given an arbitrary weight-vector $\boldsymbol{\omega} \in \mathbf{N}^n$, we can reorder the coordinates and write $\boldsymbol{\omega} = (\omega_1, ..., \omega_k, 0, ..., 0)$, where $\omega_1, ..., \omega_k$ are strictly positive. The *$\boldsymbol{\omega}$-weighted blowing-up* is the map

$$\begin{aligned} \Phi_{\boldsymbol{\omega}} : \mathbf{S}^{k-1} \times \mathbf{R}^+ \times \mathbf{R}^{n-k} &\longrightarrow \mathbf{R}^n, \\ (\bar{\mathbf{x}}, \tau, \mathbf{x}') &\longmapsto (\tau^{\boldsymbol{\omega}} \bar{\mathbf{x}}, \mathbf{x}'), \end{aligned}$$

where $\mathbf{x}' = (x_{k+1}, ..., x_n)$. The sets

$$Y = \{x_1 = ... = x_k = 0\} \quad \text{and} \quad D = \Phi^{-1}_{\boldsymbol{\omega}}(Y) = \mathbf{S}^{k-1} \times \{0\} \times \mathbf{R}^{n-k}$$

will be called the *blowing-up center* and the *exceptional divisor* of the blowing-up, respectively. The set $\widetilde{M} = \mathbf{S}^{k-1} \times \mathbf{R}^+ \times \mathbf{R}^{n-k}$ will be called the *blowed-up space*.

It is obvious that $\Phi_{\boldsymbol{\omega}}$ restricts to a diffeomorphism between $\widetilde{M} \setminus D$ and $\mathbf{R}^n \setminus Y$. The blowing-up creates a boundary component $\partial \widetilde{M} = D$.

The definition of $\boldsymbol{\omega}$-weighted blowing-up can be easily extended to the spaces with corners $\mathbf{R}^n_s$, thus defining an analytic surjective map

$$\Phi_{\boldsymbol{\omega}} : \widetilde{M} \longrightarrow \mathbf{R}^n_s.$$

It is easy to describe the effect of the blowing-up on a divisor $\mathfrak{D} \subset \mathbf{R}^n_s$. If $\mathfrak{D} \subset \mathbf{R}^n_s$ is a divisor, then the set

$$\widetilde{\mathfrak{D}} = \Phi^{-1}_{\boldsymbol{\omega}}(\mathfrak{D}) \cup D$$

is a divisor in $\widetilde{M}$. The divisor $\widetilde{\mathfrak{D}}$ will be called the *total transform* of $\mathfrak{D}$.

Remark 2.8. It follows from our definition of manifold with corners that a divisor $\mathfrak{D}\subset\mathbf{R}^n_s$ is always given by a finite union of coordinate hyperplanes, namely

$$\mathfrak{D}=\bigcup_{i\in\iota}\{x_i=0\},$$

for some sublist $\iota\subset[n, ..., 1]$.

Now, let us fix an analytic (not identically zero) vector field χ in $\mathbf{R}^n_s$. Consider the analytic vector field χ^* defined in $\widetilde{M}\setminus D$ as the pull-back of χ under the diffeomorphism

$$\Phi_{\boldsymbol{\omega}}\colon \widetilde{M}\setminus D\longrightarrow \mathbf{R}^n_s\setminus Y.$$

The following result is obtained by a straightforward computation.

PROPOSITION 2.9. *Let $m=\mu_{\boldsymbol{\omega}}(\chi)$ be the $\boldsymbol{\omega}$-multiplicity of χ (seen as a formal vector field at the origin). Then, the new vector field*

$$\widetilde{\chi}=\tau^{-m}\cdot\chi^*$$

satisfies the following conditions:

(i) *$\widetilde{\chi}$ has an analytic extension to $\widetilde{M}$ (which we still denote by $\widetilde{\chi}$)*;

(ii) *the exceptional divisor D is an invariant manifold for $\widetilde{\chi}$*;

(iii) *if χ is nondegenerate with respect to some divisor $\mathfrak{D}\subset\mathbf{R}^n_s$, then $\widetilde{\chi}$ is nondegenerate with respect to the total transformed divisor $\widetilde{\mathfrak{D}}$.*

Proof. See, e.g., [P]. □

The vector field $\widetilde{\chi}$ will be called the *strict transform* of χ.

Example 2.10. Let us see an example of a typical *dicritical* situation, where the blowing-up of a reduced vector field results into a nondegenerate vector field whose set of singularities has codimension 1. Consider the vector field in $\mathbf{R}^3_{(x,y,z)}$

$$\chi=y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},$$

and choose the weight-vector $\boldsymbol{\omega}=(0,1,1)$. The $\boldsymbol{\omega}$-weighted blowing-up (with center $Y=\{y=z=0\}$) gives the manifold $\widetilde{M}=\mathbf{R}_x\times\mathbf{S}^1_\theta\times\mathbf{R}^+_\tau$ with the exceptional divisor

$$D=\{(x,\theta,\tau)\in\widetilde{M}:\tau=0\}.$$

Since $\mu_{\boldsymbol{\omega}}(\chi)=0$, the strict transform of χ is the *radial* vector field $\widetilde{\chi}=\tau\partial/\partial\tau$, which vanishes identically on D.

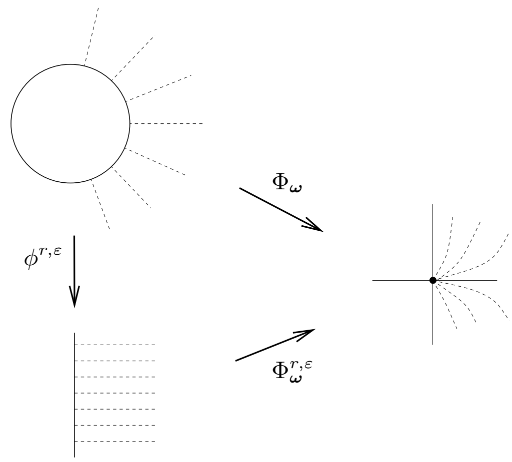


Figure 4. The x_r-directional chart (dotted lines indicate the fibration $\{d\bar{\mathbf{x}}=0\}$).

2.4. Directional charts of blowing-up

Let $\boldsymbol{\omega}=(\omega_1,...,\omega_k,0,...,0)$ be a weight-vector as above.

Given an index $1\leqslant r\leqslant k$, the x_r*-directional* $\boldsymbol{\omega}$-weighted blowing-up is the pair of analytic maps

$$\Phi_{\boldsymbol{\omega}}^{r,+}: U \longrightarrow \mathbf{R}^n\cap\{x_r\geqslant 0\} \quad \text{and} \quad \Phi_{\boldsymbol{\omega}}^{r,-}: U \longrightarrow \mathbf{R}^n\cap\{x_r\leqslant 0\},$$

with domain $U:=\mathbf{R}^{r-1}\times\mathbf{R}^+\times\mathbf{R}^{n-r}$, which are defined as follows. Write the coordinates in U as $(\tilde{x}_1,...,\tilde{x}_n)$. Then, for $\varepsilon\in\{+,-\}$, the map $\mathbf{x}=\Phi_{\boldsymbol{\omega}}^{r,\varepsilon}(\tilde{\mathbf{x}})$ is given by

$$\begin{cases} x_i=\tilde{x}_r^{\omega_i}\tilde{x}_i & \text{for } i=1,...,r-1,r+1,...,k,\\ x_r=\varepsilon\tilde{x}_r^{\omega_r}, & \\ x_j=\tilde{x}_j & \text{for } j=k+1,...,n. \end{cases}$$

PROPOSITION 2.11. *For $\varepsilon\in\{+,-\}$, there exists an analytic diffeomorphism*

$$\phi^{r,\varepsilon}: V^{r,\varepsilon}\longrightarrow U,$$

with domain $V^{r,\varepsilon}:=\{(\bar{\mathbf{x}},\tau,\mathbf{x}')\in\mathbf{S}^{k-1}\times\mathbf{R}^+\times\mathbf{R}^{n-k}:\varepsilon\bar{x}_r>0\}$, which makes the following diagram commutative:

$$\begin{array}{ccc} V^{r,\varepsilon} & \xrightarrow{\Phi_{\boldsymbol{\omega}}} & \mathbf{R}^n\cap\{\varepsilon x_r\geqslant 0\} \\ \downarrow{\scriptstyle \phi^{r,\varepsilon}} & & \downarrow{\scriptstyle \mathrm{id}} \\ U & \xrightarrow{\Phi_{\boldsymbol{\omega}}^{r,\varepsilon}} & \mathbf{R}^n\cap\{\varepsilon x_r\geqslant 0\}, \end{array}$$

where id *is the identity map.*

Proof. See [DR]. □

The pairs $(V^{r,+},\phi^{r,+})$ and $(V^{r,-},\phi^{r,-})$ will be called x_r-*directional charts* of the blowing-up.

Notice that the exceptional divisor D is mapped by $\phi^{r,\varepsilon}$ to the hyperplane $\{\tilde{x}_r=0\}$. Moreover, the union of the domains of all directional charts,

$$V^{1,+}\cup V^{1,-}\cup\ldots\cup V^{k,+}\cup V^{k,-},$$

gives an open covering of the blowed-up space $\widetilde{\mathbf{M}}=\mathbf{S}^{k-1}\times\mathbf{R}^{+}\times\mathbf{R}^{n-k}$.

2.5. Weighted trivializations and blowing-up on manifolds

Let us fix a weight-vector $\boldsymbol{\omega}=(\omega_1,\ldots,\omega_k,0,\ldots,0)$. We shall say that an analytic map $\phi\colon U\to\mathbf{R}^n$ with an open subset $U\subset\mathbf{R}^n$ as domain *preserves the* $\boldsymbol{\omega}$*-quasihomogeneous structure on* $\mathbf{R}^n$ if

$$\phi^*(H^d_{\boldsymbol{\omega}})\subset H^d_{\boldsymbol{\omega}}$$

for each natural number $d\in\mathbf{N}$.

Let M be an n-dimensional analytic manifold (with corners) and $Y\subset M$ be a submanifold of codimension k. A *trivialization atlas* for $Y\subset M$ is a collection of pairs $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in A}$, where $\{U_\alpha\}_{\alpha\in A}$ is an open covering of Y and

$$\phi_\alpha\colon U_\alpha\longrightarrow\mathbf{R}^n_s$$

is a local chart such that $\phi_\alpha(Y\cap U_\alpha)=\{0\}\times\mathbf{R}^{n-k}_{s'}$ for some $s'\leqslant s$.

We shall say that $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in A}$ is an $\boldsymbol{\omega}$-*weighted* trivialization atlas if for each pair of indices $\alpha,\beta\in A$, the transition map

$$\phi_{\alpha\beta}:=\phi_\beta\circ\phi_\alpha^{-1}\colon\phi_\alpha(U_\alpha\cap U_\beta)\longrightarrow\phi_\beta(U_\alpha\cap U_\beta)$$

preserves the ω-quasihomogeneous structure on $\mathbf{R}^n$.

Proposition 2.12. *Let* $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in A}$ *be an* $\boldsymbol{\omega}$*-weighted trivialization atlas for a submanifold* $Y\subset M$. *Then, there exists an* n*-dimensional analytic manifold* $\widetilde{M}$ *and a proper analytic surjective map*

$$\Phi\colon\widetilde{M}\longrightarrow M$$

such that the following conditions hold:

(i) Φ *induces a diffeomorphism between* $M\setminus Y$ *and* $\widetilde{M}\setminus D$, *where* $D:=\Phi^{-1}(Y)$;

(ii) *there exists a collection of local charts* $\{(\widetilde{U}_\alpha, \tilde{\phi}_\alpha)\}_{\alpha\in A}$ *in* $\widetilde{M}$, *where* $\{\widetilde{U}_\alpha\}_{\alpha\in A}$ *is an open covering of* D *and*

$$\tilde{\phi}_\alpha : \widetilde{U}_\alpha \longrightarrow \mathbf{S}^{k-1}_{s''} \times \mathbf{R}^+ \times \mathbf{R}^{n-k}_{s'}$$

is an analytic diffeomorphism (*where* $\mathbf{S}^{k-1}_t = \{\bar{\mathbf{x}} \in \mathbf{S}^{k-1} : \bar{x}_1 \geqslant 0, ..., \bar{x}_t \geqslant 0\}$ *and* $s'+s''=s$), *such that the following diagram is commutative*:

$$\begin{array}{ccc} \widetilde{U}_\alpha & \xrightarrow{\ \Phi\ } & U_\alpha \\ {\scriptstyle \tilde{\phi}_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \phi_\alpha} \\ \mathbf{S}^{k-1}_{s''} \times \mathbf{R}^+ \times \mathbf{R}^{n-k}_{s'} & \xrightarrow{\ \Phi_{\boldsymbol{\omega}}\ } & \mathbf{R}^n, \end{array}$$

where $\Phi_{\boldsymbol{\omega}}$ *is the* $\boldsymbol{\omega}$*-weighted blowing-up in* $\mathbf{R}^n$.

Proof. See [DR, Proposition II.9]. □

The map $\Phi: \widetilde{M} \to M$ will be called an $\boldsymbol{\omega}$*-weighted blowing-up of* M with center on Y, with respect to the trivialization $\{(\widetilde{U}_\alpha, \tilde{\phi}_\alpha)\}_{\alpha\in A}$.

Remark 2.13. The existence of an $\boldsymbol{\omega}$-weighted trivialization for a submanifold $Y\subset M$ can be a strong topological restriction. This condition can be defined in a more intrinsic way as the existence of a certain nested sequence of subbundles in the conormal bundle N^*Y (see, e.g., [Me, §5.15]).

2.6. Blowing-up of singularly foliated manifolds

Let $\mathbf{M}=(M, \Upsilon, \mathfrak{D}, L)$ be a singularly foliated manifold and $Y\subset M$ be a submanifold which has an $\boldsymbol{\omega}$-weighted trivialization. An $\boldsymbol{\omega}$*-weighted* blowing-up of $\mathbf{M}$ with center Y is a mapping

$$\Phi: \widetilde{\mathbf{M}} \longrightarrow \mathbf{M}$$

defined by taking the 4-tuple $\widetilde{\mathbf{M}}=(\widetilde{M}, \widetilde{\Upsilon}, \widetilde{\mathfrak{D}}, \tilde{L})$ in the following way:

(i) The mapping $\Phi: \widetilde{M}\to M$ is the $\boldsymbol{\omega}$-weighted blowing-up of M with center on Y;

(ii) The list $\widetilde{\Upsilon}$ is given by $\Upsilon\cup[n]$, where

$$n := \begin{cases} 1+\max\{i : i\in\Upsilon\}, & \text{if } \Upsilon\neq\varnothing, \\ 1, & \text{if } \Upsilon=\varnothing; \end{cases}$$

(iii) The divisor $\widetilde{\mathfrak{D}}$ is the total transform of $\mathfrak{D}$, with the tagging

$$\widetilde{\Upsilon} \ni i \longmapsto \begin{cases} D_i', & \text{if } i \in \widetilde{\Upsilon} \setminus [n], \\ \widetilde{D}, & \text{if } i = n, \end{cases}$$

where $\widetilde{D} := \Phi^{-1}(Y)$ and D_i' is the strict transform of the corresponding divisor $D_i \subset M$ (for each $i \in \Upsilon$);

(iv) The line field $\tilde{L}$ is obtained as follows: up to some refinement of the coverings, we can suppose that the line field L is given by a collection $\{(U_\beta, \chi_\beta)\}_{\beta \in B}$, where χ_β is a nondegenerate analytic vector field defined in U_β, and that there exists some subcollection of indices $A \subset B$ such that $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ is the $\boldsymbol{\omega}$-weighted trivialization of Y. For each $\alpha \in B$, we can consider the strict transform $\widetilde{\chi}_\alpha$ of χ_α (see Proposition 2.9) as an analytic vector field defined in $\widetilde{U}_\alpha = \Phi^{-1}(U_\alpha)$. Now, Proposition 2.12 implies that the collection $\{(\widetilde{U}_\alpha, \widetilde{\chi}_\alpha)\}_{\alpha \in B}$ defines a singular line field $\tilde{L}$ on $\widetilde{M}$ which satisfies our requirements (see [DR] or [P] for the details).

2.7. Axis definition and controllability

Let $\mathbf{M} = (M, \Upsilon, \mathfrak{D}, L)$ be a singularly foliated manifold of dimension 3.

As we explained in §1.3, the local strategy for the resolution of singularities at a point $p \in \mathrm{NElem}(\mathbf{M})$ is based on some invariants attached to the Newton polyhedron. This Newton polyhedron depends on the vector field χ which locally generates the line field, but also on the choice of local coordinates (x, y, z) at p. This usually creates difficulties for obtaining a global strategy for the resolution, since the information obtained from the polyhedron is coordinate-dependent.

In order to obtain *intrinsic* invariants, we have to restrict the choice of local coordinates and require that they respect some additional *structure* on the ambient space. We now introduce such a structure.

Definition 2.14. An *axis* for $\mathbf{M}$ is given by a pair $\mathrm{Ax} = (A, \mathfrak{z})$, where $A \subset M$ is an open neighborhood of the set $\mathrm{NElem}(\mathbf{M})$, and $\mathfrak{z}$ is a singular orientable analytic line field defined on A such that the following conditions hold:

(i) $\mathfrak{z}$ is $\mathfrak{D} \cap A$-preserving;

(ii) $\mathrm{Ze}(\mathfrak{z}) = \varnothing$ (where $\mathrm{Ze}(\mathfrak{z})$ is the set of singularities of $\mathfrak{z}$);

(iii) for each point $p \in A \cap \mathfrak{D}$, if we choose a local chart $(U, (x, y, z))$ such that $\mathfrak{z}$ is locally generated by $\partial/\partial z$, then

$$\mathcal{I}_p \not\subset \mathcal{J}_p,$$

where $\mathcal{I}_p \subset \mathcal{O}_p$ is the ideal which defines the germ of analytic sets $\mathrm{NElem}(\mathbf{M})_p$ and $\mathcal{J}_p \subset \mathcal{O}_p$ is the defining ideal of the set $\{x = y = 0\}$ (i.e. the leaf of the axis through the point p);

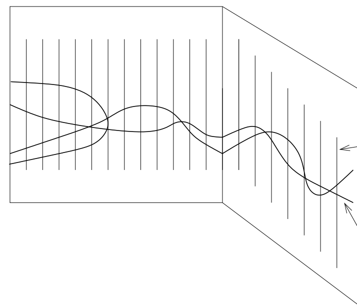


Figure 5. Axis.

(iv) for each point $p \in A \setminus \mathfrak{D}$, if we choose a local chart $(U,(x,y,z))$ such that $\mathfrak{z}$ is locally generated by $\partial/\partial z$, then

$$\chi(\mathcal{J}_p) \not\subset \mathcal{J}_p,$$

where χ is a local generator of L.

The requirement in condition (iii) is equivalent to saying that $\{x=y=0\}$ is not contained in NElem(**M**). The (stronger) requirement in condition (iv) is equivalent to saying that $\{x=y=0\}$ is not an invariant curve for the line field L.

Remark 2.15. It is not always possible to define an axis for a singularly foliated line field. For instance, if there exists a point $p \in \mathrm{NElem}(\mathbf{M})$ such that $\#\iota_p=3$, then any line field which is $\mathfrak{D}$-preserving necessarily vanishes at p. In this case, the requirement in condition (ii) of the definition cannot be satisfied.

We shall say that the singularly foliated manifold **M** is *controllable* if there exists an axis Ax as defined above. The pair $(\mathbf{M}, \mathrm{Ax})$ will be called a *controlled singularly foliated manifold.*

The next result describes a situation where an axis can always be defined. Let M be an analytic manifold of dimension 3 without boundary and let χ be a reduced analytic vector field defined in M. We consider the singularly foliated manifold $\mathbf{M}=(M,\Upsilon,\mathfrak{D},L)$, where $\Upsilon=\varnothing$, $\mathfrak{D}=\varnothing$ and $L=L_\chi$ is the analytic line field generated by χ.

Proposition 2.16. *Given a singularly foliated manifold* $\mathbf{M}=(M,\varnothing,\varnothing,L_\chi)$ *as above, there exists an axis* $\mathrm{Ax}=(A,\mathfrak{z})$ *for* **M**. *The pair* $(\mathbf{M},\mathrm{Ax})$ *is a controlled singularly foliated manifold.*

Proof. The set of nonelementary points of **M** is a 1-dimensional analytic subset $\mathrm{NElem} \subset M$. Let $S \subset \mathrm{NElem}$ be the discrete subset of points where NElem is not locally smooth.

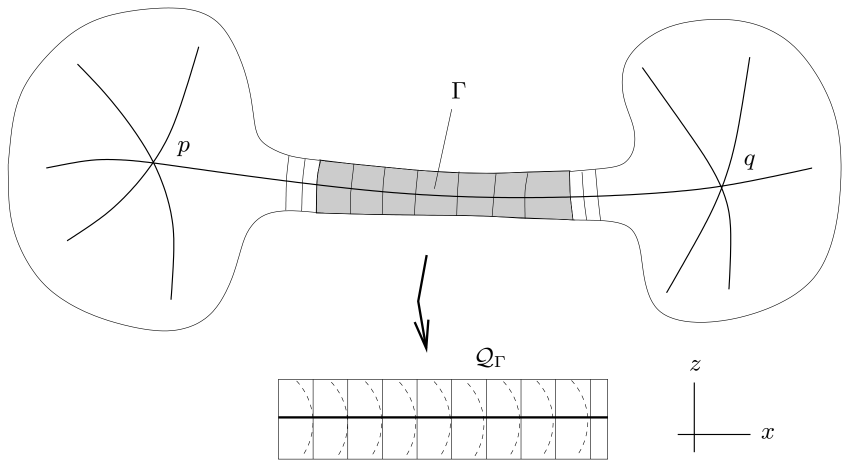


Figure 6. The definition of $Z_{\mathcal{Q}}$ on the strip $\mathcal{Q}$.

First of all, we are going to define a nonsingular vector field Z_p in an open neighborhood $U_p \subset M$ of each point $p \in S$ with the property that no trajectory of Z_p is a leaf of $L \cap U_p$. We fix arbitrary local coordinates (x, y, z) in a neighborhood of p and construct the Newton polyhedron $\mathcal{N}$ for the vector field χ with respect to these coordinates. If the support of $\mathcal{N}$ contains at least one point in the region

$$(\{-1\} \times \{0\} \times \mathbf{Z}) \cup (\{0\} \times \{-1\} \times \mathbf{Z}),$$

then it suffices to locally define Z_p as the vector field $\partial/\partial z$. Otherwise, it follows that $\{x=y=0\}$ is an invariant curve for the vector field χ. In this case, it is immediate to verify that we can choose natural numbers $s, t \in \mathbf{N}_{>0}$ such that the change of coordinates

$$\tilde{x} = x + z^s, \quad \tilde{y} = y + z^t \quad \text{and} \quad \tilde{z} = z$$

results in a new coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ where the above property holds.

Now, we are going to glue together the collection of vector fields $\{Z_p\}_{p \in S}$ in a C^∞ way along the smooth part of NElem. Let $\Gamma \subset \text{NElem} \setminus S$ be a regular analytic curve connecting two points $p, q \in S$. Possibly restricting U_p to some smaller neighborhood of p, we may assume the Z_p is transversal to $\Gamma \cap U_p$ (recall that Γ is an analytic arc). Therefore, in some neighborhood of $\Gamma \cap U_p$, we can define a 2-dimensional strip $\mathcal{Q}_p$ formed by the union of all trajectories of Z_p starting at points of $\Gamma \cap U_p$. The same argument gives us a 2-dimensional strip $\mathcal{Q}_q$ with base $\Gamma \cap U_q$.

Using the tubular neighborhood theorem, we can glue together these two strips in a C^∞ way, as shown in Figure 6. Therefore, we get a global 2-dimensional strip $\mathcal{Q}$ with base Γ. Using partitions of unity, it is easy to define a nonsingular C^∞ vector field $Z_\mathcal{Q}$ in an open neighborhood of $\mathcal{Q}$ such that the following conditions hold:

(i) $Z_\mathcal{Q}$ is tangent to the strip $\mathcal{Q}$;

(ii) $Z_\mathcal{Q} \cap U_p = Z_p$ and $Z_\mathcal{Q} \cap U_q = Z_q$;

(iii) no trajectory of $Z_\mathcal{Q}$ is left invariant by χ.

Putting together all these local constructions, we finally obtain a nonsingular C^∞ vector field Z defined in an open neighborhood $A \subset M$ of NElem($\mathbf{M}$) which has the following property:

(P) No trajectory of Z is invariant by the vector field χ.

Using Grauert's embedding theorem [G], we can analytically embed the manifold M in $\mathbf{R}^k$, for some sufficiently large $k \in \mathbf{N}$. Doing so, the vector field Z can be seen as a map $Z: A \to \mathbf{R}^k$, and it is clear that property (P) is an open property for the Whitney topology on $C^\infty(\mathbf{R}^k, \mathbf{R}^k)$. Therefore, using Weierstrass' approximation theorem (in the version of [G]), we can approximate Z by an analytic nonsingular vector field $\widetilde{Z}: A \to TM$ which also has property (P). This proves the proposition. □

3. Newton polyhedron and adapted coordinates

3.1. Adapted local charts

Let $(\mathbf{M}, \mathrm{Ax})$ be a controlled singularly foliated manifold, with $\mathbf{M} = (M, \Upsilon, \mathfrak{D}, L)$ and $\mathrm{Ax} = (A, \mathfrak{z})$.

A local chart $(U, (x, y, z))$ centered at a point $p \in A$ will be called an *adapted local chart* if the following conditions hold:

- $\mathfrak{z}$ is locally generated by $\partial/\partial z$;
- if $p \in \mathfrak{D}$ and $\iota_p = [i]$, then $D_i = \{x = 0\}$;
- if $p \in \mathfrak{D}$ and $\iota_p = [i, j]$ (with $i > j$), then $D_i = \{x = 0\}$ and $D_j = \{y = 0\}$;

where ι_p is the incidence list defined in (6).

Notice that an adapted local chart can always be defined at a point $p \in A$. The condition $p \in A \cap \mathfrak{D}$ automatically implies that $\#\iota_p \in \{1, 2\}$, by Remark 2.15.

Despite the fact that the definition of adapted local chart is given for all points in A, we shall be mostly concentrated (at least until the end of §4) on points lying in $A \cap \mathfrak{D}$.

In Figure 7 we represent the two possible configurations with the corresponding position of the divisors.

Proposition 3.1. *Let $(U, (x, y, z))$ and $(U', (x', y', z'))$ be two adapted local charts at a point $p \in A$. Then, the transition map has the form*

$$x' = F(x, y), \quad y' = G(x, y) \quad \textit{and} \quad z' = f(x, y) + z w(x, y, z),$$

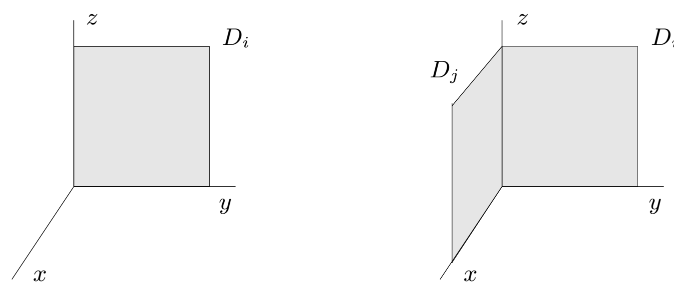


Figure 7. Adapted local charts (with $i>j$).

where w is a unit and

$$\frac{\partial(F,G)}{\partial(x,y)}(0,0)\neq 0.$$

More specifically, if $\#\iota_p=1$ then F and G have the particular form

$$F(x,y)=xu(x,y) \quad \textit{and} \quad G(x,y)=g(x)+yv(x,y),$$

where u and v are units and $g(0)=0$. Similarly, if $\#\iota_p=2$ then

$$F(x,y)=xu(x,y) \quad \textit{and} \quad G(x,y)=yv(x,y)$$

for some units u and v.

Proof. The change of coordinates should map the vector field $\partial/\partial z$ into the vector field $U\cdot\partial/\partial z'$ (for some unit $U\in\mathcal{O}_p$). Moreover, if $\#\iota_p\geqslant 1$, it maps the divisor $\{x=0\}$ into $\{x'=0\}$. If $\#\iota_p=2$, the divisor $\{y=0\}$ should also be mapped to $\{y'=0\}$. □

3.2. Newton map and Newton data

Let $(\mathbf{M},\mathrm{Ax})$ be a controlled singularly foliated manifold, with $\mathbf{M}=(M,\Upsilon,\mathfrak{D},L)$ and $\mathrm{Ax}=(A,\mathfrak{z})$.

We fix a point $p\in A$ and an adapted local chart $(U,(x,y,z))$ centered at p. Our goal is to define the Newton polyhedron of $(\mathbf{M},\mathrm{Ax})$ at p with respect to the coordinates (x,y,z).

First of all, we choose an analytic vector field χ which generates L at U. Next, we expand χ in the *logarithmic basis*: Consider the meromorphic functions

$$f:=\chi(\ln x)=\frac{\chi(x)}{x},\quad g:=\chi(\ln y)=\frac{\chi(y)}{y}\quad\text{and}\quad h:=\chi(\ln z)=\frac{\chi(z)}{z},\tag{7}$$

where χ acts as a derivation on $\mathbf{R}\{x,y,z\}$. Then, we can write

$$\chi=fx\frac{\partial}{\partial x}+gy\frac{\partial}{\partial y}+hz\frac{\partial}{\partial z}.$$

Remark 3.2. If χ is $\{x{=}0\}$-preserving (respectively, $\{y{=}0\}$- and $\{z{=}0\}$-preserving) then f (respectively, g and h) is an analytic germ in $\mathbf{R}\{x,y,z\}$.

We can write the Laurent series expansion of the functions (f,g,h) given in (7) as

$$(f,g,h)=\sum_{\mathbf{v}\in\mathbf{Z}^3}(f_{\mathbf{v}},g_{\mathbf{v}},h_{\mathbf{v}})\cdot x^{v_1}y^{v_2}z^{v_3},$$

where $(f_{\mathbf{v}},g_{\mathbf{v}},h_{\mathbf{v}})$ is a vector in $\mathbf{R}^3$ for each integer vector $\mathbf{v}{=}(v_1,v_2,v_3)\in\mathbf{Z}^3$. The *Newton map* for χ at p, relative to the chart $(U,(x,y,z))$, is the map

$$\begin{aligned}\Theta\colon \mathbf{Z}^3 &\longrightarrow \mathbf{R}^3,\\ \mathbf{v}&\longmapsto (f_{\mathbf{v}},g_{\mathbf{v}},h_{\mathbf{v}}).\end{aligned}$$

The *support* of Θ is given by

$$\operatorname{supp}(\Theta)=\{\mathbf{v}\in\mathbf{Z}^n:\Theta(\mathbf{v})\neq 0\}.$$

Remark 3.3. The Newton map Θ has the following properties:

- $\operatorname{supp}(\Theta)\subset\mathbf{N}^3\cup(\{-1\}\times\mathbf{N}^2)\cup(\mathbf{N}\times\{-1\}\times\mathbf{N})\cup(\mathbf{N}^2\times\{-1\})$;
- $\mathbf{v}\in(\{-1\}\times\mathbf{N}^2)\Rightarrow\Theta(\mathbf{v})\in\mathbf{R}\times\{0\}\times\{0\}$;
- $\mathbf{v}\in(\mathbf{N}\times\{-1\}\times\mathbf{N})\Rightarrow\Theta(\mathbf{v})\in\{0\}\times\mathbf{R}\times\{0\}$;
- $\mathbf{v}\in(\mathbf{N}^2\times\{-1\})\Rightarrow\Theta(\mathbf{v})\in\{0\}\times\{0\}\times\mathbf{R}$.

The *Newton polyhedron* for $(\mathbf{M},\mathrm{Ax})$ at p, relative to the chart $(U,(x,y,z))$, is the convex polyhedron in $\mathbf{R}^3$ given by

$$\mathcal{N}=\operatorname{conv}(\operatorname{supp}(\Theta))+\mathbf{R}^3_+,$$

where $\operatorname{conv}(\cdot)$ is the convex closure operation and the "+" operator denotes the usual Minkowski sum of convex polyhedrons.

LEMMA 3.4. *The Newton polyhedron is independent of the choice of the local generator of L.*

Proof. Indeed, if χ and χ' are two local generators, we know that $\chi'{=}U\chi$ for some unit $U\in\mathbf{R}\{x,y,z\}$. Going back to the definition of the Newton polyhedron, it is clear that the corresponding polyhedrons $\mathcal{N}$ and $\mathcal{N}'$ will coincide. □

It is obvious that different choices of the local coordinates (x,y,z) lead to different Newton polyhedrons. Later on, we shall see that certain essential properties of $\mathcal{N}$ are preserved by the action of the group of coordinate changes defined by Proposition 3.1.

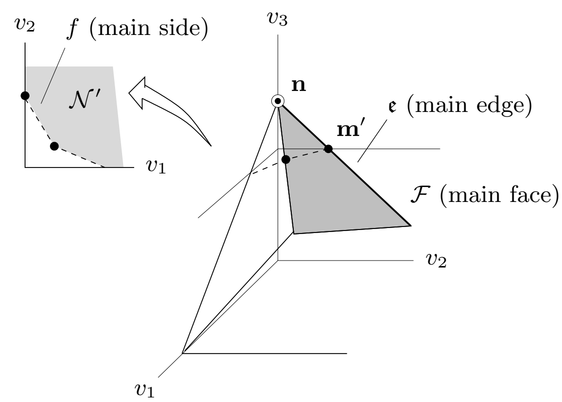


Figure 8. The main vertex, the main edge and the derived polygon for $\mathbf{n}$.

From now on, we shall adopt the usual language of the theory of convex polyhedrons, and refer to the *vertices*, *edges* and *faces* of $\mathcal{N}$ (the faces will always be 2-*dimensional*).

Given a face $F \subset \mathcal{N}$, there exists a weight-vector $\boldsymbol{\omega} \in \mathbf{N}^3$ and an integer $\mu \in \mathbf{Z}$ such that

$$F = \mathcal{N} \cap \{\mathbf{v} \in \mathbf{R}^3 : \langle \boldsymbol{\omega}, \mathbf{v} \rangle = \mu\}.$$

Notice that if this property is satisfied for a pair $(\boldsymbol{\omega}, \mu)$, then it is satisfied on the entire positive ray $R = \{t \cdot (\boldsymbol{\omega}, \mu) : t > 0\}$. The *weight-vector* and the *multiplicity* associated with F are given by the unique pair $(\boldsymbol{\omega}, \mu) \in R$ such that $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is a nonzero vector of natural numbers satisfying $\gcd(\omega_1, \omega_2, \omega_3) = 1$.

Definition 3.5. The triple $\Omega = ((x, y, z), \iota_p, \Theta)$ will be called a *Newton data* for the controlled singularly foliated manifold $(\mathbf{M}, \mathrm{Ax})$ (centered) at the point p.

For notational simplicity, we shall write *vertices*, *edges* and *faces* of Ω when referring to the corresponding objects of the Newton polyhedron $\mathcal{N}$. We shall also refer to the support of the Newton map Θ simply as $\mathrm{supp}(\Omega)$.

3.3. Derived polygon and displacements

Let us fix a Newton data $\Omega = ((x, y, z), \iota_p, \Theta)$ at a point $p \in A$, and let $\mathcal{N}$ be the corresponding Newton polyhedron. The *derived polygon* associated with a vertex $\mathbf{n} \in \mathcal{N}$ is given by

$$\mathcal{N}'(\mathbf{n}) := \mathcal{N} \cap \left\{(v_1, v_2, v_3) \in \mathbf{R}^3 : v_3 = n_3 - \tfrac{1}{2}\right\}.$$

Thus, $\mathcal{N}'(\mathbf{n})$ is a convex polygon contained in the plane $\left\{\mathbf{v} : v_3 = n_3 - \frac{1}{2}\right\}$ (see Figure 8).

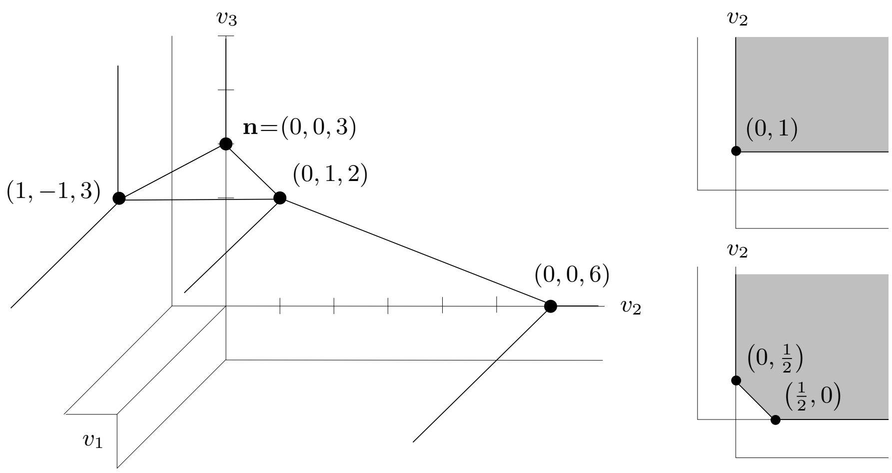


Figure 9. The Hironaka characteristic polygon and the derived polygon are distinct.

Remark 3.6. The derived polygon has some similarities with the *characteristic polygon* introduced by Hironaka is his proof of the resolution of singularities for excellent surfaces (we refer to [H3] for the precise definition of this polygon). The following example shows that these two notions are distinct in the context of vector fields: Consider the germ of vector fields

$$\chi=(z^3x+xyz^2)\frac{\partial}{\partial x}+xz^3\frac{\partial}{\partial y}+y^7\frac{\partial}{\partial z}.$$

The associated Newton polyhedron is shown in Figure 9 (left). Let us choose the vertex $\mathbf{n}=(0,0,3)$ (this is the minimal vertex of $\mathcal{N}$ with respect to the lexicographical ordering in $\mathbf{R}^3$). Then, the derived polygon and the Hironaka characteristic polyhedron are given respectively by

$$\mathcal{N}'=\mathcal{N}\cap\{\mathbf{v}\in\mathbf{R}^3:v_3=\tfrac{5}{2}\}\quad\text{and}\quad\mathcal{N}''=\mathcal{N}\cap\{\mathbf{v}\in\mathbf{R}^3:v_3=2\}.$$

The resulting polygons are depicted in the right part of Figure 9.

PROPOSITION 3.7. *Suppose that the vertex* $\mathbf{n}$ *is such that* $n_3\geqslant 1$. *Then, the derived polygon* $\mathcal{N}'(\mathbf{n})$ *is nonempty.*

Proof. Indeed, suppose by contradiction that $\mathcal{N}'(\mathbf{n})=\varnothing$. Then, since $n_3\geqslant 1$, the Newton polyhedron $\mathcal{N}$ should be contained in the region $\{(v_1,v_2,v_3)\in\mathbf{Z}^3:v_3\geqslant 1\}$. According to the definition of $\mathcal{N}$, this would imply that the line field L is locally generated by a vector field χ which is degenerate (because the ideal $\mathcal{I}_\chi(z)$ would be divisible by z^2). This contradicts our assumptions. □

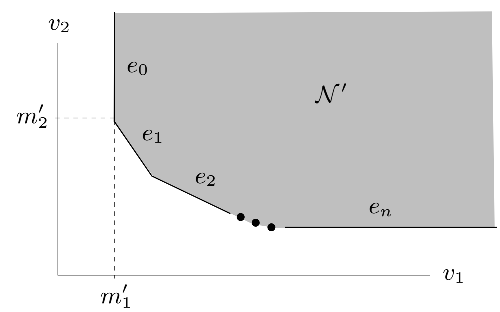


Figure 10. The derived polygon.

For the rest of this subsection, let us assume that the derived polygon $\mathcal{N}'(\mathbf{n})$ is nonempty.

The *main derived vertex* of $\mathcal{N}'(\mathbf{n})$ is the minimal vertex $\mathbf{m}'(\mathbf{n})$ of $\mathcal{N}'(\mathbf{n})$ with respect to the lexicographical ordering. We write

$$\mathbf{m}'(\mathbf{n}) = \left(m_1'(\mathbf{n}), m_2'(\mathbf{n}), n_3 - \tfrac{1}{2}\right).$$

The *main edge* associated with the vertex $\mathbf{n}$ is the unique edge $\mathfrak{e}(\mathbf{n}) \subset \mathcal{N}$ which contains the segment $\overline{\mathbf{n}, \mathbf{m}'(\mathbf{n})}$.

Proposition 3.8. *The rational numbers $m_1'(\mathbf{n})$ and $m_2'(\mathbf{n})$ always belong to the finite grid*

$$\frac{1}{2(n_3+1)!}\mathbf{Z}.$$

Proof. Indeed, the main edge associated with $\mathbf{n}$ has the form $\mathfrak{e}(\mathbf{n}) = \overline{\mathbf{n}, \mathbf{v}}$, for some vertex $\mathbf{v} = (v_1, v_2, v_3)$ such that $-1 \leqslant v_3 < n_3$. Then, it is clear that

$$m_1'(\mathbf{n}) = n_1 + \frac{v_1 - n_1}{2(n_3 - v_3)} \quad \text{and} \quad m_2'(\mathbf{n}) = n_2 + \frac{v_2 - n_2}{2(n_3 - v_3)}.$$

Now, it suffices to remark that the denominator of these fractions fractions always lies in the range $\{1, ..., 2(n_3+1)!\}$. □

We picture $\mathcal{N}'(\mathbf{n})$ in the 2-dimensional plane as in Figure 10, with the horizontal axis corresponding to the v_1-coordinate. Using this representation, we enumerate the sides of $\mathcal{N}'(\mathbf{n})$ from left to right as $e_0, e_1, ..., e_n$, with e_0 being the infinite vertical side and e_n being the infinite horizontal side.

The *main side* $f(\mathbf{n})$ of $\mathcal{N}'(\mathbf{n})$ is defined as follows (see Figure 10):

$$f(\mathbf{n}) := \begin{cases} e_0, & \text{if } m_1'(\mathbf{n}) > 0, \\ e_1, & \text{if } m_1'(\mathbf{n}) = 0. \end{cases}$$

By the definition of $\mathcal{N}'(\mathbf{n})$, to each each side $e\in\mathcal{N}'(\mathbf{n})$ there corresponds a unique face F of $\mathcal{N}$ such that

$$F\cap\mathcal{N}'(\mathbf{n})=e.$$

The *main face* associated with the vertex $\mathbf{n}$ is the unique face $\mathcal{F}(\mathbf{n})\subset\mathcal{N}$ such that

$$\mathcal{F}(\mathbf{n})\cap\mathcal{N}'(\mathbf{n})=f(\mathbf{n})$$

(see Figure 8). By construction, the edge $\mathfrak{e}(\mathbf{n})$ can be uniquely written as

$$\mathfrak{e}(\mathbf{n})=\{\mathbf{n}+t(\Delta,-1):t\in I\},$$

where $I\subset\mathbf{R}$ is a compact interval and $\Delta\in\mathbf{Q}^2\setminus\{(0,0)\}$ is a nonzero vector of rational numbers. Using this, the derived vertex $\mathbf{m}'(\mathbf{n})$ can be rewritten as

$$\mathbf{m}'(\mathbf{n})=\left(n_1+\tfrac{1}{2}\Delta_1, n_2+\tfrac{1}{2}\Delta_2, n_3-\tfrac{1}{2}\right).$$

Similarly, the main side $f(\mathbf{n})$ of $\mathcal{N}'(\mathbf{n})$ can be uniquely written as

$$f(\mathbf{n})=\{\mathbf{m}'(\mathbf{n})+t(C,-1,0):t\in I\},$$

where $I\subset\mathbf{R}$ is an interval and C is a number in $\overline{\mathbf{Q}}_{\geqslant 0}:=\mathbf{Q}_{\geqslant 0}\cup\{\infty\}$.

Remark 3.9. Observe that $C=\infty$ and $C=0$ correspond to the cases where the main side is the infinite horizontal and the infinite vertical side, respectively (see Figure 11).

We will call $\Delta(\mathbf{n}):=\Delta$ and $C(\mathbf{n}):=C$ the *vertical* and the *horizontal displacement vector* associated with the vertex $\mathbf{n}$, respectively.

3.4. Regular-nilpotent configurations and main vertex

Let us keep the notation of the previous subsection. In this subsection, we further assume that the base point $p\in A$ belongs to the divisor $\mathfrak{D}$.

The *higher vertex* is the minimal point $\mathbf{h}\in\mathcal{N}$ with respect to the lexicographical ordering in $\mathbf{R}^3$. It is immediate to see that the minimal point always exists and that it is a vertex of $\mathcal{N}$ (see Remark 3.3).

PROPOSITION 3.10. *The higher vertex* $\mathbf{h}=(h_1,h_2,h_3)$ *has the following properties*:
(i) *if* $\#\iota_p=1$, *then* $h_1=0$ *and* $h_2,h_3\geqslant -1$;
(ii) *if* $\#\iota_p=2$, *then* $h_1=0$, $h_2\geqslant 0$ *and* $h_3\geqslant -1$.

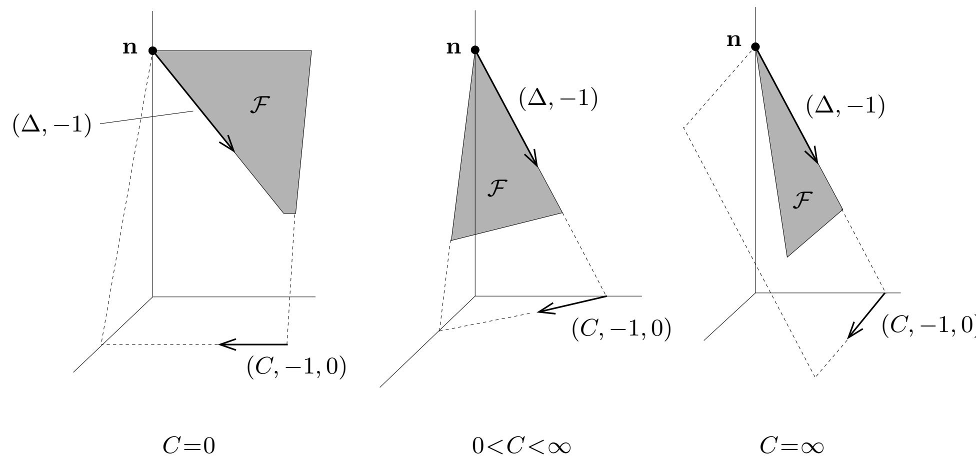


Figure 11. The vertical and horizontal displacements.

Proof. To prove (i), we observe that the surface $\{x=0\}$ is preserved by the vector field χ if and only if

$$\operatorname{supp}(\mathcal{N})\cap\{-1\}\times\mathbf{N}^2=\varnothing.$$

This is equivalent to saying that $h_1\geqslant 0$ and $h_2, h_3\geqslant -1$. However, if $h_1\geqslant 1$, the ideal $\mathcal{I}_\chi(x)\in\mathbf{R}\{x,y,z\}$, which is generated by (fx, gxy, hxz), would be divisible by x^2. This would contradict the hypothesis that χ is a nondegenerate vector field (see Definition 2.2).

The proof of (ii) is analogous. □

The main edge associated with the higher vertex is given by

$$\mathfrak{e}(\mathbf{h})=\overline{\mathbf{h},\mathbf{n}}, \tag{8}$$

where $\mathbf{n}$ is also a vertex of $\mathcal{N}$. It follows from Proposition 3.7 that this edge always exists if $h_3\geqslant 1$. We define $\mathfrak{e}(\mathbf{h}):=\varnothing$ if the derived polygon $\mathcal{N}'(\mathbf{h})$ is empty.

We shall say that the Newton data Ω is in a *nilpotent configuration* if the following three conditions are satisfied:

(i) $\#\iota_p=1$;

(ii) $\mathbf{h}=(0,-1,h_3)$ for some integer $h_3\in\mathbf{N}$;

(iii) $\mathbf{n}=(0,0,n_3)$ for some integer $n_3<h_3$.

If one of these conditions fails, we shall say that Ω is in a *regular configuration*.

Remark 3.11. As we shall see in §4, the treatment of nilpotent configurations constitutes one of the points where the method of resolution of singularities for vector fields differs *essentially* from the usual methods of resolution of singularities for functions and

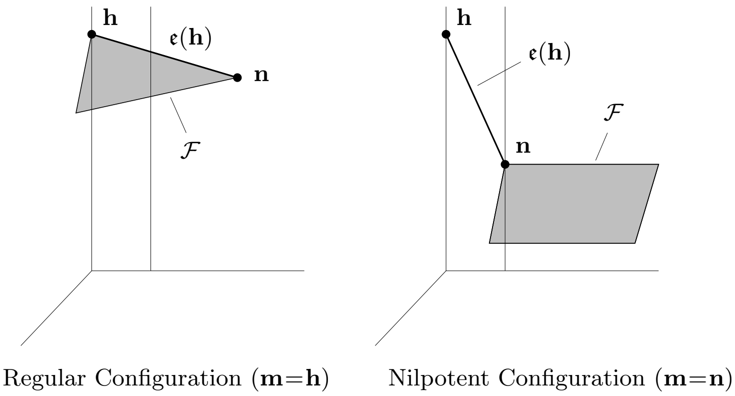


Figure 12. The regular and nilpotent configurations.

analytic sets. At several points during our proof, we will have to address the delicate issue of the *transition* between regular and nilpotent configurations.

The *main vertex* $\mathbf{m}$ of Ω is chosen as follows:

$$\mathbf{m} := \begin{cases} \mathbf{h}, & \text{if } \Omega \text{ is in a regular configuration,} \\ \mathbf{n}, & \text{if } \Omega \text{ is in an nilpotent configuration,} \end{cases}$$

(where $\mathbf{n}$ is defined by (8)). The corresponding vertical and horizontal displacements

$$\Delta := \Delta(\mathbf{m}) \quad \text{and} \quad C := C(\mathbf{m})$$

will be called the *vertical displacement vector* and the *horizontal displacement* of Ω.

The face $\mathcal{F}:=\mathcal{F}(\mathbf{m})$ and the edge $\mathfrak{e}:=\mathfrak{e}(\mathbf{m})$ will be called the *main face* and the *main edge* associated with Ω, respectively. The polygon $\mathcal{N}'=\mathcal{N}'(\mathbf{m})$ will be called the *main derived polygon*.

3.5. The class $\mathbf{New}^{i,\mathbf{m}}_{\Delta,C}$

Let $\Omega=((x,y,z),\iota_p,\Theta)$ be a Newton data for $(\mathbf{M},\mathrm{Ax})$ at a point $p\in A\cap\mathfrak{D}$. We shall say that Ω belongs to the *class* $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$ if the following conditions hold:

(i) $\#\iota_p=i$;

(ii) $\mathbf{m}$ is the main vertex of Ω;

(iii) Δ is the vertical displacement vector of Ω;

(iv) C is the horizontal displacement of Ω.

The union of all the classes of Newton data will be denoted simply by New.

Let us consider the Lie group $\mathcal{G}$ of all polynomial changes of coordinates in $\mathbf{R}^3$ which have the form

$$\tilde{x}=x, \quad \tilde{y}=y+g(x) \quad \text{and} \quad \tilde{z}=z+f(x,y), \tag{9}$$

where $f\in\mathbf{R}[x,y]$ and $g\in\mathbf{R}[x]$ are real polynomials. The group operation is the composition, and the inverse of the map (9) is simply given by

$$x=\tilde{x}, \quad y=\tilde{y}-g(\tilde{x}) \quad \text{and} \quad z=\tilde{z}-f(\tilde{x},\tilde{y}-g(\tilde{x})). \tag{10}$$

For simplicity, we shall denote the map (9) by $(f,g)\in\mathcal{G}$ and its inverse by $(f,g)^{-1}$. We define also the subgroups $\mathcal{G}^1=\mathcal{G}$ and $\mathcal{G}^2=\{(f,g)\in\mathcal{G}:g=0\}$.

Remark 3.12. The Lie algebra associated with $\mathcal{G}$ is the algebra $\mathfrak{G}$ of all polynomial vector fields of the form

$$G(x)\frac{\partial}{\partial y}+F(x,y)\frac{\partial}{\partial z},$$

with polynomials $G\in\mathbf{R}[x]$ and $F\in\mathbf{R}[x,y]$.

There is a natural action of the Lie group $\mathcal{G}$ on the class of Newton data New, given as follows: The action of a map $(f,g)\in\mathcal{G}$ in the data $\Omega=((x,y,z),\iota,\Theta)$ is the Newton data given by

$$((\tilde{x},\tilde{y},\tilde{z}),\tilde{\iota},\widetilde{\Theta}),$$

where $(\tilde{x},\tilde{y},\tilde{z})$ are the coordinates given by (9), the list $\tilde{\iota}$ is the incidence list at the point $\tilde{p}=(\tilde{x},\tilde{y},\tilde{z})^{-1}(0)$, and $\widetilde{\Theta}$ is the Newton map for $(\mathbf{M},\mathrm{Ax})$ at $\tilde{p}$, relative to this new adapted local chart. We denote this action simply by $(f,g)\cdot\Omega$.

Remark 3.13. In the cases that we will consider more often, we have $f(0)=g(0)=0$. In this case, $\tilde{p}=p$, i.e. the Newton data $\widetilde{\Omega}$ is centered at the same point as Ω. If this is not the case, we tacitly assume that the point $\tilde{p}=(\tilde{x},\tilde{y},\tilde{z})^{-1}(0)$ lies in the domain of definition of the local adapted chart $(U,(x,y,z))$.

3.6. The subgroups $\mathcal{G}_{\Delta,C}$

Recall that the *support* of a polynomial $H\in\mathbf{R}[\mathbf{x}]$ is the subset

$$\operatorname{supp}(H)=\{\mathbf{v}\in\mathbf{Z}^n:\mathbf{x}^{\mathbf{v}} \text{ is a nonzero monomial of } H\}.$$

According to the *support* of the polynomials f and g given in (9), we shall now define several subgroups in $\mathcal{G}$.

Given $\Delta=(\Delta_1,\Delta_2)\in\mathbf{Q}^2_{\geqslant 0}$ and $C\in\overline{\mathbf{Q}}_{\geqslant 0}$, we define $\mathcal{G}_{\Delta,C}$ as the subgroup of all maps $(f,g)\in\mathcal{G}$ such that the following conditions hold:

(i) the support $S_f=\operatorname{supp}(f)$ is contained in the set

$$\begin{aligned} \{(a,b)\in\Delta+s(C,-1):s\in\overline{\mathbf{Q}}_{\geqslant 0}\}\cap\mathbf{N}^2, &\quad \text{if } \Delta_1=0,\\ \{\Delta\}\cap\mathbf{N}^2, &\quad \text{if } \Delta_1>0;\end{aligned}$$

(ii) the support $S_g=\operatorname{supp}(g)$ is contained in the set

$$\begin{aligned} \{C\}\cap\mathbf{N}, &\quad \text{if } \Delta_1=0,\\ \varnothing, &\quad \text{if } \Delta_1>0.\end{aligned}$$

We further define the subgroups $\mathcal{G}^1_{\Delta,C}$ and $\mathcal{G}^2_{\Delta,C}$ as

$$\mathcal{G}^1_{\Delta,C}=\mathcal{G}_{\Delta,C} \quad\text{and}\quad \mathcal{G}^2_{\Delta,C}=\mathcal{G}_{\Delta,C}\cap\{(f,g)\in\mathcal{G}_{\Delta,C}:g=0\}.$$

Remark 3.14. In the above definition, we have the following *extreme* cases for $\mathcal{G}_{\Delta,C}$:

- if $C=\infty$, then $g=0$ and $f=\xi x^{\Delta_1}y^{\Delta_2}$, where the constant $\xi\in\mathbf{R}$ necessarily vanishes if $\Delta\notin\mathbf{N}^2$;
- if $\Delta_1=0$ and $C=0$, then $f\in\mathbf{R}[y]$ is a polynomial in y of degree at most δ_2, and $g=\eta$, for some constant $\eta\in\mathbf{R}$;
- if $\Delta=(0,0)$ and $C=\infty$, then $g=0$ and $f=\xi$, for some real constant $\xi\in\mathbf{R}$.

In the last two cases, the change of coordinates (9) correspond to translations $\tilde{y}=y+\eta$, $\tilde{z}=z+f(y)$ and $\tilde{y}=y$, $\tilde{z}=z+\xi$, respectively.

It will be useful to consider the following decomposition of the group $\mathcal{G}$: Define the subgroup

$$\mathcal{G}_\Delta:=\{(f,g)\in\mathcal{G}_{\Delta,C}:g=0,f=\xi x^{\Delta_1}y^{\Delta_2},\xi\in\mathbf{R}\}$$

(the constant ξ necessarily vanishes if $\Delta\notin\mathbf{N}^2$), and the normal subgroup

$$\mathcal{G}^+_{\Delta,C}=\{(f,g)\in\mathcal{G}_{\Delta,C}:\Delta\notin\operatorname{supp}(f)\},$$

which will be called the subgroup of *edge-preserving maps*. It is easy to see that

$$\mathcal{G}^+_{\Delta,C}\cap\mathcal{G}_\Delta=\{0\} \quad\text{and}\quad \mathcal{G}_{\Delta,C}=\mathcal{G}_\Delta\circ\mathcal{G}^+_{\Delta,C}=\mathcal{G}^+_{\Delta,C}\circ\mathcal{G}_\Delta.$$

In other words, $\mathcal{G}_{\Delta,C}$ is the *semi-direct product* of $\mathcal{G}_\Delta$ and $\mathcal{G}^+_{\Delta,C}$. Similar decompositions hold for the subgroups $\mathcal{G}^1$ and $\mathcal{G}^2$.

Later on, we shall need the following remark.

Remark 3.15. For a map $(f,g)\in\mathcal{G}^+_{\Delta,C}$, the support of f is such that

$$S_f\subset\{(a,b)\in\mathbf{N}^2 : a\geqslant C\tau(\Delta_2)\},$$

where

$$\tau(\Delta_2):=\begin{cases}1, & \text{if } \Delta_2\in\mathbf{N},\\ \Delta-\lfloor\Delta\rfloor, & \text{otherwise.}\end{cases}$$

3.7. Action of $\mathcal{G}$ via adjoint map

The action of the group $\mathcal{G}$ on a Newton data Ω can be studied via the associated Lie algebra $\mathfrak{G}$. Indeed, a map $(f,g)\in\mathcal{G}$ is the time-one map of the flow associated with the vector field $\Gamma_{f,g}\in\mathfrak{G}$ given by

$$\Gamma_{f,g}=g(x)\frac{\partial}{\partial y}+f(x,y)\frac{\partial}{\partial z}.$$

Therefore, if Ω is associated with the vector field χ, the Newton data $(f,g)\cdot\Omega$ can be obtained from the transformed vector field

$$((f,g))_*\chi=\chi+\frac{1}{2}[\Gamma_{f,g},\chi]+\frac{1}{6}[\Gamma_{f,g},[\Gamma_{f,g},\chi]]+\dots=\sum_{n=0}^{\infty}\frac{1}{n!}(\operatorname{ad}(\Gamma_{f,g}))^n\chi, \tag{11}$$

because $e^{\operatorname{ad}(\cdot)}=\operatorname{Ad}(\operatorname{Exp}(\cdot))$, where $\operatorname{Ad}(\cdot)$ and $\operatorname{ad}(\cdot)$ are the adjoint map and its differential, respectively.

Using this remark, we can see how the action of a map in $\mathcal{G}_{\Delta,C}$ modifies the *multiplicity* of a vector field. Let $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)\in\mathbf{N}^3$ be a weight-vector such that $\gcd(\omega_1,\omega_2,\omega_3)=1$ and

$$\langle\boldsymbol{\omega},(-1,\Delta_1,\Delta_2)\rangle=\langle\boldsymbol{\omega},(0,-1,C)\rangle=0$$

(with the convention that $\omega_1=0$ if $C=\infty$). Then, $\boldsymbol{\omega}$ is the weight-vector associated with the main face $\mathcal{F}$ of the Newton polyhedron $\mathcal{N}(\Omega)$.

Recall from §2.3 that each analytic vector field χ has an associated $\boldsymbol{\omega}$-multiplicity

$$\mu_{\boldsymbol{\omega}}(\chi):=\max\{k\in\mathbf{Z}:\chi(H^d_{\boldsymbol{\omega}})\subset H^{d+k}_{\boldsymbol{\omega}}\text{ for all } d\in\mathbf{N}\}.$$

Lemma 3.16. *If $(f,g)\in\mathcal{G}_{\Delta,C}$ is nonzero, then $\mu_{\boldsymbol{\omega}}(\Gamma_{f,g})=0$.*

Proof. This follows directly from the definition of the group $\mathcal{G}_{\Delta,C}$. □

It follows that, if $\mu_{\boldsymbol{\omega}}(\chi)=m$ and (f,g) is nonzero, then $\mu_{\boldsymbol{\omega}}([\Gamma_{f,g},\chi])=m$. As a consequence, we have the following result.

Corollary 3.17. *Choose* $(f,g)\in\mathcal{G}_{\Delta,C}$. *Then, the vector field* $\widetilde{\chi}=(f,g)_*\chi$ *is such that*

$$\mu_{\boldsymbol{\omega}}(\widetilde{\chi})=\mu_{\boldsymbol{\omega}}(\chi).$$

Proof. It suffices to use Lemma 3.16 and formula (11). □

In the same way, we can prove that the coordinate change associated with (f,g) always preserves the $\boldsymbol{\omega}$-quasihomogeneous structure on $\mathbf{R}^3$ (see §2.5).

Let us now study the action of $\mathcal{G}_{\Delta,C}$ on a single *differential monomial* given by

$$m=x^{v_1}y^{v_2}z^{v_3}\left(\alpha x\frac{\partial}{\partial x}+\beta y\frac{\partial}{\partial y}+\gamma z\frac{\partial}{\partial z}\right),$$

with constants $\mathbf{v}=(v_1,v_2,v_3)\in\mathbf{Z}^3$ and $\alpha,\beta,\gamma\in\mathbf{R}$. The corresponding Newton map Θ is such that $\operatorname{supp}(\Theta)=\{\mathbf{v}\}$.

For the particular case of a map $(f,g)\in\mathcal{G}_{\Delta,C}$ of the form $(f,g)=(\xi x^{\delta_1}y^{\delta_2},0)$, the coordinate change $\tilde{y}=y+g(x)$ and $\tilde{z}=z+f(x,y)$ maps m to the vector field

$$\widetilde{m}=x^{v_1}y^{v_2}(z-\xi x^{\delta_1}y^{\delta_2})^{v_3}\left(\alpha x\frac{\partial}{\partial x}+\beta y\frac{\partial}{\partial y}+(\gamma z+\xi((\alpha\delta_1+\beta\delta_2)-\gamma)x^{\delta_1}y^{\delta_2})\frac{\partial}{\partial z}\right)$$

(where we drop the tildes). In particular, it is easy to see that the Newton map $\widetilde{\Theta}$ associated with $\widetilde{m}$ has support contained in the set

$$\{\mathbf{u}\in\mathbf{Z}^3:\mathbf{u}=\mathbf{v}+t(\delta_1,\delta_2,-1),t\geqslant 0\}.$$

Similarly, for a map (f,g) of the form $(f,g)=(0,\eta x^C)$, we get (dropping the tildes)

$$\widetilde{m}=x^{v_1}(y-\eta x^C)^{v_2}z^{v_3}\left(\alpha x\frac{\partial}{\partial x}+(\beta y+\eta(C\alpha-\beta))\frac{\partial}{\partial y}+\gamma z\frac{\partial}{\partial z}\right),$$

and the the Newton map $\widetilde{\Theta}$ associated with $\widetilde{m}$ has support contained in the set

$$\{\mathbf{u}\in\mathbf{Z}^3:\mathbf{u}=\mathbf{v}+s(C,-1,0),s\geqslant 0\}.$$

Now, an arbitrary map $(f,g)\in\mathcal{G}_{\Delta,C}$ can be written as the composition of a finite number of maps of the form $(\xi x^{\delta_1}y^{\delta_2},0)$ and $(0,\eta x^C)$. Therefore, the above computations give the following result.

Lemma 3.18. *Consider the differential monomial* m *given above. Then, for an arbitrary pair* (Δ,C) *and for an arbitrary map* $(f,g)\in\mathcal{G}_{\Delta,C}$, *the Newton data for the vector field* $(f,g)_*m$ *has its support contained in the set*

$$\{\mathbf{u}\in\mathbf{Z}^3:\mathbf{u}=\mathbf{v}+t(\Delta_1,\Delta_2,-1)+s(C,-1,0),t\geqslant 0,s\geqslant 0\}$$

(*see Figure* 13).

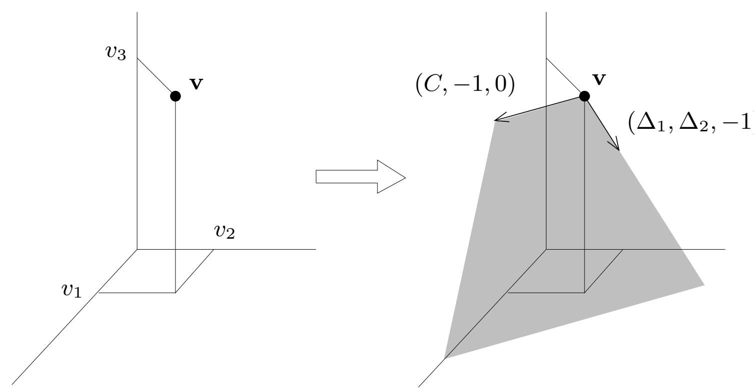


Figure 13. The support of $(f,g)_*m$ for a differential monomial m.

More generally, we can consider the action of (f,g) on an arbitrary vector field χ as follows. Write the expansion of χ as

$$\chi = \sum_{i\in I} m_i,$$

where, for each $i\in I$, m_i is a differential monomial whose Newton data has support at $\mathbf{v}_i\in\mathbf{Z}^3$. For a map $(f,g)\in\mathcal{G}_{\Delta,C}$, we clearly have

$$(f,g)_*\chi = \sum_{i\in I}(f,g)_*m_i,$$

and this gives the following result.

Corollary 3.19. *Let χ be as above. For an arbitrary pair (Δ,C) and for an arbitrary map $(f,g)\in\mathcal{G}_{\Delta,C}$, the Newton data for the vector field $(f,g)_*\chi$ has its support contained in the set*

$$\bigcup_{i\in I}\{\mathbf{u}\in\mathbf{Z}^3:\mathbf{u}=\mathbf{v}_i+t(\Delta_1,\Delta_2,-1)+s(C,-1,0), t\geqslant 0, s\geqslant 0\}.$$

Remark 3.20. In [AGV], the authors use these kind of coordinate changes to study normal forms of *quasihomogeneous functions*. In [H3], Hironaka uses similar transformations in his definition of well and very well preparations of function germs.

4. Local theory at NElem$\cap\mathfrak{D}$

Let $(\mathbf{M},\mathrm{Ax})$ be a controlled singularly foliated manifold. Throughout this section, we fix a divisor point $p\in A\cap\mathfrak{D}$, an adapted local chart $(U,(x,y,z))$ for $(\mathbf{M},\mathrm{Ax})$ at p, and let $\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$ be the corresponding Newton data.

Proposition 4.1. *The main vertex* $\mathbf{m}=(m_1,m_2,m_3)$ *associated with* Ω *is such that* $m_1=0$ *and* $m_2\in\{-1,0\}$.

Proof. Suppose by contradction that the main vertex $\mathbf{m}$ does not satisfy the above requirements. Then, one of the following conditions holds:

(a) $m_1=-1$;

(b) $m_1\geqslant 1$;

(c) $m_2\geqslant 1$.

In case (a), it follows from the definition of the Newton polyhedron that the plane $\{x=0\}$ is not invariant. This contradicts the definition of an adapted local chart at p and the hypothesis that p belongs to $A\cap\mathfrak{D}$.

In case (b), choose a nondegenerate vector field χ which is a local generator for the line field L. Then, the condition $m_1\geqslant 1$ implies that the ideal $\mathcal{I}_\chi(x)\subset\mathcal{O}_p$ is divisible by x^2. This contradicts Definition 2.2.

Thus, in case (c), we may assume that $m_1=0$ and $m_2\geqslant 1$. This clearly implies that

$$\operatorname{supp}(\Omega)\cap(\{0\}\times\{-1,0\}\times\mathbf{R})=\varnothing. \tag{12}$$

If we write the vector field χ as

$$\chi=fx\frac{\partial}{\partial x}+gy\frac{\partial}{\partial y}+hz\frac{\partial}{\partial z}$$

(with $fx,gy,hz\in\mathbf{R}\{x,y,z\}$), then (12) is equivalent to saying that the functions f, g and h vanish identically along the vertical line $l:=\{x=y=0\}$. A simple computation shows that it is equivalent to asserting that the Jacobian matrix $D\chi|_l$ has the form

$$D\chi|_l=\begin{pmatrix}0&0&0\\ *&0&0\\ *&*&0\end{pmatrix},$$

where the $*$'s denote some arbitrary real numbers. It follows that the line l is contained in the set of nonelementary points NElem($\mathbf{M}$), which contradicts Definition 2.14. □

4.1. Stable Newton data and final situations

In the next definitions, we consider the action of the transformation group $\mathcal{G}^i_{\Delta,C}$ on the Newton data Ω. The following notions will be essential in the sequel to study the effect of the *translations* in the blowing-up chart.

We say that Ω is *stable* if

$$(f,g)\cdot\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$$

for all $(f,g)\in\mathcal{G}^i_{\Delta,C}$ with $f(0)=g(0)=0$. In other words, Ω is stable if the action of $\mathcal{G}^i_{\Delta,C}$ preserves the main vertex, the value of the vertical displacement vector Δ and the value of the horizontal displacement C.

A weaker notion of stability will also be useful. We say that Ω is *edge-stable* if for each map $(f,g)\in\mathcal{G}^i_{\Delta,C}$ with $f(0)=g(0)=0$ there exists a constant $\widetilde{C}\in\overline{\mathbf{Q}}_{\geqslant 0}$ such that

$$(f,g)\cdot\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,\widetilde{C}}.$$

Remark 4.2. Intuitively, the notions of stable and edge-stable Newton data can be seen as weaker versions of the notion of *maximal contact* introduced in the work of Hironaka [H1]. As said above, the main goal is to take into account the effect of the translations in the blowing-up chart.

More precisely, for a stable (respectively, edge-stable) Newton data, one guarantees that the main invariant strictly decreases after a conveniently chosen blowing-up map followed by any translation of the form $(\tilde{y}=y+\eta, \tilde{z}=z+\xi)$ (respectively, $(\tilde{z}=z+\xi)$). We refer to §4.8 for the precise statements.

In the context of vector fields, the usual notion of maximal contact is too strong and often leads to *divergent* formal objects. For instance, the computation of the maximal contact variety (in the sense of [H1]) for the *Euler vector field*

$$x^2\frac{\partial}{\partial x}+(y-x)\frac{\partial}{\partial y}$$

leads to the formal power series $V=\left\{y-\sum_{n=1}^{\infty}(n-1)!x^n\right\}$, which has zero as radius of convergence.

Using these concepts, we can now identify when the Newton data Ω is centered at an *elementary point* $p\in\mathrm{Elem}(\mathbf{M})$.

First of all, we introduce the following notion. We shall say that Ω is in a *final situation* if, looking at the higher vertex $\mathbf{h}\in\mathcal{N}$ (see definition in §3.4) and the associated edge $\mathfrak{e}(\mathbf{h})$, one of the following conditions is satisfied (see Figure 14):

(i) the vertex $\mathbf{h}=(h_1,h_2,h_3)$ is such that $h_1=0$ and either

$$\text{(a)}\ (h_2,h_3)=(0,0),\quad \text{(b)}\ (h_2,h_3)=(-1,0),\quad \text{or}\quad \text{(c)}\ (h_2,h_3)=(0,-1);$$

(ii) the edge $\mathfrak{e}(\mathbf{h})$ is given by $[(0,-1,k),(0,0,-1)]$ for some $k\geqslant 1$;

(iii) the edge $\mathfrak{e}(\mathbf{h})$ is given by $[(0,-1,k),(0,0,0)]$ for some $k\geqslant 1$;

(iv) the edge $\mathfrak{e}(\mathbf{h})$ is given by $[(0,-1,1),(0,1,-1)]$ and Ω is edge-stable.

The following result justifies the above nomenclature.

PROPOSITION 4.3. *If the Newton data Ω is in a final situation then it is centered at an elementary point $p\in\mathrm{Elem}(\mathbf{M})$.*

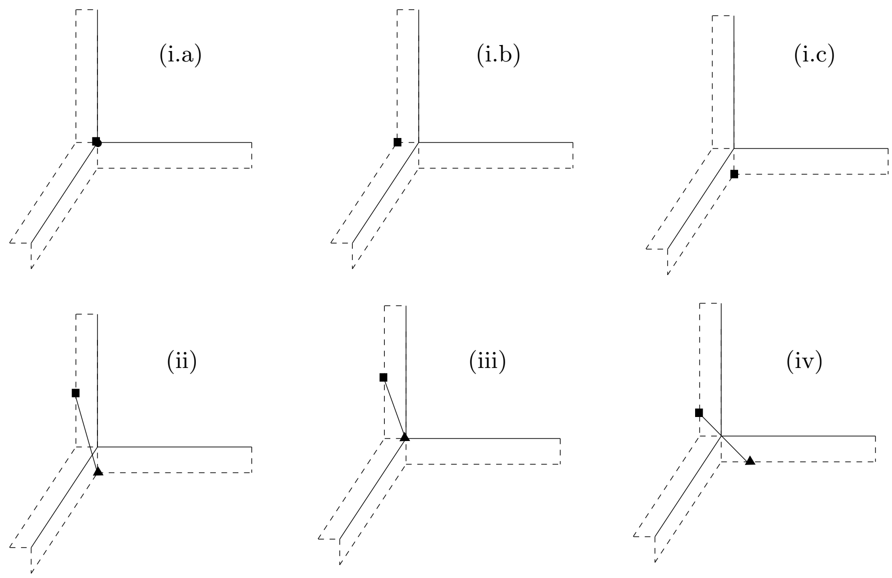


Figure 14. The final situations.

Proof. Consider a vector field χ which locally generates the line field L, in a neighborhood of p. If $\chi(p)\neq 0$ we are done. Otherwise, we can write the linear part $D\chi(p)$ as the matrix

$$\begin{pmatrix} \lambda & 0 & 0 \\ * & a & b \\ * & c & d \end{pmatrix},$$

where $\lambda, a, b, c, d\in\mathbf{R}$ and the $*$'s denote some arbitrary real constants. We consider the following cases:

(a) $\lambda\neq 0$;

(b) $\lambda=0$.

In case (a), it is clear that λ belongs to the spectrum of $D\chi(p)$, and therefore χ is elementary.

In case (b), it clearly suffices to prove the following claim.

Claim. The matrix

$$B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is not nilpotent (it is obvious that $B\neq 0$).

To prove the claim, suppose initially that $b=0$ or $c=0$. Then, it follows from the definition of a final situation that $(a,d)\neq(0,0)$, and therefore B contains at least one nonzero eigenvalue.

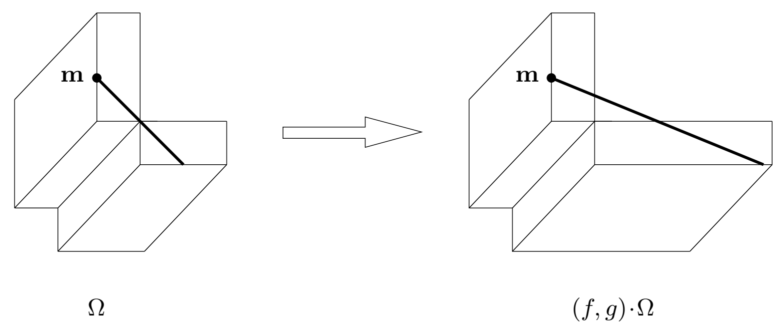


Figure 15. The transition from Ω to $(f,g)\cdot\Omega$.

Suppose now that $b\neq 0$ and $c\neq 0$. It follows that $\Delta=(0,1)$ and $\mathbf{m}=(0,-1,1)$. Assume by contradiction that B is nilpotent and consider the $\mathcal{G}^i_{\Delta,C}$-map $(f,g)=((a/b)y,0)$, which corresponds to the coordinate change

$$\tilde{y}=y \quad \text{and} \quad \tilde{z}=z+(a/b)y$$

(see Figure 15). It is easy to see that the Newton data $(f,g)\cdot\Omega$ (associated with the local chart $(U,(x,\tilde{y},\tilde{z}))$) is such that the corresponding matrix B is given by

$$B=\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$

This implies that $(f,g)\cdot\Omega$ belongs to the class $\mathrm{New}^{\boldsymbol{i},\mathbf{m}}_{\tilde{\Delta},\tilde{C}}$, for some vertical displacement vector $\tilde{\Delta}>_{\mathrm{lex}}(0,1)$. This contradicts the assumption that Ω is edge-stable. The claim is proved. □

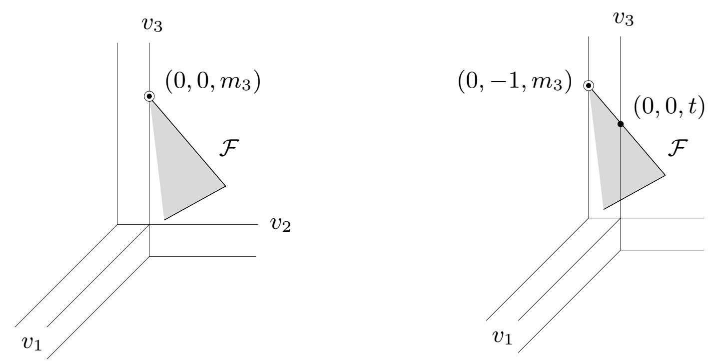
Remark 4.4. The above result has the following partial converse (which we will not need in the sequel). If Ω is an edge-stable Newton data centered at an elementary point $p\in\mathrm{Elem}(\mathbf{M})$, then Ω is necessarily in a final situation.

4.2. The local invariant

Let us now introduce the main invariant used in the local strategy of the resolution of singularities. First of all, we prove the following result.

LEMMA 4.5. *Suppose that p belongs to* $\mathrm{NElem}(\mathbf{M})$. *Then the main derived polygon $\mathcal{N}'$ is nonempty.*

Proof. According to Proposition 3.7, it suffices to prove that $m_3\geqslant 1$. But this is a direct consequence of the fact that the Newton data Ω is not in a final situation. □



$\mathrm{inv}{=}(m_3, 0, m_3, \ldots)$ $\mathrm{inv}{=}(\lfloor t{+}1\rfloor, -1, m_3, \ldots)$

Figure 16. The invariant $\mathrm{inv}(\Omega)$.

Let us suppose that p belongs to $\mathrm{NElem}(\mathbf{M})$. Writing the main vertex as $\mathbf{m}{=}(m_1, m_2, m_3)$ and the vertical displacement vector as $\Delta{=}(\Delta_1, \Delta_2)$, we define the *virtual height* associated with Ω as the natural number

$$\mathfrak{h} := \begin{cases} \lfloor m_3{+}1{-}1/\Delta_2 \rfloor, & \text{if } m_2 = -1 \text{ and } \Delta_1 = 0, \\ m_3, & \text{if } m_2 = 0 \text{ or } \Delta_1 > 0. \end{cases}$$

For $m_2{=}{-}1$ and $\Delta_1{=}0$, the virtual height $\mathfrak{h}$ is the smallest integer which is strictly greater than the height of the point of intersection between the main edge and the vertical plane $\{\mathbf{v}{=}(v_1, v_2){\in}\mathbf{R}^2{:}v_2{=}0\}$ (as shown in Figure 16).

We refer the reader to §4.13 for an example which motivates the use of the notion of virtual height.

Definition 4.6. The *primary invariant* is the vector

$$\mathrm{inv}_1 := (\mathfrak{h}, m_2{+}1, m_3).$$

The *secondary invariant* is the vector

$$\mathrm{inv}_2 = (\#\iota_p{-}1, \lambda\Delta_1, \lambda \max\{0, \Delta_2\}),$$

where $\lambda{:=}2(m_3{+}1)!$. The *invariant* associated with the Newton data Ω is the pair

$$\mathrm{inv}(\Omega) := (\mathrm{inv}_1, \mathrm{inv}_2).$$

Remark 4.7. It follows from the assumption $\#\iota_p{\geqslant}1$, from the choice of λ and from Proposition 3.8 that the vector $\mathrm{inv}(\Omega)$ always belongs to $\mathbf{N}^6$.

4.3. Regular-nilpotent transitions

Let us introduce the following notation. Given a subset $A\subset\mathbf{Z}^3$, let $\Omega|_A=((x,y,z),\iota_p,\Theta|_A)$ be the Newton data which is obtained from Ω by considering the *restricted* Newton map

$$\Theta|_A:\mathbf{Z}^3\longrightarrow\mathbf{R}^3$$

defined by

$$\Theta|_A(\mathbf{v})=\begin{cases}\Theta(\mathbf{v}), & \text{if } \mathbf{v}\in A,\\ 0, & \text{if } \mathbf{v}\in\mathbf{Z}^3\setminus A.\end{cases}$$

If Ω is associated with a vector field χ (which is a local generator of the line field at p), we denote by $\chi|_A$ the vector field associated with $\Omega|_A$. Notice that $\chi|_A$ is possibly a degenerate vector field.

LEMMA 4.8. *The Newton data Ω is edge-stable if and only if for all $(f,0)\in\mathcal{G}_\Delta$, there exists a constant $\widetilde{C}\in\overline{\mathbf{Q}}_{\geqslant 0}$ such that*

$$(f,0)\cdot\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,\widetilde{C}}.$$

Proof. It suffices to notice that each map $(f,g)\in\mathcal{G}^i_{\Delta,C}$ can be uniquely written as a composition

$$(f,g)=(f_0,0)\circ(\tilde{f},\tilde{g}),$$

where $(f_0,0)$ belongs to $\mathcal{G}_\Delta$ and $(\tilde{f},\tilde{g})$ is a map belonging to the normal subgroup

$$\mathcal{G}^{i,+}_{\Delta,C}\lhd\mathcal{G}^i_{\Delta,C}.$$

Now, Corollary 3.19 implies that, if we denote the main edge associated with Ω by $\mathfrak{e}$, the Newton data $\widetilde{\Omega}:=(\tilde{f},\tilde{g})\cdot\Omega$ is such that

$$\widetilde{\Omega}|_{\mathfrak{e}}=\Omega|_{\mathfrak{e}}.$$

Moreover, $\mathfrak{e}$ is also the main edge of $\widetilde{\Omega}$. This concludes the proof. □

Recall that the Newton data can be either in a regular or in a nilpotent configuration (see §3.4). Let us say that Ω is in a *potentially nilpotent situation* if it is in a regular configuration but

$$\mathbf{m}=(0,-1,m_3)\quad\text{and}\quad\Delta=(0,1)$$

for some $m_3\geqslant 1$. The next result describes some basic aspects of the action of a $\mathcal{G}^i_{\Delta,C}$-map on Ω.

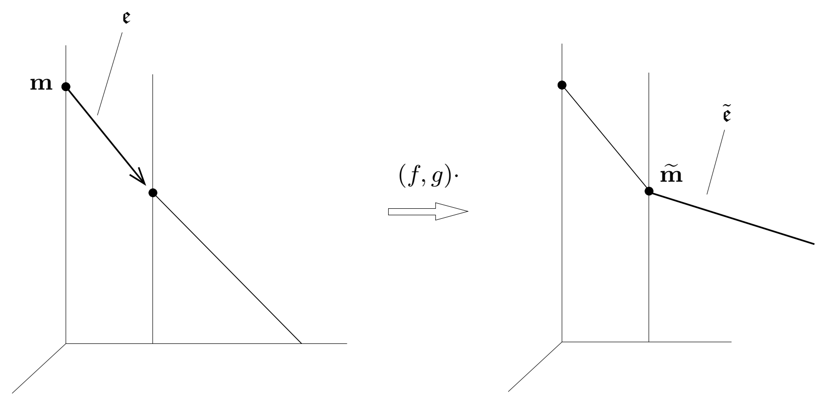


Figure 17. The regular-nilpotent transition.

LEMMA 4.9. *Given* $(f,g)\in\mathcal{G}^i_{\Delta,C}$, *the Newton data* $\widetilde{\Omega}:=(f,g)\cdot\Omega$ *belongs to the class* $\mathrm{New}^{i,\widetilde{\mathbf{m}}}_{\tilde{\Delta},\widetilde{C}}$, *where two cases may occur*:

(i) *if* Ω *is not in a potentially nilpotent situation then*

$$\widetilde{\mathbf{m}}=\mathbf{m} \quad \textit{and} \quad (\tilde{\Delta},\widetilde{C})\geqslant_{\mathrm{lex}}(\Delta,C); \tag{13}$$

in particular, if Ω *is in a regular* (*respectively*, *nilpotent*) *configuration*, *then* $\widetilde{\Omega}$ *is also in a regular* (*respectively*, *nilpotent*) *configuration*;

(ii) *if* Ω *is in a potentially nilpotent situation then either*

(ii.a) $\widetilde{\mathbf{m}}=\mathbf{m}$ *and* $(\tilde{\Delta},\widetilde{C})\geqslant_{\mathrm{lex}}(\Delta,C)$, *or*

(ii.b) $\widetilde{\Omega}$ *is in a nilpotent configuration and*

$$\widetilde{\mathbf{m}}=(0,0,m_3-1) \quad \textit{and} \quad \tilde{\Delta}>_{\mathrm{lex}}\Delta.$$

Proof. It suffices to use Corollary 3.19. □

In case (ii.b), we say that the data Ω is in a *hidden nilpotent configuration* and that the transformation $\Omega\to\widetilde{\Omega}$ is a *regular-nilpotent* transition.

In view of Lemma 4.8, a hidden nilpotent configuration may be detected just by the action of the subgroup $\mathcal{G}_\Delta$.

4.4. Resonant configurations

This is a rather technical subsection, whose main goal is to characterize those types of Newton data (called *resonant configurations*) for which the action of the group $\mathcal{G}$ is not effective. This characterization is essential to prove the uniqueness of the local strategy for the resolution of singularities at points $p\in\mathrm{NElem}(\mathbf{M})\cap\mathfrak{D}$.

We remark in passing that the occurrence of these resonant configurations has no analogues in the theory of resolution of singularities for functions and analytic sets.

Suppose that Ω is a Newton data associated with an adapted local chart $(U,(x,y,z))$ with center at a divisor point $p\in\mathfrak{D}\cap A$ such that $p\in\mathrm{NElem}(\mathbf{M})$.

Let us study how *effective* the action of the group $\mathcal{G}^i_{\Delta,C}$ is on the *support* of Ω. Recall that the support is given by

$$\mathrm{supp}(\Omega):=\{\mathbf{v}\in\mathbf{Z}^3:\Theta(\mathbf{v})\neq 0\},$$

where $\Theta:\mathbf{Z}^3\to\mathbf{R}^3$ is the Newton map associated with Ω. To state our next result, we need the following definition. Let $\mathfrak{e}$ be the main edge of Ω. We shall say that Ω is in *c-resonant configuration* (for some $c\in\mathbf{N}$) if there exists a map $(f,g)\in\mathcal{G}^i_{\Delta,c}\setminus\mathcal{G}_\Delta$ such that

$$\mathrm{supp}((f,g)\cdot\Omega|_{\mathfrak{e}})\subset\mathfrak{e}.$$

In other words, there is a map $(f,g)\in\mathcal{G}^i_{\Delta,c}\setminus\mathcal{G}_\Delta$ (i.e. *not* of the form $(\xi x^{\Delta_1}y^{\Delta_2},0)$) whose action on the restricted Newton data $\Omega|_{\mathfrak{e}}$ results into a Newton data which still has the support on $\mathfrak{e}$.

LEMMA 4.10. *Suppose that $\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$ is a Newton data which is in a c-resonant configuration. Then, Ω is not edge-stable. Moreover, $i=1$ and $\Delta=(0,s)$ for some $s>0$. Considering the associated vector field χ, one of the following situations occurs:*

(i) $\Delta=(0,1)$ *and the restriction of χ to the main edge is given by*

$$\chi|_{\mathfrak{e}}=(z+\lambda y)^m\left(\alpha\left(x\frac{\partial}{\partial x}+cy\frac{\partial}{\partial y}+cz\frac{\partial}{\partial z}\right)+\beta(z+\lambda y)\frac{\partial}{\partial y}+\gamma(z+\lambda y)\frac{\partial}{\partial z}\right)$$

for some $m\geqslant 1$, $\lambda\in\mathbf{R}$ and $(\alpha,\beta,\gamma)\in\mathbf{R}^3$ such that $\beta\neq 0$ and $(\alpha,\gamma+\lambda\beta)\neq(0,0)$;

(ii) $\Delta=(0,1/\tau)$ *for some $\tau\in\mathbf{N}_{\geqslant 2}$, and*

$$\chi|_{\mathfrak{e}}=z^{\tau m}\left(\alpha\left(x\frac{\partial}{\partial x}+cy\frac{\partial}{\partial y}+\frac{c}{\tau}z\frac{\partial}{\partial z}\right)+\beta z^\tau\frac{\partial}{\partial y}+\gamma z\frac{\partial}{\partial z}\right)$$

for some $m\geqslant 1$ and $(\alpha,\beta,\gamma)\in\mathbf{R}^3$ such that $\beta\neq 0$ and $(\alpha,\gamma)\neq(0,0)$.

Proof. If $\Delta=(\Delta_1,\Delta_2)$ for some $\Delta_1>0$, then $\mathcal{G}^i_{\Delta,c}=\mathcal{G}_\Delta$ by definition, and nothing has to be proved.

Let us assume that Ω is edge-stable. Up to an x-directional blowing-up with weight-vector $\boldsymbol{\omega}=k\cdot(1,c,sc)$ (where $k\in\mathbf{N}$ is chosen in such a way that $\boldsymbol{\omega}\in\mathbf{N}^3$), we can write

$$\chi|_{\mathfrak{e}}=F(y,z)x\frac{\partial}{\partial x}+G(y,z)\frac{\partial}{\partial y}+H(y,z)\frac{\partial}{\partial z},$$

where F, G and H are $(1,s)$-quasihomogeneous functions of degrees M, $M+1$ and $M+s$, respectively, for some rational number $M\in\mathbf{Q}$.

After this blowing-up, the map (f,g) is transformed into an element of the group $\mathcal{G}^i_{(0,s),0}$. Keeping the same notation for this map, we can write

$$(f,g)=(a_0+a_1y+...+a_ky^k,\eta),\quad \text{with } a_0,...,a_k,\eta\in\mathbf{R}, \tag{14}$$

where $k:=\lfloor s\rfloor$ and $(f,g)\neq(a_sy^s,0)$. Our problem reduces to finding conditions on F, G and H such that there is one such map for which

$$\operatorname{supp}((f,g)\cdot\Omega|_{\mathfrak{e}})\subset\mathfrak{e}. \tag{15}$$

Let us consider the four possible cases:

(1) $s\in\mathbf{N}$ and $s\geqslant 2$;

(2) $s=1$;

(3) $s=1/\tau$ with $\tau\in\mathbf{N}$ and $\tau\geqslant 2$;

(4) $s\notin\mathbf{N}\cup 1/\mathbf{N}$.

In case (1), two possible expressions for F, G and H may appear:

(1.a) if the main vertex has the form $\mathbf{m}=(0,-1,m)$, then

$$F(y,z)=F_0(y,z),\quad G(y,z)=G_0(y,z)\quad\text{and}\quad H(y,z)=y^{s-1}H_0(y,z), \tag{16}$$

where F_0, G_0 and H_0 are $(1,s)$-quasihomogeneous functions of degree $ms-1$, ms and ms, respectively; moreover, $G_0(0,z)=\beta z^m$, for some $\beta\neq 0$;

(1.b) if the main vertex has the form $\mathbf{m}=(0,-1,m)$, then

$$F(y,z)=F_1(y,z),\quad G(y,z)=G_1(y,z)y\quad\text{and}\quad H(y,z)=H_1(y,z)z+H_2(y,z)y^s, \tag{17}$$

where F_1, G_1, H_1 and H_2 are $(1,s)$-quasihomogeneous functions of degree ms.

Assuming that condition (15) holds, it is easy to see that, in expression (16), $F_0\equiv 0$, and G_0 and H_0 should be a power of a common $(1,s)$-quasihomogeneous form of degree s:

$$G_0(y,z)=\beta(z+\lambda y^s)^m\quad\text{and}\quad H_0(y,z)=\gamma(z+\lambda y^s)^m$$

for some $\beta,\lambda\in\mathbf{R}^*$ and $\gamma\in\mathbf{R}$. Looking only at the function G_0, we see that the only possible map (f,g) satisfying our requirements is given by

$$f=-\lambda((y+\eta)^s-y^s)\quad\text{and}\quad g=\eta\quad\text{for some } \eta\neq 0.$$

In this case, the restricted vector field $\chi|_{\mathfrak{e}}$ is transformed to (dropping the tildes)

$$(z+\lambda y^s)^m\left(\beta\frac{\partial}{\partial y}+\left(-s\lambda\beta y^s+(\gamma+s\lambda\beta)(y-\eta)^{s-1}\right)\frac{\partial}{\partial z}\right).$$

Therefore, since $s\geqslant 2$ and (15) is assumed, the relation $\gamma+s\lambda\beta=0$ necessarily holds. But if we consider the original expression of the vector field $\chi|_{\mathfrak{e}}$ and apply the $\mathcal{G}_\Delta$-map $(f',g')=(\lambda y^s,0)$, we get (dropping again the tildes)

$$z^m\left(\beta\frac{\partial}{\partial y}+\left(\gamma+s\lambda\beta)y^{s-1}\right)\frac{\partial}{\partial z}\right)=z^m\left(\beta\frac{\partial}{\partial y}\right)$$

(i.e. the support of $\Omega|_{\mathfrak{e}}$ has a single point). This implies that Ω is not edge-stable, yielding a contradiction.

Similarly, for the expression (17), a simple computation shows that if there is a nonzero (f,g) which fixes the support of $\Omega|_{\mathfrak{e}}$, then F_1, G_1 and H should necessarily be powers of a $(1,s)$-quasihomogeneous form of degree s, namely

$$F_1(y,z)=\alpha(z+\lambda y^s)^m,\quad G_1(y,z)=\beta(z+\lambda y^s)^m\quad\text{and}\quad H(y,z)=\gamma(z+\lambda y^s)^{m+1}$$

for some $\lambda\in\mathbf{R}$. Moreover, since $(f,g)\neq(a_s y^s,0)$, we conclude from the general expression (17) that $G_1=0$. Therefore, $\chi|_{\mathfrak{e}}$ has the general form

$$\chi|_{\mathfrak{e}}=(z+\lambda y^s)^m\left(\alpha x\frac{\partial}{\partial x}+\gamma(z+\lambda y^s)\frac{\partial}{\partial z}\right).$$

Notice, however, that this expression implies that Ω is not edge-stable. In fact, applying the map $(f',g')=(\lambda y^s,0)$, we get a new Newton data $(f',g')\cdot\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta',C'}$ with $\Delta'>_{\mathrm{lex}}\Delta$. This gives a contradiction.

If we suppose that $s=1$, then the reasoning used above leads us to the general expression

$$\chi|_{\mathfrak{e}}=(z+\lambda y)^m\left(\alpha x\frac{\partial}{\partial x}+\beta(z+\lambda y)\frac{\partial}{\partial y}+\gamma(z+\lambda y)\frac{\partial}{\partial z}\right),$$

with $\lambda\in\mathbf{R}$ and $(\alpha,\beta,\gamma)\in\mathbf{R}^3\setminus\{(0,0,0)\}$. If we apply the map $(f_1,g_1)=(\lambda y,0)$, we see that if $\beta=0$ or $(\alpha,\gamma+\lambda\beta)=(0,0)$ then Ω is not edge-stable. If this is not the case, we are precisely (up to blowing-up) in the configuration listed in item (i) of the statement.

Suppose now that $s=1/\tau$, with $\tau\in\mathbf{N}$ and $\tau\geqslant 2$. Then, $(f,g)=(\eta,\xi)$ for some $\xi,\eta\in\mathbf{R}$ and we obtain the general expression

$$\chi|_{\mathfrak{e}}=z^{\tau m}\left(\alpha x\frac{\partial}{\partial x}+\beta z^\tau\frac{\partial}{\partial y}+\gamma z\frac{\partial}{\partial z}\right),$$

with $m\geqslant 1$ and $(\alpha,\beta,\gamma)\in\mathbf{R}^3$. Since $\mathrm{supp}(\Omega)\cap\mathfrak{e}$ has at least two points, we see that $\beta\neq 0$ and $(\alpha,\gamma)\neq(0,0)$. This is precisely (up to blowing-up) the configuration listed in item (ii) of the statement.

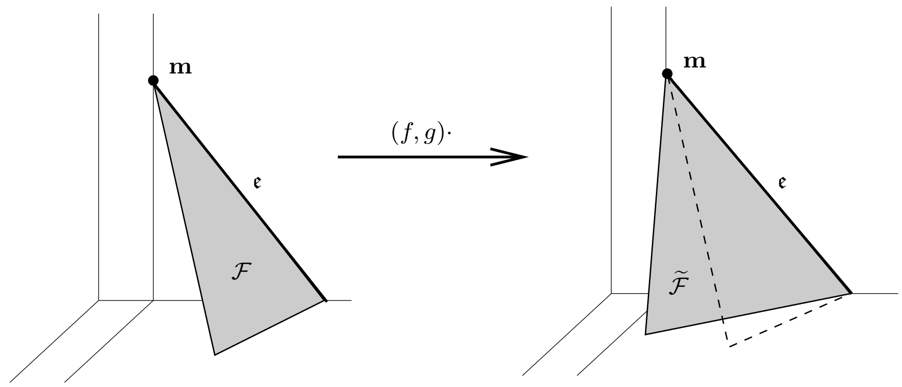


Figure 18. The action of a map $(f,g)\in\mathcal{G}^i_{\Delta,c}\setminus\mathcal{G}_\Delta$ with $c<C$.

It remains to study the case where $s\notin\mathbf{N}\cup 1/\mathbf{N}$. Here, under the assumption (15), we get

$$\chi|_{\mathfrak{e}} = z^{m/s}\left(\alpha x\frac{\partial}{\partial x}+\beta y\frac{\partial}{\partial y}+\gamma z\frac{\partial}{\partial z}\right)$$

for some $(\alpha,\beta,\gamma)\in\mathbf{R}^3\setminus\{(0,0,0)\}$. However, this expression implies that $\operatorname{supp}(\Omega)$ and $\mathfrak{e}$ intersect in a single point, yielding again a contradiction. □

Notice that the configurations (i) and (ii) of the previous lemma do not represent edge-stable Newton data.

Indeed, the item (ii) of the lemma is obviously excluded because it represents a nilpotent configuration with higher vertex $\mathbf{h}=(0,-1,\tau(m+1))$ and associated edge $\mathfrak{e}(\mathbf{h})$ given by

$$\mathfrak{e}(\mathbf{h}) = \overline{\mathbf{h},\mathbf{n}},$$

where $\mathbf{n}=(0,0,\tau m)$. In this case, it follows immediately from the definition that the main vertex associated with Ω is $\mathbf{n}$ and not $\mathbf{h}$. The same reasoning can be used to exclude the item (i) of the lemma with $\lambda=0$.

The item (i) with $\lambda\neq 0$ is also excluded because it is not edge-stable. Indeed, the coordinate change $\tilde{z}=z+\lambda y$ causes a regular-nilpotent transition (see Lemma 4.9).

The following result is an immediate consequence of Lemma 4.10.

Proposition 4.11. *Let* $\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$ *be an edge-stable Newton data. Consider a map* $(f,g)\in\mathcal{G}^i_{\Delta,c}\setminus\mathcal{G}_\Delta$ *with* $c<C$. *Then,* $(f,g)\cdot\Omega$ *necessarily belongs to* $\mathrm{New}^{i,\mathbf{m}}_{\Delta,c}$.

In other words, the map $(f,g)\in\mathcal{G}^i_{\Delta,c}\setminus\mathcal{G}_\Delta$ with $c<C$ acts *effectively* on $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$.

Using the same computations made in the proof of Lemma 4.10, we can immediately prove the following result, which gives a more precise description of the action of $\mathcal{G}^i_{\Delta,c}\setminus\mathcal{G}_\Delta$ on the main edge of Ω.

Lemma 4.12. *Suppose that* $\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$ *is an edge-stable Newton data. Then, for each* $(f,g)\in\mathcal{G}^{i}_{\Delta,c}\setminus\mathcal{G}_{\Delta}$, *the Newton data*

$$\widetilde{\Omega}=(f,g)\cdot(\Omega|_{\mathfrak{e}})$$

(*i.e. the action of* (f,g) *in* $\Omega|_{\mathfrak{e}}$) *is such that the following conditions hold*:

(i) *for each pair* $(k,l)\in\mathrm{supp}(f)$, *we have* $\mathrm{supp}(\widetilde{\Omega})\cap(\mathfrak{e}+(k,l,-1))\neq\varnothing$;

(ii) *for each* $c\in\mathrm{supp}(g)$, *we have* $\mathrm{supp}(\widetilde{\Omega})\cap(\mathfrak{e}+(c,-1,0))\neq\varnothing$.

4.5. Basic edge-preparation and basic face-preparation

To state the next lemma, we consider the set

$$\mathrm{New}^{i,\mathbf{m}}_{\Delta}:=\bigcup_{C}\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$$

containing all the classes of Newton data with fixed values for $(i,\mathbf{m},\Delta)$.

Lemma 4.13. *Suppose that the Newton data* $\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$ *is centered at a point* $p\in\mathrm{NElem}(\mathbf{M})$. *Then, if* Ω *is not edge-stable*, *there exists a unique map* $(f,0)\in\mathcal{G}_{\Delta}$ *such that*

$$(f,0)\cdot\Omega\notin\mathrm{New}^{i,\mathbf{m}}_{\Delta}. \tag{18}$$

Proof. Choose an arbitrary map $(f,0)\in\mathcal{G}_{\Delta}$ satisfying (18) and define $\widetilde{\Omega}:=(f,0)\cdot\Omega$ (there exists at least one such map, by Lemma 4.8). Let $\widetilde{\mathbf{m}}$ and $\tilde{\Delta}$ be the main vertex and the vertical displacement vector associated with $\widetilde{\Omega}$.

Suppose, first of all, that $\widetilde{\mathbf{m}}\neq\mathbf{m}$. Then, it follows from Lemma 4.9 that Ω is in a hidden nilpotent configuration and that $\Omega\to\widetilde{\Omega}$ is a regular-nilpotent transition.

In these conditions, we know that $\mathbf{m}=(0,-1,m_3)$ and $\Delta=(0,1)$. Therefore, $m_3\geqslant2$ (because otherwise Ω would be associated with an elementary point p, by Proposition 4.3) and moreover f has the particular form $f=\xi y$ for some constant $\xi\in\mathbf{R}$.

Let us suppose that there exists another map $(f',0)=(\xi' y,0)$ such that $\widetilde{\Omega}'=(f',0)\cdot\Omega$ does not belong to $\mathrm{New}^{i,\mathbf{m}}_{\Delta}$. Then, the composition $(\tilde{f},0)=(f',0)\circ(f,0)^{-1}$ is such that $\tilde{f}$ is given by $\tilde{f}=(\xi'-\xi)y$ and it maps $\widetilde{\Omega}$ to $\widetilde{\Omega}'$.

We claim that $\tilde{f}=0$. Assume on the contrary that $\xi\neq\xi'$. If $\mathfrak{e}$ is the main edge associated with Ω, then

$$\chi|_{\mathfrak{e}}=F(y,z)x\frac{\partial}{\partial x}+G(y,z)\frac{\partial}{\partial y}+H(y,z)\frac{\partial}{\partial z},$$

where F, G and H are homogeneous polynomials of degrees m_3-1, m_3 and m_3, respectively. The hypothesis that $\mathbf{m}=(0,-1,m_3)$ implies that $G(y,z)=\varrho z^{m_3}+\dots$ for some nonzero constant $\varrho\in\mathbf{R}$.

If we apply the change of coordinates $\tilde{z}=z+\xi y$ to $\chi|_{\mathfrak{e}}$, we get a vector field

$$\widetilde{\chi}|_{\mathfrak{e}} = \tilde{f}x\frac{\partial}{\partial x}+\tilde{g}\frac{\partial}{\partial y}+\widetilde{H}\frac{\partial}{\partial \tilde{z}},$$

with

$$\tilde{f}=F, \quad \tilde{g}=G \quad \text{and} \quad \widetilde{H}=H+\xi G.$$

The assumption that $\widetilde{\Omega}$ is in a nilpotent configuration implies that $\tilde{z}=0$ is a root of multiplicity $\geqslant m_3-1$ of $\tilde{g}(1,\tilde{z})$. This is equivalent to saying that $z=-\xi$ is a root of multiplicity $\geqslant m_3-1$ of $G(1,z)$. Let us split the proof into two cases:

(a) $\widetilde{\Omega}'$ is in a regular configuration;

(b) $\widetilde{\Omega}'$ is in a nilpotent configuration.

In case (a), the same computations made in the previous paragraph imply that the polynomial $G(1,z)$ should have $z=-\xi'$ as a root of multiplicity $\geqslant m_3$. This is absurd, since $m_3\geqslant 2$ and therefore $m_3+(m_3-1)>m_3$.

In case (b), we conclude that $z=-\xi'$ should also be a root of $G(1,z)$ of multiplicity m_3-1. This implies that $2(m_3-1)\leqslant m_3$, i.e. $m_3\leqslant 2$.

Since we assume that $m_3\geqslant 2$, it remains to treat the case (b) with $m_3=2$. Here, $\widetilde{\chi}|_{\mathfrak{e}}$ is necessarily given (dropping the tildes) by

$$\widetilde{\chi}|_{\mathfrak{e}} = \varrho z(z+\beta y)\frac{\partial}{\partial y}+z\left(\alpha x\frac{\partial}{\partial x}+\gamma z\frac{\partial}{\partial z}\right)$$

for some $(\alpha,\beta,\gamma)\in\mathbf{R}^3\setminus\{(0,0,0)\}$ and $\varrho\neq 0$. If we apply the coordinate change $z'=z+\eta y$ (where $\eta:=\xi-\xi'$), we get (dropping the primes)

$$\varrho(z-\eta y)(z+(\beta-\eta)y)\left(\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial z}\right)+(z-\eta y)\left(\alpha x\frac{\partial}{\partial x}+\gamma(z-\eta y)\frac{\partial}{\partial z}\right).$$

We now use the assumption that $\widetilde{\Omega}'$ is in a nilpotent configuration. Looking at the coefficients of $\partial/\partial x$ and $\partial/\partial y$, this implies that $\alpha=0$ and $\eta=\beta$. Therefore, the coefficient of $\partial/\partial z$ has the form

$$(z-\eta y)(\varrho\eta z+\gamma(z-\eta y))$$

and, since this expression should be equal to $\gamma' z^2$ (for some nonzero real constant γ'), we conclude that necessarily $\eta=0$. This proves the claim.

We now prove the lemma in the simpler case where $\widetilde{\mathbf{m}}=\mathbf{m}$. Here, the vertical displacement vector $\tilde{\Delta}$ is such that

$$\tilde{\Delta}>_{\text{lex}}\Delta.$$

Suppose that there exists another map $(f',0)\in\mathcal{G}_\Delta$ such that the Newton data

$$\widetilde{\Omega}' := (f',0)\cdot\Omega$$

also has a displacement vector $\tilde{\Delta}' >_{\mathrm{lex}} \Delta$. We claim that the map

$$(\tilde{f},0) := (f',0)\circ(f,0)^{-1}\in\mathcal{G}_\Delta,$$

which sends $\widetilde{\Omega}$ to $\widetilde{\Omega}'$, is necessarily the identity. In fact, if $\widetilde{\chi}$ denotes the vector field which is associated with $\widetilde{\Omega}$, then

$$\widetilde{\chi}|_{\mathfrak{e}} = y^{m_2} z^{m_3}\left(\alpha x\frac{\partial}{\partial x}+\beta y\frac{\partial}{\partial y}+\gamma z\frac{\partial}{\partial z}\right)$$

for some $(\alpha,\beta,\gamma)\in\mathbf{R}^3\setminus\{(0,0,0)\}$ such that $\alpha=\gamma=0$ if $m_2=-1$.

If we write $\tilde{f}=\xi x^{\Delta_1}y^{\Delta_2}$, then the map $(\tilde{f},0)$ transforms $\widetilde{\chi}|_{\mathfrak{e}}$ to

$$y^{m_2}(z-\xi x^{\Delta_1}y^{\Delta_2})^{m_3}\left(\alpha x\frac{\partial}{\partial x}+\beta y\frac{\partial}{\partial y}+(\gamma z-\xi(\gamma-\alpha\Delta_1-\beta\Delta_2)x^{\Delta_1}y^{\Delta_2})\frac{\partial}{\partial z}\right). \qquad (19)$$

Since $m_3\geqslant 1$, the assumption $\tilde{\Delta}' >_{\mathrm{lex}} \Delta$ necessarily implies that $\xi=0$. □

The map $(f,0)\in\mathcal{G}_\Delta$, which is defined by Proposition 4.13, will be called the *basic edge-preparation map* associated with Ω.

Remark 4.14. The expression obtained in (19) has the following simple consequence, which will be needed in §4.11. Suppose that the Newton data Ω is such that $\Delta_1>0$ and

$$\mathrm{supp}(\Omega|_{\mathfrak{e}})\cap\{\mathbf{v}\in\mathbf{Z}^3 : v_3 = m_3-1\}=\varnothing.$$

Then, Ω is edge-stable. Indeed, for all maps $(f,0)\in\mathcal{G}_\Delta$ we know that $(f,0)\cdot\Omega$ has the same main vertex of Ω (because no regular-nilpotent transition can occur, since $\Delta_1>0$). Moreover, the expression (19) implies that $\mathrm{supp}((f,0)\cdot\Omega|_{\mathfrak{e}})$ contains at least two points. Therefore, $(f,0)\cdot\Omega$ has the same main edge as Ω.

LEMMA 4.15. *Suppose that* $\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$ *is an edge-stable Newton data. Then*

$$(f,0)\cdot\Omega$$

also belongs to $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$, *for all maps* $(f,0)\in\mathcal{G}_\Delta$.

Proof. If $C=\infty$ or $C=0$ then nothing has to be proven. If $0<C<\infty$ then the main face $\mathcal{F}$ of the Newton polyhedron $\mathcal{N}$ associated with Ω contains at least one vertex $\mathbf{v}\in\operatorname{supp}(\Omega)$ which is not in the main edge $\mathfrak{e}$ and is such that

$$(\mathbf{v}-\Delta)\cap\mathcal{N}=\varnothing. \tag{20}$$

Indeed, choose some arbitrary vertex $\mathbf{v}'\in\operatorname{supp}(\Omega)\cap\mathcal{F}\setminus\mathfrak{e}$ (there exists at least one such vertex, because $0<C<\infty$). If $\mathbf{v}'$ satisfies (20), then choose $\mathbf{v}=\mathbf{v}'$. Otherwise, there necessarily exists some $\varepsilon>0$ such that the segment $\{\mathbf{v}-t\Delta:t\in[0,-\varepsilon]\}$ is an edge of $\mathcal{F}$. Then, it suffices to choose $\mathbf{v}$ to be the other extreme of that edge, i.e. $\mathbf{v}:=\mathbf{v}'-\varepsilon\Delta$.

Now, if we choose the vertex $\mathbf{v}\in\operatorname{supp}(\Omega)$ as above, it is clear that $\widetilde{\Omega}(\mathbf{v})=\Omega(\mathbf{v})$. Therefore, since $\mathbf{v}$ and $\mathfrak{e}$ are affinely independent and Ω is edge-stable, we conclude that $\widetilde{\Omega}\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$. □

In the next lemma, we consider the action of the subgroup $\mathcal{G}^{i,+}_{\Delta,C}$ of edge-preserving maps (see §3.6).

Lemma 4.16. *Suppose that* $\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$ *is an edge-stable Newton data which is not stable. Then, there exists a unique edge-preserving map* $(f,g)\in\mathcal{G}^{i,+}_{\Delta,C}$ *such that*

$$(f,g)\cdot\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,\widetilde{C}}$$

for some $\widetilde{C}>C$.

Proof. To prove the existence part, let $(f,g)\in\mathcal{G}^{i}_{\Delta,C}$ be such that $(f,g)\cdot\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,\widetilde{C}}$. Then, we can uniquely decompose (f,g) as $(f_2,0)\circ(f_1,g_1)$, where $(f_2,0)$ belongs to $\mathcal{G}_\Delta$ and $(f_1,g_1)\in\mathcal{G}^{i,+}_{\Delta,C}$ is an edge-preserving map.

We claim that $\Omega_1:=(f_1,g_1)\cdot\Omega$ belongs to $\mathrm{New}^{i,\mathbf{m}}_{\Delta,\widetilde{C}}$ for some $\widetilde{C}>C$. Indeed, suppose, by contradiction, that this is not the case. Then, Ω_1 is an edge-stable Newton data in $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$. Using Lemma 4.15, we conclude that $(f_2,0)\cdot\Omega_1$ also belongs to $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$. This yields a contradiction.

To prove the uniqueness of $(f,g)\in\mathcal{G}^{i,+}_{\Delta,C}$, consider two maps (f_1,g_1) and (f_2,g_2) in $\mathcal{G}^{i,+}_{\Delta,C}$ such that

$$\Omega_j:=(f_j,g_j)\cdot\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,C_j}$$

for some $C_j>C$, $j=1,2$. Then, if we define the composed map

$$(\tilde{f},\tilde{g}):=(f_2,g_2)\circ(f_1,g_1)^{-1}\in\mathcal{G}^{i,+}_{\Delta,C},$$

we get $\Omega_2=(\tilde{f},\tilde{g})\cdot\Omega_1$. Using Proposition 4.11, we conclude that $(\tilde{f},\tilde{g})=(0,0)$. □

Given an edge-stable Newton data Ω, the map $(f,g)\in\mathcal{G}^{i,+}_{\Delta,C}$ which is defined by Lemma 4.16 will be called the *basic face-preparation map* associated with Ω.

4.6. Formal adapted charts and invariance of $(\mathbf{m}, \Delta, C)$

For a fixed adapted local chart $(U,(x,y,z))$ at a divisor point $p\in\mathfrak{D}\cap A$, and a choice of local generator for the line field L, there is an associated Newton data Ω. In §3, we have seen how to associate certain *combinatorial quantities* $(\mathbf{m}, \Delta, C)$ to the Newton data.

A natural question is how these combinatorial quantities depend on the choice of the adapted local chart. Our present goal is to answer this question.

First of all, we need to slightly extend the concept of adapted local chart (see §3.1).

A *formal adapted chart* at a divisor point $p\in\mathfrak{D}\cap A$ is a triple (x,y,z) formed by elements of the formal completion $\widehat{\mathcal{O}}_p\supset\mathcal{O}_p$ (with respect to the Krull topology) such that the following conditions hold:

(i) the formal functions x, y and z are independent at p (i.e. their residue class generates $\widehat{\mathbf{m}}_p/\widehat{\mathbf{m}}_p^2$);

(ii) $\mathfrak{z}$ is locally generated by $\partial/\partial z$;

(iii) if $\iota_p=[i]$ then $D_i=\{x=0\}$;

(iv) if $\iota_p=[i,j]$ (with $i>j$) then $D_i=\{x=0\}$ and $D_j=\{y=0\}$.

It is immediate that the construction of §3.2 can be carried out in the present setting. Thus, up to a choice of a local generator χ for the line field at p, there exists a well-defined *formal Newton map*

$$\Theta\colon \mathbf{Z}^3 \longrightarrow \mathbf{R}^3$$

for $(\mathbf{M},\mathrm{Ax})$ at p, relative to (x,y,z). We call the triple $((x,y,z),\iota_p,\Theta)$ a *formal Newton data* for $(\mathbf{M},\mathrm{Ax})$ at p. We define the classes $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$ exactly as above.

Given two formal adapted charts (x,y,z) and $(\tilde{x},\tilde{y},\tilde{z})$ at p, the transition map is given by

$$\tilde{x}=xu(x,y),\quad \tilde{y}=g(x)+yv(x,y)\quad\text{and}\quad \tilde{z}=f(x,y)+zw(x,y,z),\tag{21}$$

where $g\in\mathbf{R}[[x]]$, $u,v,f\in\mathbf{R}[[x,y]]$ and $w\in\mathbf{R}[[x,y,z]]$ are such that $g(0)=f(0)=0$, u, v and w are units, and $g=0$ if $\#\iota_p=2$. The set of all such changes of coordinates forms a group, which we denote by $\widehat{\mathrm{G}}$. An element of $\widehat{\mathrm{G}}$ will be shortly denoted by (f,g,u,v,w). We consider also the subgroups

$$\widehat{\mathrm{G}}^1=\widehat{\mathrm{G}}\quad\text{and}\quad \widehat{\mathrm{G}}^2=\{(f,g,u,v,w)\in\widehat{\mathrm{G}}:g=0\}.$$

Remark 4.17. The Lie algebra associated with the group $\widehat{\mathrm{G}}$ is formed by all formal vector fields having the form

$$xu(x,y)\frac{\partial}{\partial x}+(g(x)+yv(x,y))\frac{\partial}{\partial y}+(f(x,y)+zw(x,y,z))\frac{\partial}{\partial z},$$

where u,v,w are units and $g(0)=f(0)=0$.

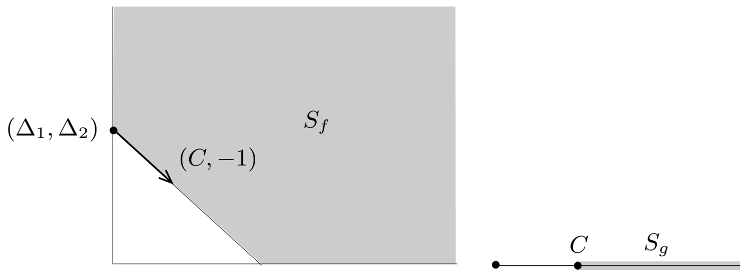


Figure 19. The supports of S_f and S_g in the definition of $\widehat{\mathrm{G}}^i_{\Delta,C}$.

We denote by $\widehat{\mathrm{G}}^i_{\Delta,C}$ the subgroup of all maps $(f,g,u,v,w)\in\widehat{\mathrm{G}}^i$ such that the supports of the maps f and g satisfy the following conditions (see Figure 19):

$$S_f\subset\{(a,b)\in\mathbf{N}^2:\langle(1,C),(a,b)-\Delta\rangle\geqslant 0 \text{ and } (a,b)\geqslant_{\mathrm{lex}}\Delta\}\quad\text{and}\quad S_g\subset\{c\in\mathbf{N}:c\geqslant C\}.$$

If $C=\infty$, the former condition is replaced by $S_f\subset\{(a,b)\in\mathbf{N}^2:b\geqslant\Delta_2\}$. We shall say that $\widehat{\mathrm{G}}^i_{\Delta,C}$ is the group of (Δ,C)-*face maps*.

The following lemma relates the groups $\widehat{\mathrm{G}}^i_{\Delta,C}$ and $\mathcal{G}^i_{\Delta,C}$.

LEMMA 4.18. *There exists a normal subgroup $\widehat{\mathrm{G}}^{i,+}_{\Delta,C}\lhd\widehat{\mathrm{G}}^i_{\Delta,C}$ such that the quotient $\widehat{\mathrm{G}}^i_{\Delta,C}/\widehat{\mathrm{G}}^{i,+}_{\Delta,C}$ is naturally isomorphic to $\mathcal{G}^i_{\Delta,C}$. We shall say that $\widehat{\mathrm{G}}^{i,+}_{\Delta,C}$ is the subgroup of* (Δ,C)-face preserving maps.

Proof. We define explicitly the subgroup $\widehat{\mathrm{G}}^{i,+}_{\Delta,C}$ as follows:

(a) if $C\in\{0,\infty\}$ then $(f,g,u,v,w)\in\widehat{\mathrm{G}}^i_{\Delta,C}$ belongs to $\widehat{\mathrm{G}}^{i,+}_{\Delta,C}$ if and only if

$$S_f\subset\{(a,b)\in\mathbf{N}^2:(a,b)>_{\mathrm{lex}}\Delta\}\quad\text{and}\quad S_g\subset\{c\in\mathbf{N}:c>C\};$$

(b) if $0<C<\infty$ then $(f,g,u,v,w)\in\widehat{\mathrm{G}}^i_{\Delta,C}$ belongs to $\widehat{\mathrm{G}}^{i,+}_{\Delta,C}$ if and only if

$$S_f\subset\{(a,b)\in\mathbf{N}^2:\langle(1,C),(a,b)-\Delta\rangle>0\}.$$

It is immediate to verify that this gives a normal subgroup of $\widehat{\mathrm{G}}^i_{\Delta,C}$. Moreover,

$$\mathcal{G}^i_{\Delta,C}\cap\widehat{\mathrm{G}}^{i,+}_{\Delta,C}=\{0\}\quad\text{and}\quad\widehat{\mathrm{G}}^i_{\Delta,C}=\widehat{\mathrm{G}}^{i,+}_{\Delta,C}\circ\mathcal{G}^i_{\Delta,C}=\mathcal{G}^i_{\Delta,C}\circ\widehat{\mathrm{G}}^{i,+}_{\Delta,C}$$

(i.e. $\widehat{\mathrm{G}}^i_{\Delta,C}$ is the semi-direct product of $\mathcal{G}^i_{\Delta,C}$ and $\widehat{\mathrm{G}}^{i,+}_{\Delta,C}$). □

The group $\widehat{\mathrm{G}}^i$ acts in an obvious way on the set of formal Newton data. Given $\Omega\in\mathrm{New}$, we denote by $\widehat{\mathrm{G}}^i\cdot\Omega$ its orbit under this action. We adopt a similar notation for the action of the subgroups $\widehat{\mathrm{G}}^i_{\Delta,C}$ and $\widehat{\mathrm{G}}^{i,+}_{\Delta,C}$.

Lemma 4.19. *Given a formal Newton data* $\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$, *the orbit*

$$\widehat{\mathrm{G}}^{i,+}_{\Delta,C}\cdot\Omega$$

lies entirely in the class $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$. *If we further assume that* Ω *is* stable *then the orbit*

$$\widehat{\mathrm{G}}^{i}_{\Delta,C}\cdot\Omega$$

also lies in the class $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$.

Proof. This is an immediate corollary of Lemma 4.18, Corollary 3.19 and the definition of a stable Newton data. □

As a consequence, we obtain the following result on the invariance of the quantities $(\mathbf{m},\Delta,C)$.

Proposition 4.20. *Let* $\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$ *and* $\widetilde{\Omega}\in\mathrm{New}^{i,\widetilde{\mathbf{m}}}_{\tilde{\Delta},\widetilde{C}}$ *be two stable Newton data which lie on the same* $\widehat{\mathrm{G}}^i$*-orbit. Then* $(\mathbf{m},\Delta,C)=(\widetilde{\mathbf{m}},\tilde{\Delta},\widetilde{C})$.

Proof. Let $(f,g,u,v,w)\in\widehat{\mathrm{G}}^i$ be the map such that $(f,g,u,v,w)\cdot\Omega=\widetilde{\Omega}$. We shall prove that (f,g,u,v,w) belongs to the subgroup $\widehat{\mathrm{G}}^i_{\Delta,C}$.

We define $P\subset\overline{\mathbf{Q}}^2\times\overline{\mathbf{Q}}$ as the subset of all pairs (Δ_0,C_0) such that (f,g,u,v,w) belongs to the subgroup $\widehat{\mathrm{G}}^i_{\Delta_0,C_0}$.

Since the union $\bigcup_{\Delta,C}\widehat{\mathrm{G}}^i_{\Delta,C}$ exhausts $\widehat{\mathrm{G}}^i$, we know that P is nonempty. Let us fix an element $(\bar{\Delta},\overline{C})\in P$. Using Lemma 4.18, we can uniquely write

$$(f,g,u,v,w)=(\bar{f},\bar{g})\circ(f_1,g_1,u_1,v_1,w_1),$$

with $(\bar{f},\bar{g})\in\mathcal{G}^i_{\bar{\Delta},\overline{C}}$ and $(f_1,g_1,u_1,v_1,w_1)\in\widehat{\mathrm{G}}^{i,+}_{\bar{\Delta},\overline{C}}$. From the discussion in §3.6, we can further write the decomposition

$$(\bar{f},\bar{g})=(f_0,0)\cdot(f_1,g_1), \tag{22}$$

with $(f_0,0)\in\mathcal{G}^i_{\bar{\Delta}}$ and $(f_1,g_1)\in\mathcal{G}^{i,+}_{\bar{\Delta},\overline{C}}$.

First of all, let us assume by contradiction that $\mathbf{m}\neq\widetilde{\mathbf{m}}$. Then, we immediately see that either $\widetilde{\Omega}$ or Ω is in a hidden nilpotent configuration, and that the action of the map $(\bar{f},\bar{g})$ (or its inverse) causes regular-nilpotent transition. This contradicts the hypothesis that both Ω and $\widetilde{\Omega}$ are stable.

Assuming that $\mathbf{m}=\widetilde{\mathbf{m}}$, let us suppose by contradiction that $\Delta>_{\mathrm{lex}}\tilde{\Delta}$. Then, the pair $(\tilde{\Delta},C_0)$ necessarily lies in the set P (for some constant C_0). Moreover, in the corresponding decomposition (22) for $(\bar{\Delta},\overline{C}):=(\tilde{\Delta},C_0)$, one has

$$(f_0,0)=(\xi x^{\tilde{\Delta}_1}y^{\tilde{\Delta}_2},0)\quad\text{for some constant } \xi\neq0.$$

However, using the above decomposition of (f,g,u,v,w), we immediately see that

$$(f_0,0)^{-1}\cdot\widetilde{\Omega}\notin \mathrm{New}^{i,\widetilde{\mathbf{m}}}_{\tilde{\Delta},\widetilde{C}},$$

and this contradicts the hypothesis that $\widetilde{\Omega}$ is stable.

Finally, we assume by contradiction that $(\mathbf{m},\Delta)=(\widetilde{\mathbf{m}},\tilde{\Delta})$ and $C>\widetilde{C}$. We prove the following statement.

Claim. There exists a constant $C_0<C$ such that the pair (Δ,C_0) lies in P. Moreover, the decomposition (22) for $(\bar{\Delta},\overline{C}):=(\Delta,C_0)$ is such that the map

$$(f_1,g_1)\in\mathcal{G}^{i,+}_{\Delta,C_0}$$

is nonzero.

Indeed, if the claim is false, the map (f,g,u,v,w) should lie in $\widehat{\mathrm{G}}^{i}_{\Delta,C}$ and Lemma 4.19 would imply that $\widetilde{\Omega}$ also lies in $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$. This contradicts the assumption that $C>\widetilde{C}$.

Using the above claim and Proposition 4.11, we conclude that $(f_1,g_1)\cdot\Omega$ belongs to $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C_0}$. Consequently, $\widetilde{\Omega}$ also lies in $\mathrm{New}^{i,m}_{\Delta,C_0}$ (i.e. $\widetilde{C}=C_0$). Taking the inverse map, we see that

$$(f_1,g_1)^{-1}\cdot\widetilde{\Omega}\notin \mathrm{New}^{i,\mathbf{m}}_{\Delta,C_0}.$$

This contradicts the hypothesis that $\widetilde{\Omega}$ is stable. The proposition is proved. □

4.7. Stabilization of adapted charts

The main goal of this subsection is to prove that one can always find a stable Newton data for $(\mathbf{M},\mathrm{Ax})$ at a nonelementary point p lying on the divisor $\mathfrak{D}$.

PROPOSITION 4.21. *Let $p\in\mathfrak{D}\cap A$ be a divisor point belonging to* $\mathrm{NElem}(\mathbf{M})$. *Then, there exists an* analytic *adapted local chart $(U,(x,y,z))$ at p such that the associated Newton data $\Omega=((x,y,z),\iota_p,\Theta)$ is stable.*

Proposition 4.21 will be an immediate consequence of the following result.

PROPOSITION 4.22. (Stabilization of adapted charts) *Let $p\in\mathfrak{D}\cap A$ be a divisor point belonging to* $\mathrm{NElem}(\mathbf{M})$ *and let $(U,(x,y,z))$ be an analytic adapted local chart at p. Then, there exists an analytic change of coordinates*

$$\tilde{y}=y+g(x)\quad\textit{and}\quad\tilde{z}=z+f(x,y),\quad\textit{where } f\in\mathbf{R}\{x,y\} \textit{ and } g\in\mathbf{R}\{x\},$$

with $f(0)=g(0)=0$, such that the Newton data associated with the new adapted local chart $(U,(x,\tilde{y},\tilde{z}))$ is stable.

We shall prove this proposition using two lemmas which describe the stabilization of Newton data. We shall say that a map $(f,g,u,v,w)\in\widehat{\mathrm{G}}^i$ is a *stabilization map* for Ω if $f(0)=g(0)=0$ and $(f,g,u,v,w)\cdot\Omega$ is a stable Newton data. Similarly, we say that $(f,g,u,v,w)\in\widehat{\mathrm{G}}^i$ is an *edge-stabilization map* for Ω if $f(0)=g(0)=0$ and $(f,g,u,v,w)\cdot\Omega$ is an edge-stable Newton data.

LEMMA 4.23. *Let Ω be the Newton data for a divisor point $p\in\mathfrak{D}\cap A$ belonging to* NElem($\mathbf{M}$). *Then, there exists an edge-stabilization map for Ω which has the form*

$$(f,0,1,1,1)\in\widehat{\mathrm{G}}^i$$

for some $f\in\mathbf{R}\{x,y\}$.

Proof. Define $\Omega_0:=\Omega$ and consider the sequence

$$\Omega_0,\quad \Omega_1=(f_0,0)\cdot\Omega_0,\quad ...,\quad \Omega_{n+1}=(f_n,0)\cdot\Omega_n,\quad ..., \tag{23}$$

where each Ω_{n+1} is obtained by applying the basic edge-preparation map

$$(f_n,0)=(\xi_n x^{a_n}y^{b_n},0)$$

to Ω_n (see §4.5).

If there exists a finite natural number n such that Ω_n is edge-stable, then we are done. In fact, the polynomial map

$$f(x,y)=\sum_{i=0}^{n-1}\xi_i x^{a_i}y^{b_i}$$

is such that $(f,0,1,1,1)\cdot\Omega$ is edge-stable.

Otherwise, $\{(a_n,b_n)\}_{n\geqslant 0}$ is an infinite sequence, strictly increasing for the lexicographical ordering. Up to discarding a finite initial segment of the sequence (23), we may assume that all Ω_n have the same main vertex. In fact, it follows from Lemma 4.9 that a regular-nilpotent transition in this sequence can occur after *at most* m_3+1 basic edge-preparation maps (where $\mathbf{m}=(m_1,m_2,m_3)$ is the main vertex of Ω_0).

Therefore we may assume that, for each $n\geqslant 0$,

$$\Omega_n\in\mathrm{New}^{i,\mathbf{m}}_{(a_n,b_n)}$$

and $(a_{n+1},b_{n+1})>_{\mathrm{lex}}(a_n,b_n)$. Two cases can occur (see Figure 20):

(i) $a_n\to\infty$ as $n\to\infty$;

(ii) there exist two natural numbers $a,N\in\mathbf{N}$, with $a\geqslant\Delta_1$, such that

$$a_n=a\quad\text{for all } n\geqslant N,$$

and $b_n\to\infty$ as $n\to\infty$.

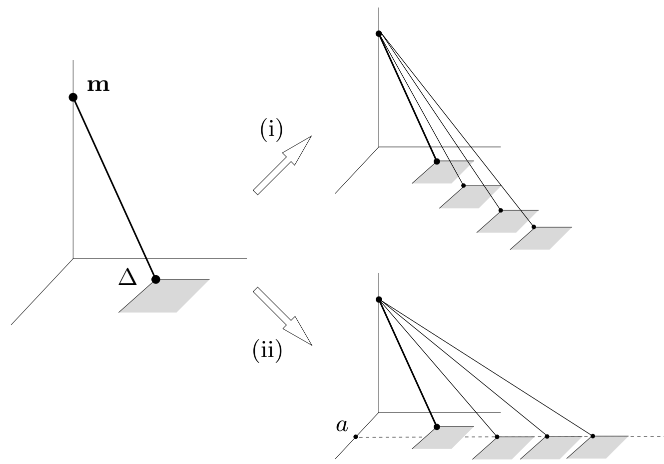


Figure 20. Cases (i) and (ii) for the edge-stabilization sequence.

We first prove that case (i) yields a contradiction. In fact, if we consider the formal series

$$f(x,y)=\sum_{i=0}^{\infty}\xi_i x^{a_i}y^{b_i}, \tag{24}$$

it is clear that $\widetilde{\Omega}=(f,0,1,1,1)\cdot\Omega$ belongs to $\mathrm{New}^{i,\mathbf{m}}_{(\infty,\infty)}$.

If we write the vector field associated with Ω as

$$\chi=F(x,y,z)\frac{\partial}{\partial x}+G(x,y,z)\frac{\partial}{\partial y}+H(x,y,z)\frac{\partial}{\partial z},$$

then the change of coordinates $\tilde{z}=z+f(x,y)$ gives the formal vector field

$$\widetilde{\chi}=\tilde{f}(x,y,\tilde{z})\frac{\partial}{\partial x}+\tilde{g}(x,y,\tilde{z})\frac{\partial}{\partial y}+\widetilde{H}(x,y,\tilde{z})\frac{\partial}{\partial z},$$

where $\tilde{f}(x,y,\tilde{z})=F(x,y,\tilde{z}-f(x,y))$, $\tilde{g}(x,y,\tilde{z})=G(x,y,\tilde{z}-f(x,y))$ and

$$\widetilde{H}(x,y,\tilde{z})=H(x,y,\tilde{z}-f(x,y))+\frac{\partial f}{\partial x}(x,y)\tilde{f}(x,y,\tilde{z})+\frac{\partial f}{\partial y}(x,y)\tilde{g}(x,y,\tilde{z}).$$

If we write $\mathbf{m}=(m_1,m_2,m_3)$ then the condition $\widetilde{\Omega}\in\mathrm{New}^{i,\mathbf{m}}_{(\infty,\infty)}$ implies that there exists a factorization

$$\begin{pmatrix}\tilde{f}\\ \tilde{g}\\ \widetilde{H}\end{pmatrix}=\tilde{z}^{m_3}\begin{pmatrix}\tilde{f}_1\\ \tilde{g}_1\\ \tilde{z}\widetilde{H}_1\end{pmatrix},$$

for some formal germs F_1, G_1 and H_1. Going back to the original variables, we get

$$\begin{pmatrix} F \\ G \\ H \end{pmatrix} = (z+f(x,y))^{m_3} \begin{pmatrix} F_1 \\ G_1 \\ (z+f(x,y))H_1 \end{pmatrix},$$

where (F_1, G_1, H_1) is given by

$$\begin{pmatrix} F_1 \\ G_1 \\ H_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\partial f/\partial x & -\partial f/\partial y & 1 \end{pmatrix} \begin{pmatrix} \tilde{f}_1 \\ \tilde{g}_1 \\ \widetilde{H}_1 \end{pmatrix}.$$

Recall now that $m_3 \geqslant 1$ (because Ω is centered at a point $p \in \mathrm{NElem}(\mathbf{M})$). Therefore, we can apply Corollary A.2 to conclude that $f(x,y)$ is necessarily an analytic function.

As a consequence, the Newton data $\widetilde{\Omega} = (f,0)\cdot\Omega$ is analytic. Notice, however, that the associated vector field $\widetilde{\chi}$ violates the condition of being nondegenerate with respect to the divisor (see Definition 2.2). Indeed, the ideal $\mathcal{I}_{\widetilde{\chi}}(\tilde{z})$ is generated by

$$\mathcal{I}_{\widetilde{\chi}}(\tilde{z}) = (\tilde{z}\tilde{f}, \tilde{z}\tilde{g}, \widetilde{H}),$$

and therefore $\mathcal{I}_{\widetilde{\chi}}(\tilde{z})$ is divisible by $\tilde{z}$. Since $\{z=0\}$ is not a component of the divisor, we get a contradiction to the assumption that $\widetilde{\chi}$ is nondegenerate.

Suppose now that (ii) holds. Then, the formal map f given in (24) can be written in the form

$$f(x,y) = f_\delta(y)x^\delta + f_{\delta+1}(y)x^{\delta+1} + \ldots + f_a(y)x^a, \quad \text{with } f_\delta, \ldots, f_a \in \mathbf{R}[[y]], \tag{25}$$

where $\delta := \Delta_1$. We claim that $f_\delta, f_{\delta+1}, \ldots, f_a$ are analytic germs.

Indeed, let us apply the change of coordinates $\tilde{z} = z + f(x,y)$. Keeping the same notation as above, we get the formal vector field

$$\tilde{f}(x,y,\tilde{z})\frac{\partial}{\partial x} + \tilde{g}(x,y,\tilde{z})\frac{\partial}{\partial y} + \widetilde{H}(x,y,\tilde{z})\frac{\partial}{\partial z},$$

which is associated with the (formal) Newton data $\widetilde{\Omega} = (f,0)\cdot\Omega$. From the hypothesis, we know that $\widetilde{\Omega}$ belongs to $\mathrm{New}^{i,\mathbf{m}}_{\tilde{\Delta},\widetilde{C}}$, for some $\tilde{\Delta} = (\tilde{a}, \tilde{b})$ such that $\tilde{a} > a$. This is equivalent to saying that, if we consider the homomorphic images $[\tilde{f}]$, $[\tilde{g}]$ and $[\widetilde{H}]$ in the quotient ring $\widehat{R}_a = \mathbf{R}[[x,y,z]]/(x^a)\mathbf{R}[[x,y,z]]$, we get

$$\begin{pmatrix} [\tilde{f}] \\ [\tilde{g}] \\ [\widetilde{H}] \end{pmatrix} = [\tilde{z}]^{m_3} \begin{pmatrix} [\tilde{f}_1] \\ [\tilde{g}_1] \\ [\tilde{z}\widetilde{H}_1] \end{pmatrix}.$$

Going back to the original functions F, G and H and using the same reasoning as above, we conclude, by Corollary A.3 of Appendix A, that the germ $[z-f(x,y)]\in\widehat{R}_a$ belongs to the homomorphic image of a convergent germ.

Therefore, using the expression (25), we conclude that $f_\delta, f_{\delta+1}, ..., f_a$ are convergent germs, and the function f given by (25) is analytic. This proves the claim.

Let us call the map $(f,0)\in\widehat{G}^i$ the *extended edge-preparation step* associated with Ω. If the Newton data

$$\Omega^{(1)} := (f,0)\cdot\Omega$$

is edge-prepared, then we are done. Otherwise, we can start all over again, defining a new extended edge-preparation step $(f^{(1)},0)$ associated with $\Omega^{(1)}$, and setting $\Omega^{(2)}=(f^{(1)},0)\cdot\Omega^{(1)}$, $\Omega^{(3)}=(f^{(2)},0)\cdot\Omega^{(2)}$, and so on.

Suppose that we can iterate this procedure infinitely many times. Then, we get a sequence of Newton data $\{\Omega^{(n)}\}_{n\geqslant 0}$, where each element of the sequence is obtained from its predecessor by an extended edge-preparation step $(f^{(n)},0)$, for some analytic germ $f^{(n)}\in\mathbf{R}\{x,y\}$ such that

$$f^{(n)} = O\big(x^{\Delta_1^{(n)}}\big), \tag{26}$$

for some strictly increasing sequence $\{\Delta_1^{(n)}\}_{n\geqslant 0}$ of natural numbers. We claim that there exists a finite $n\in\mathbf{N}$ such that $\Omega^{(n)}$ is edge-prepared.

Indeed, suppose by contradiction that this sequence is infinite. Then, it follows from the expression (26) that the composed map

$$\mathbf{f}_n = f^{(n)}\circ f^{(n-1)}\circ ...\circ f^{(1)}\circ f$$

converges in the Krull topology, as $n\to\infty$, to a formal map $\tilde{f}\in\mathbf{R}[[x,y]]$. Moreover, the formal Newton data $\widetilde{\Omega}:=(\tilde{f},0)\cdot\Omega$ belongs to $\mathrm{New}^{i,\mathbf{m}}_{(\infty,\infty)}$. Using the same reasoning as for item (i), we get a contradiction. □

LEMMA 4.24. *Let Ω be an analytic Newton data centered at a divisor point $p\in\mathfrak{D}\cap A$ belonging to* NElem($\mathbf{M}$). *Assume that Ω is edge-stable. Then, there exists a stabilization map for Ω which has the form*

$$(f,g,1,1,1)\in\widehat{\mathrm{G}}^i$$

for some convergent germs $g\in\mathbf{R}\{x\}$ and $f\in\mathbf{R}\{x,y\}$ with $f(0)=g(0)=0$.

Proof. Define $\Omega_0:=\Omega$ and consider the sequence

$$\Omega_0,\quad \Omega_1=(f_0,g_0)\cdot\Omega_0,\quad ...,\quad \Omega_{n+1}=(f_n,g_n)\cdot\Omega_n,\quad ...,$$

where Ω_{n+1} is obtained by applying the basic face-preparation map (f_n,g_n) to Ω_n (see §4.5). Notice that, for each $n\geqslant 0$,

$$\Omega_n\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,C_n}$$

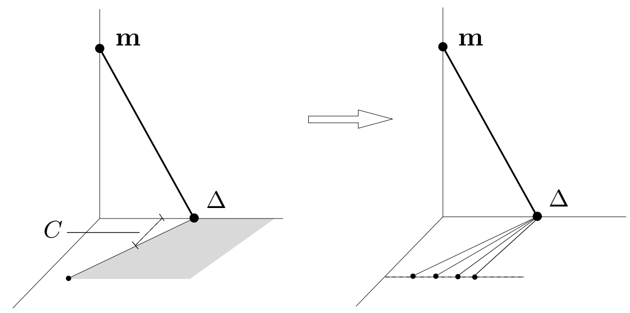


Figure 21. The sequence of basic face-preparations.

and $C_0 < C_1 < C_2 < \ldots$ is a strictly increasing sequence of rational numbers.

Notice that the rational numbers C_n always belong to the discrete lattice

$$\frac{1}{((m_3+1)\Delta)!}\mathbf{Z}$$

(see Remark 3.15). Therefore $C_n \to \infty$ as $n \to \infty$.

If there exists a finite natural number n such that Ω_n is stable, then we are done. In fact, the composed map

$$(\mathbf{f}_n, \mathbf{g}_n) := (f_n, g_n) \circ \ldots \circ (f_0, g_0)$$

is a polynomial map and $(\mathbf{f}_n, \mathbf{g}_n, 1, 1, 1) \in \widehat{\mathrm{G}}^i$ is such that $(\mathbf{f}_n, \mathbf{g}_n, 1, 1, 1) \cdot \Omega$ is stable.

Otherwise, $\{\Omega_n\}_{n \geqslant 0}$ forms an infinite sequence, and the condition that $(f_n, g_n) \in \mathcal{G}^i_{\Delta, C_n}$ implies that the sequence of composed maps $\{(\mathbf{f}_n, \mathbf{g}_n)\}_{n \geqslant 0}$ converges (in the Krull topology) to a pair of formal maps (f, g) such that

$$(f, g, 1, 1, 1) \in \widehat{\mathrm{G}}^i.$$

Moreover, $f \in \mathbf{R}[[x, y]]$ can be written in the form

$$f(x, y) = f_0(x) + f_1(x) y + \ldots + f_{b-1}(x) y^{b-1},$$

with each f_i belonging to $\mathbf{R}[[x]]$ and $b = \lceil \Delta_2 \rceil$. Notice that

$$\widetilde{\Omega} := (f, g, 1, 1, 1) \cdot \Omega$$

is a formal Newton data which belongs to $\mathrm{New}^{i,\mathbf{m}}_{\Delta,\infty}$.

We claim that $(f, g, 1, 1, 1)$ is an analytic map.

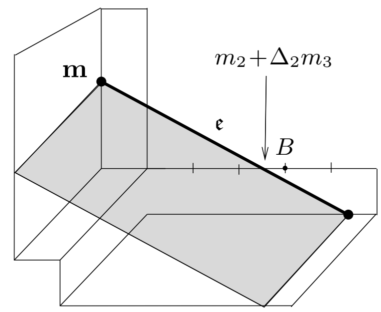


Figure 22. The number B.

Suppose initially that $g=0$. Let us write the vector field associated with $\widetilde{\Omega}$ as

$$\tilde{f}(x,\tilde{y},\tilde{z})\frac{\partial}{\partial x}+\tilde{g}(x,\tilde{y},\tilde{z})\frac{\partial}{\partial \tilde{y}}+\widetilde{H}(x,\tilde{y},\tilde{z})\frac{\partial}{\partial z}.$$

From the hypothesis, we know that the coefficients $\tilde{f}(x,\tilde{y},0)$, $\tilde{g}(x,\tilde{y},0)$ and $\widetilde{H}(x,\tilde{y},0)$ are such that

$$\tilde{f}(x,\tilde{y},0),\widetilde{H}(x,\tilde{y},0)\in(\tilde{y}^B)\mathbf{R}[[x,y]] \quad \text{and} \quad \tilde{g}(x,\tilde{y},0)\in(\tilde{y}^{B+1})\mathbf{R}[[x,y]], \tag{27}$$

where $B:=\lceil m_2+\Delta_2 m_3\rceil$ (see Figure 22). Since Ω is centered at a point $p\in\mathrm{NElem}(\mathbf{M})$, we know also that $B\geqslant 1$ (because otherwise Ω would be in a final situation, contradicting Proposition 4.3). Hence, we can use the same reasoning as in the proof of Lemma 4.23 to show that f is analytic.

Let us suppose now that $g\neq 0$. Then, from the definition of $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$, we know that $\{y=0\}$ is not a local irreducible component of the divisor at the point p.

Moreover, it follows from the condition (27) that the coefficients $\tilde{f}$, $\tilde{g}$ and $\widetilde{H}$ belong to the ideal $(\tilde{y},\tilde{z})$. This is equivalent to saying that the analytic coefficients F, G and H of the original vector field χ are contained in the ideal

$$J=(y+g(x),z+f(x,-g(x)).$$

Since $\{y=0\}$ is not a divisor component and χ is nondegenerate, the ideal J is necessarily the defining ideal of an irreducible 1-dimensional component of the germ $\mathrm{Ze}(\chi)_p$ (the analytic set of zeros of χ).

In other words, the prime ideal J is an element of the irreducible primary decomposition of $I:=\mathrm{rad}(F,G,H)$. Therefore, it follows from Lemma A.1 that the functions $g(x)$ and $f(x,-g(x))$ are necessarily analytic.

Now, we can decompose the map $(f,g,1,1,1)$ in a unique way as

$$(f,g,1,1,1) = (\tilde{f},0,1,1,1)\circ(f_1,g_1,1,1,1),$$

where $f_1(x):=f(x,-g(x))$, $g_1(x):=g(x)$ and

$$\tilde{f} = \tilde{f}_0(x)+...+\tilde{f}_{b-1}(x)y^{b-1} \in \mathbf{R}[[x,y]]$$

is a conveniently chosen formal map. Since $(f_1,g_1,1,1,1)\cdot\Omega$ is analytic, we conclude as above that $(\tilde{f},0,1,1,1)$ is also analytic. This completes the proof of the lemma. □

Proof of Proposition 4.22. We define the stabilization map $(f,g,1,1,1)\in\widehat{\mathrm{G}}^i$ as the composition

$$(f,g,1,1,1) := (f_2,g_2,1,1,1)\circ(f_1,0,1,1,1),$$

where $(f_1,0,1,1,1)$ is the edge-stabilization map given by Lemma 4.23 and $(f_2,g_2,1,1,1)$ is the face stabilization map given by Proposition 4.24. □

We denote by $\mathrm{St}\,\Omega$ the *stabilized* Newton data defined by the above construction. The transition from Ω to $\mathrm{St}\,\Omega$ will be called the *stabilization* of the Newton data.

We remark that the notion of stable Newton data at a point p is *independent* of the choice of the local generator for the line field from which this data is defined (see Lemma 3.4).

For this reason, and for notational simplicity, we often say that an adapted local chart $(U,(x,y,z))$ at p is *stable* (respectivley, *edge-stable*) whenever the corresponding Newton data $\Omega=((x,y,z),\iota_p,\Theta)$ is stable (respectively, *edge-stable*), where Θ is defined by fixing some arbitrary choice of local generator for the line field at p.

Remark 4.25. Given a *formal* Newton data Ω at a point $p\in\mathrm{NElem}(\mathbf{M})$, it is easy to see that the condition $\Delta_1>0$ immediately implies that the *formal curve* $\{x=z=0\}$ lies entirely in the set $\mathrm{NElem}(\mathbf{M})$.

Note also that the condition of nondegeneracy for the local generator χ of the line field guarantees that $\mathrm{NElem}(\mathbf{M})\cap\{z=0\}$ is an analytic set of dimension at most equal to 1, and therefore $\{x=z=0\}$ is necessarily an *analytic* curve.

However, these conditions *do not* imply that the formal coordinates (x,y,z) are analytic. This is the reason why we needed extra arguments to prove the analyticity of the stabilization map at the end of the proof of Lemma 4.24.

4.8. Newton invariant and local resolution of singularities

Let $(\mathbf{M},\mathrm{Ax})$ be a controlled singularly foliated manifold, and let $p\in\mathfrak{D}\cap A$ be a divisor point belonging to $\mathrm{NElem}(\mathbf{M})$.

In Proposition 4.22, we have proved that there always exists an (analytic) adapted local chart $(U,(x,y,z))$ for $(\mathbf{M},\mathrm{Ax})$ at p such that the associated Newton data Ω is stable.

The *Newton invariant* for $(\mathbf{M},\mathrm{Ax})$ at p is the vector of natural numbers

$$\mathrm{inv}(\mathbf{M},\mathrm{Ax},p)=\mathrm{inv}(\Omega)\in\mathbf{N}^6,$$

where $\mathrm{inv}(\Omega)$ is given by Definition 4.6.

Let $\mathcal{N}$ be the Newton polyhedron associated with Ω. The *weight-vector* for $(\mathbf{M},\mathrm{Ax})$ at p is the nonzero vector $\boldsymbol{\omega}\in\mathbf{N}^3$ such that $\gcd(\omega_1,\omega_2,\omega_3)=1$ and

$$\mathcal{F}=\mathcal{N}\cap\{\mathbf{v}\in\mathbf{R}^3:\langle\boldsymbol{\omega},\mathbf{v}\rangle=\mu\}\quad\text{for some } \mu\in\mathbf{Z},$$

where $\mathcal{F}$ is the main face of $\mathcal{N}$. The integer μ in the formula is called the *face order* for $(\mathbf{M},\mathrm{Ax})$ at p.

Remark 4.26. If Ω belongs to the class $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$, then we can explicitly compute that $\boldsymbol{\omega}=k\,\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is defined by

$$\boldsymbol{\alpha}=\begin{cases}(1,0,\Delta_1), & \text{if } C=0,\\ (0,1,\Delta_2), & \text{if } C=\infty,\\ (1,C,C\Delta_2), & \text{if } 0<C<\infty,\end{cases}$$

and $k\in\mathbf{N}$ is the least natural number such that $k\boldsymbol{\alpha}$ belongs to $\mathbf{N}^3$.

The *local blowing-up center* associated with $(\mathbf{M},\mathrm{Ax})$ at p is the submanifold $Y_p\subset U$ defined by

$$Y_p=\begin{cases}\{x=y=z=0\}, & \text{if } \boldsymbol{\omega}=(*,*,*),\\ \{x=z=0\}, & \text{if } \boldsymbol{\omega}=(*,0,*),\\ \{y=z=0\}, & \text{if } \boldsymbol{\omega}=(0,*,*),\end{cases}$$

where the $*$'s denote nonzero natural numbers.

LEMMA 4.27. *The local blowing-up center Y_p lies in* $\mathrm{NElem}(\mathbf{M})$.

Proof. Let us consider the case where $Y_p=\{x=z=0\}$, which corresponds to the second case in the above definition (the reasoning for the third case is analogous). Fixing some local generator χ for the line field at p, and writing $\boldsymbol{\omega}=(\omega_1,0,\omega_3)$ with $\omega_1,\omega_3\in\mathbf{N}_{>0}$, we have

$$\chi=F\frac{\partial}{\partial x}+G\frac{\partial}{\partial y}+H\frac{\partial}{\partial z},$$

where F, G and H are analytic germs with $\boldsymbol{\omega}$-multiplicity given by $\mu-\omega_1$, μ and $\mu-\omega_3$, respectively. Consider now a translation of coordinates $(\tilde{x},\tilde{y},\tilde{z})=(x,y-\eta,z)$, for some constant $\eta\in\mathbf{R}$, and let $\widetilde{\chi}=\widetilde{F}\partial/\partial\tilde{x}+\widetilde{G}\partial/\partial\tilde{y}+\widetilde{H}\partial/\partial\tilde{z}$ be the resulting local generator of the line field. Since $\omega_2=0$, it is obvious that the germs $\widetilde{F}$, $\widetilde{G}$ and $\widetilde{H}$ have $\boldsymbol{\omega}$-multiplicity at least $\mu-\omega_1$, μ and $\mu-\omega_3$, respectively. Therefore, the point $\tilde{p}\in Y_p$, which is the center of the new coordinates $(\tilde{x},\tilde{y},\tilde{z})$, also belongs to $\mathrm{NElem}(\mathbf{M})$. □

PROPOSITION 4.28. *Let $(U,(x,y,z))$ and $(\widetilde{U},(\tilde{x},\tilde{y},\tilde{z}))$ be two stable adapted local charts at p. Then, the corresponding numbers*

$$\mathrm{inv}(\mathbf{M},\mathrm{Ax},p), \quad \boldsymbol{\omega} \quad \textit{and} \quad \mu,$$

associated with these two charts, are equal. Moreover, the respective local blowing-up centers Y_p and $\widetilde{Y}_p$ coincide on $U\cap\widetilde{U}$. Finally, the transition map

$$\phi(x,y,z)=(\tilde{x},\tilde{y},\tilde{z})$$

preservers the $\boldsymbol{\omega}$-quasihomogeneous structure on $\mathbf{R}^3$.

Proof. The first part of the statement follows from Proposition 4.20.

In order to prove the second part, it suffices to remark that the transition map ϕ has the form

$$\tilde{x}=x\,u, \quad \tilde{y}=g(x)+y\,v \quad \text{and} \quad \tilde{z}=f(x,y)+z\,w,$$

and the map (f,g,u,v,w) is a member of the subgroup $\widehat{\mathrm{G}}^i_{\Delta,C}$ (by the proof of Proposition 4.20). Using the explicit definition of this subgroup and Remark 4.26, we immediately conclude that ϕ preservers the $\boldsymbol{\omega}$-quasihomogeneous structure on $\mathbf{R}^3$. □

Let Ω be a stable Newton data for $(\mathbf{M},\mathrm{Ax})$ at p, associated with an adapted local chart $(U,(x,y,z))$.

The *local blowing-up* for $(\mathbf{M},\mathrm{Ax})$ at p is the $\boldsymbol{\omega}$-weighted blowing-up of

$$\Phi:\widetilde{\mathbf{M}}\longrightarrow\mathbf{M}\cap U$$

with center on Y_p, with respect to the trivialization given by $(U,(x,y,z))$.

The apparent arbitrariness in the choice of $(U,(x,y,z))$ can be removed as follows. Consider two local blowing-ups at p,

$$\Phi_i:\widetilde{\mathbf{M}}_i\longrightarrow\mathbf{M}\cap U_i, \quad i=1,2,$$

associated with distinct stable adapted charts $(U_i,(x_i,y_i,z_i))$, $i=1,2$.

Using Proposition 4.28, it follows that (up to restricting each U_i to some smaller neighborhood of p), there exists an isomorphism $\Psi:\widetilde{\mathbf{M}}_1\to\widetilde{\mathbf{M}}_2$ (in the obvious sense of isomorphism between singularly foliated manifolds) which makes the following diagram commutative:

$$\begin{array}{ccc} \widetilde{\mathbf{M}}_1 & \xrightarrow{\Psi} & \widetilde{\mathbf{M}}_2 \\ {\scriptstyle\Phi_1}\downarrow & & \downarrow{\scriptstyle\Phi_2} \\ \mathbf{M}\cap U & \xrightarrow{\text{id}} & \mathbf{M}\cap U, \end{array}$$

where id is the identity map and $U=U_1\cap U_2$.

The main theorem of this section can now be stated as follows.

THEOREM 4.29. (Local resolution of singularities) *Let* $(\mathbf{M}, \mathrm{Ax})$ *be a controlled singularly foliated manifold and let* $p\in\mathfrak{D}\cap A$ *be a divisor point in* $\mathrm{NElem}(\mathbf{M})$*. Consider the* local blowing-up *for* $(\mathbf{M}, \mathrm{Ax})$ *at* p,

$$\Phi\colon \widetilde{\mathbf{M}} \longrightarrow \mathbf{M}\cap U,$$

with respect to some stable adapted chart $(U,(x,y,z))$*. Then, there exists an axis*

$$\widetilde{\mathrm{Ax}} = (\tilde{A}, \tilde{\mathfrak{z}})$$

for $\widetilde{\mathbf{M}}$ *such that each point* $\tilde{p}\in\Phi^{-1}(p)\cap\tilde{A}$ *belonging to* $\mathrm{NElem}(\widetilde{\mathbf{M}})$ *is such that*

$$\mathrm{inv}(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}}, \tilde{p}\,) <_{\mathrm{lex}} \mathrm{inv}(\mathbf{M}, \mathrm{Ax}, p).$$

The proof of Theorem 4.29 will be given in §4.20.

4.9. Directional blowing-ups

Let us fix a stable adapted chart $(U,(x,y,z))$ at a divisor point $p\in\mathrm{NElem}(\mathbf{M})$ and consider the corresponding $\boldsymbol{\omega}$-weighted local blowing-up

$$\Phi\colon \widetilde{\mathbf{M}} \longrightarrow \mathbf{M}\cap U,$$

as defined in the previous subsection.

Theorem 4.29 will be proved by studying the effect of this blowing-up in the x-, y-, and z-directional charts (see §2.4).

Let $\Omega=((x,y,z),\iota_p,\Theta)$ be the Newton data associated with the adapted local chart $(U,(x,y,z))$ (for some choice of local generator χ of L). It will be convenient to look at the directional blowing-ups as transformations on the Newton map Θ. For this, we consider the following matrices.

(i) x-directional transformation matrices:

$$B_x = \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_x = \begin{pmatrix} 1/\omega_1 & 0 & 0 \\ -\omega_2/\omega_1 & 1 & 0 \\ -\omega_3/\omega_1 & 0 & 1 \end{pmatrix}.$$

(ii) y-directional transformation matrices:

$$B_y = \begin{pmatrix} 1 & 0 & 0 \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_y = \begin{pmatrix} 1 & -\omega_1/\omega_2 & 0 \\ 0 & -1/\omega_2 & 0 \\ 0 & -\omega_3/\omega_2 & 1 \end{pmatrix}.$$

(iii) z-directional transformation matrices:

$$B_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \omega_1 & \omega_2 & \omega_3 \end{pmatrix} \quad \text{and} \quad M_z = \begin{pmatrix} 1 & 0 & -\omega_1/\omega_3 \\ 0 & 1 & -\omega_2/\omega_3 \\ 0 & 0 & 1/\omega_3 \end{pmatrix}.$$

We consider also the permutation matrices

$$I = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The *directional blowing-ups* of Θ are the Newton maps $\mathrm{Bl}_x\Theta$, $\mathrm{Bl}_y\Theta$ and $\mathrm{Bl}_z\Theta$ given respectively by

$$\begin{aligned} \mathrm{Bl}_x\Theta(B_x\mathbf{v}-\mu\mathbf{e}_1) &= \varepsilon^{\upsilon_1} M_x\Theta(\mathbf{v}) && (\text{defined for } \omega_1>0), \\ \mathrm{Bl}_y\Theta(IB_y\mathbf{v}-\mu\mathbf{e}_1) &= \varepsilon^{\upsilon_2} IM_y\Theta(\mathbf{v}) && (\text{defined for } \omega_2>0), \\ \mathrm{Bl}_z\Theta(JB_z\mathbf{v}-\mu\mathbf{e}_1) &= \varepsilon^{\upsilon_3} JM_z\Theta(\mathbf{v}) && (\text{defined for } \omega_3>0), \end{aligned}$$

where $\varepsilon\in\{-1,1\}$ and $\mathbf{v}\in\mathbf{Z}^3$. The directional blowing-ups of the Newton data Ω are defined as follows:

(i) *x-directional blowing-up*: $\mathrm{Bl}_x\Omega=((\bar{x},\bar{y},\bar{z}),\bar{\iota},\mathrm{Bl}_x\Theta)$,

(ii) *y-directional blowing-up*: $\mathrm{Bl}_y\Omega=((\bar{x},\bar{y},\bar{z}),\bar{\iota},\mathrm{Bl}_y\Theta)$,

(iii) *z-directional blowing-up*: $\mathrm{Bl}_z\Omega=((\bar{x},\bar{y},\bar{z}),\bar{\iota},\mathrm{Bl}_z\Theta)$,

where $\bar{\iota}=\iota_p\cup[n]$ (with $n=1+\max\{i:i\in\Upsilon\}$ for $\Upsilon\neq\varnothing$ and $n=1$ for $\Upsilon=\varnothing$) and $(\bar{x},\bar{y},\bar{z})$ is a chart respectively defined by the following singular changes of coordinates

$$\begin{aligned} &x\text{-directional blowing-up:} && x=\varepsilon\bar{x}^{\omega_1}, && y=\bar{x}^{\omega_2}\bar{y} && \text{and} && z=\bar{x}^{\omega_3}\bar{z}, \\ &y\text{-directional blowing-up:} && x=\bar{x}^{\omega_1}\bar{y}, && y=\varepsilon\bar{x}^{\omega_2} && \text{and} && z=\bar{x}^{\omega_3}\bar{z}, \\ &z\text{-directional blowing-up:} && x=\bar{x}^{\omega_1}\bar{y}, && y=\bar{x}^{\omega_2}\bar{z} && \text{and} && z=\varepsilon\bar{x}^{\omega_3}, \end{aligned}$$

followed by a division by $\bar{x}^{\mu}$. Notice that there exists a cyclic permutation of coordinates in the y- and z-directional blowing-ups (corresponding to the permutation matrices I and J).

In Figure 23, we give an illustration of the *movement* of the Newton polyhedron which is caused by these maps.

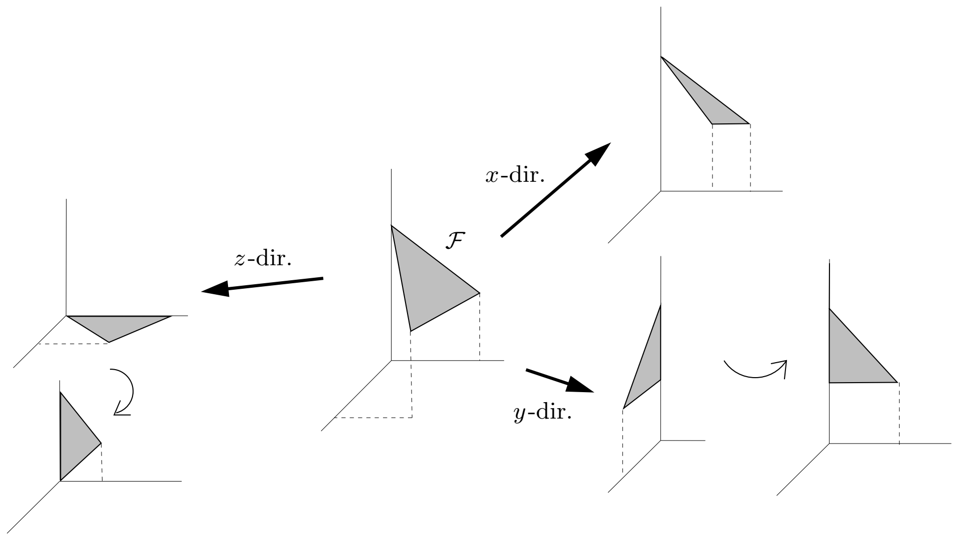


Figure 23. The directional blowing-ups.

4.10. x-directional blowing-up

Let $(\mathbf{M}, \mathrm{Ax})$ be a controlled singularly foliated manifold and let $p \in \mathfrak{D} \cap A$ be a divisor point in $\mathrm{NElem}(\mathbf{M})$.

Let $\Omega \in \mathrm{New}^{\boldsymbol{i},\mathbf{m}}_{\Delta,C}$ be a stable Newton data at p, with coordinates (x, y, z). In this subsection, we assume that the corresponding weight-vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is such that $\omega_1 > 0$. Geometrically, this assumption means that the local blowing-up center is distinct from the axis $\{y = z = 0\}$.

The *x-directional translation group* is defined by

$$\mathcal{G}^{\mathrm{tr}}_x := \begin{cases} \mathcal{G}^1_{(0,0),0}, & \text{if } \Delta_1 = 0, \\ \mathcal{G}^2_{(0,0),0}, & \text{if } \Delta_1 > 0. \end{cases}$$

In other words, if $\Delta_1 > 0$ then $\mathcal{G}^{\mathrm{tr}}_x$ is the group of all translations

$$\tilde{z} = z + \xi$$

for some real constant $\xi \in \mathbf{R}$. If $\Delta_1 = 0$ then $\mathcal{G}^{\mathrm{tr}}_x$ is the group of all translations

$$\tilde{y} = y + \eta \quad \text{and} \quad \tilde{z} = z + \xi$$

for some real constants $\eta, \xi \in \mathbf{R}$. In this subsection, we shall prove the following result.

Proposition 4.30. *Given a stable Newton data* Ω, *let* $\bar{\Omega}=\mathrm{Bl}_x\Omega$ *be its* x*-directional blowing-up. Then, for each* $(\xi,\eta)\in\mathcal{G}_x^{\mathrm{tr}}$, *either the translated Newton data*

$$\bar{\Omega}_{\xi,\eta} := (\xi,\eta)\cdot\bar{\Omega}$$

is centered at an elementary point $\tilde{p}\in\mathrm{Elem}(\widetilde{\mathbf{M}})$ *or*

$$\mathrm{inv}(\widetilde{\Omega}_{\xi,\eta}) <_{\mathrm{lex}} \mathrm{inv}(\Omega),$$

where $\widetilde{\Omega}_{\xi,\eta}=\mathrm{St}\,\bar{\Omega}_{\xi,\eta}$ *is the stabilization of* $\bar{\Omega}_{\xi,\eta}$.

The proof of Proposition 4.30 will be given at the end of §4.15 and will depend on several lemmas. First of all, let us look at the effect of Bl_x on the main face $\mathcal{F}$.

Lemma 4.31. *Let* $\bar{\Omega}:=\mathrm{Bl}_x\Omega$ *be the* x*-directional blowing-up of* Ω. *Then, there exists a bijective correspondence*

$$\mathrm{supp}(\Omega)\cap\mathcal{F}\longrightarrow\mathrm{supp}(\bar{\Omega})\cap\{0\}\times\mathbf{Z}^2,$$
$$\mathbf{v}=(v_1,v_2,v_3)\longmapsto\pi_x(\mathbf{v})=(0,v_2,v_3),$$

such that the corresponding Newton maps $\bar{\Theta}$ *and* Θ *satisfy* $\bar{\Theta}[\pi_x(\mathbf{v})]=M_x\Theta[\mathbf{v}]$.

Proof. This is immediate from the definition of Bl_x. □

The matrices M_x and B_x which appear in the definition of the x-directional blowing-up Bl_x can be written as products $B_x=B_x^2\,B_x^1$ and $M_x=M_x^2\,M_x^1$, where

$$B_x^1=\begin{pmatrix}\omega_1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\quad\text{and}\quad M_x^1=\begin{pmatrix}1/\omega_1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

and

$$B_x^2=\begin{pmatrix}1 & \omega_2 & \omega_3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\quad\text{and}\quad M_x^2=\begin{pmatrix}1 & 0 & 0\\ -\omega_2 & 1 & 0\\ -\omega_3 & 0 & 1\end{pmatrix}.$$

Therefore, the map Bl_x can be written as the composition $\mathrm{Bl}_x=\mathrm{Bl}_x^2\circ\mathrm{Bl}_x^1$, where

$$\mathrm{Bl}_x^1\Theta(B_x^1\mathbf{v})=\varepsilon^{v_1}M_x^1\Theta(\mathbf{v})\quad\text{and}\quad\mathrm{Bl}_x^2\Theta(M_x^2\mathbf{v}-\mu\mathbf{e}_1)=M_x^2\Theta(\mathbf{v}).$$

Notice that the maps Bl_x^1 and Bl_x^2 correspond, respectively, to the singular changes of coordinates

$$x=\varepsilon\,\bar{x}^{\omega_1},\quad y=\bar{y}\quad\text{and}\quad z=\bar{z},$$

and

$$x=\bar{x},\quad y=\bar{x}^{\omega_2}\bar{y}\quad\text{and}\quad z=\bar{x}^{\omega_3}\bar{z},$$

followed by a division by $\bar{x}^{\mu}$ (for $\varepsilon\in\{-1,1\}$).

4.11. The effect of a ramification

The expressions given in the previous subsection show that an x-directional blowing-up can always be written as a composition of a *ramification* $x=\varepsilon\bar{x}^{\omega_1}$ followed by a sequence of *homogeneous blowing-ups*.

Example 4.32. For $\boldsymbol{\omega}=(2,3,2)$, the x-directional blowing-up can be decomposed as the ramification $(x,y,z)=(\bar{x}^2,\bar{y},\bar{z})$ followed by the sequence of blowing-ups

$$(\bar{x},\bar{y},\bar{z})=(x_1,x_1y_1,x_1z_1),\quad (x_1,y_1,z_1)=(x_2,x_2y_2,x_2z_2)\ \text{ and }\ (x_2,y_2,z_2)=(x_3,x_3y_3,z_3).$$

Notice that the last blowing-up has its center on the curve $Y=\{x_2=y_2=0\}$.

The example from §1.4 shows that the use of ramifications is unavoidable in order to obtain a complete resolution of singularities for vector fields.

Our present goal is to study the effect of a ramification on the Newton data. If Ω belongs to the class $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$, it is obvious that $\bar{\Omega}=\mathrm{Bl}^1_x\Omega$ belongs to the class $\mathrm{New}^{i,\mathbf{m}}_{\bar{\Delta},\bar{C}}$, where

$$\bar{\Delta}=(\omega_1\Delta_1,\Delta_2)\quad\text{and}\quad \bar{C}=\omega_1C.$$

Moreover, we have the following result.

LEMMA 4.33. *For each map* $(f,g)\in\mathcal{G}^i_{\Delta,C}$ *there exists a unique map* $(\bar{f},\bar{g})\in\mathcal{G}^i_{\bar{\Delta},\bar{C}}$ *which makes the following diagram commutative*:

$$\begin{array}{ccc} \Omega & \xrightarrow{(f,g)\cdot} & (f,g)\cdot\Omega \\ {\scriptstyle \mathrm{Bl}^1_x}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Bl}^1_x} \\ \bar{\Omega} & \xrightarrow{(\bar{f},\bar{g})\cdot} & (\bar{f},\bar{g})\cdot\bar{\Omega}. \end{array}$$

Proof. We can explicitly define $\bar{f}(x,y)=f(\varepsilon x^{\omega_1},y)$ and $\bar{g}(x)=g(\varepsilon x^{\omega_1})$. □

The next lemma implies that the stability property is preserved by the transformation Bl^1_x.

LEMMA 4.34. *Suppose that* Ω *is stable. Then* $\bar{\Omega}=\mathrm{Bl}^1_x\Omega$ *is also a stable Newton data.*

Proof. First of all, let us prove that $\bar{\Omega}$ is edge-stable. For this, assume by contradiction that there exists a map $(\bar{f},0)\in\mathcal{G}^i_{\bar{\Delta}}$ such that

$$(\bar{f},0)\cdot\bar{\Omega}\notin\mathrm{New}^{i,\mathbf{m}}_{\bar{\Delta},\bar{C}}. \tag{28}$$

We must treat the following two cases:

(a) $\Delta_1=0$;

(b) $\Delta_1>0$.

In case (a), it follows that $\bar{f}$ is a function of y only. Therefore, Lemma 4.33 implies that there exists a map $(f,0)\in\mathcal{G}_\Delta$ such that $(f,0)\cdot\Omega\notin\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$. This contradicts the fact that Ω is stable.

In case (b), note that $C=0$. Write $\Delta_1=p_1/q_1$ and $\Delta_2=p_2/q_2$, with $\gcd(p_i,q_i)=1$ $(i=1,2)$. Then, by Remark 4.26, the weight-vector is given by $\boldsymbol{\omega}=(q_1,0,p_1)$. Let us split the discussion into two subcases:

(b.1) $q_1=1$;

(b.2) $q_1\geqslant 2$.

In case $(b.1)$, the map Bl^1_x is just the identity map and we are done.

In case $(b.2)$, we notice that the support of the Newton data Ω is such that

$$\mathrm{supp}(\Omega|_{\mathfrak{e}})\subset\mathbf{m}+nt\left(\frac{p_1}{q_1},\frac{p_2}{q_2},-1\right)\quad\text{for } n\in\mathbf{N},$$

where t is the least common multiple of q_1 and q_2. In particular, $t\geqslant 2$. Therefore, we get

$$\mathrm{supp}(\Omega|_{\mathfrak{e}})\cap\{\mathbf{v}\in\mathbf{Z}^3: v_3=m_3-1\}=\varnothing,$$

and the same property holds for $\mathrm{supp}(\bar{\Omega}|_{\bar{\mathfrak{e}}})$. Remark 4.14 now implies that $\bar{\Omega}$ is edge-stable.

Now suppose, by contradiction, that there exists a map $(\hat{f},\hat{g})\in\mathcal{G}^i_{\bar{\Delta},\bar{C}}\setminus\mathcal{G}^i_{\bar{\Delta}}$ such that

$$(\hat{f},\hat{g})\cdot\bar{\Omega}\notin\mathrm{New}^{i,\mathbf{m}}_{\bar{\Delta},\bar{C}}.$$

Let us look at the action of the inverse map $(\hat{f},\hat{g})^{-1}$ on the restricted Newton data $\bar{\Omega}|_{\mathfrak{e}}$. Notice that

$$(\hat{f},\hat{g})^{-1}\cdot\bar{\Omega}|_{\mathfrak{e}}$$

is precisely the restriction of $\bar{\Omega}$ to the main face $\bar{\mathcal{F}}$.

Looking at the points on the support of $\bar{\Omega}|_{\bar{\mathcal{F}}}$ and using Lemma 4.12, we can easily see that $(\hat{f},\hat{g})^{-1}$ should necessarily be of the form

$$(\hat{f},\hat{g})^{-1}=(f(x^{\omega_1},y),g(x^{\omega_1}))$$

for some $f\in\mathbf{R}[x,y]$ and $g\in\mathbf{R}[x]$. Using Lemma 4.33, this implies that Ω is not stable, yielding a contradiction. □

4.12. The x-directional projected group and the group $\mathcal{G}_{\Delta,C}$

Let us now introduce another subgroup of $\mathcal{G}$, which will be mainly used for studying the effect of the translations on the x-directional blowing-up $\mathrm{Bl}_x\Omega$.

The *x-directional projected group* adapted to $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$ is defined by

$$\mathrm{Pr}\mathcal{G}_x := \begin{cases} \mathcal{G}^1_{(0,\Delta_2),\infty}, & \text{if } \Delta_1 > 0, \\ \mathcal{G}^1_{(0,\Delta_2),0}, & \text{if } \Delta_1 = 0. \end{cases}$$

In other words, if $\Delta_1>0$ then each $(f,g)\in\mathrm{Pr}\mathcal{G}_x$ has the form

$$g=0 \quad \text{and} \quad f=\xi y^{\Delta_2},$$

where the constant $\xi\in\mathbf{R}$ necessarily vanishes if $\Delta_2\notin\mathbf{N}$. If $\Delta_1=0$ then each $(f,g)\in\mathrm{Pr}\mathcal{G}_x$ has the form

$$g(x)=\eta \quad \text{and} \quad f(x,y)=a_0+a_1y+...+a_by^b,$$

where $b:=\lfloor\Delta_2\rfloor$ and $\eta, a_0, ..., a_b\in\mathbf{R}$ are constants.

LEMMA 4.35. *Suppose that $\omega_1=1$. Then, given a map $(f,g)\in\mathrm{Pr}\mathcal{G}_x$, there exists a unique map $(f_{\boldsymbol\omega},g_{\boldsymbol\omega})\in\mathcal{G}^1_{\Delta,C}$ which makes the following diagram commutative*:

$$\begin{array}{ccc} \mathrm{New} & \xrightarrow{(f_{\boldsymbol\omega},g_{\boldsymbol\omega})\cdot} & \mathrm{New} \\ {\scriptstyle \mathrm{Bl}_x}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Bl}_x} \\ \mathrm{New} & \xrightarrow{(f,g)\cdot} & \mathrm{New}. \end{array}$$

Proof. Suppose first that $\Delta_1=0$ and that $(f,g)=(\xi y^k,\eta)$ for some constants $\eta,\xi\in\mathbf{R}$ and $0\leqslant k\leqslant b$. The change of coordinates which is associated with (f,g) is

$$\tilde{y}=y+\eta \quad \text{and} \quad \tilde{z}=z+\xi y^k.$$

Now, if we apply the blowing-up map $(X,Y,Z)=(x,x^{\omega_2}y,x^{\omega_3}z)$ on both sides of these equalities and simplify common powers of X, we get

$$\widetilde{Y}=Y+\eta X^{\omega_2} \quad \text{and} \quad \widetilde{Z}=Z+\xi X^{\omega_3-k\omega_2}Y$$

(notice that $\omega_3\geqslant k\omega_2$). Therefore it suffices to define

$$(f_{\boldsymbol\omega},g_{\boldsymbol\omega}):=(\xi X^{\omega_3-k\omega_2}Y,\eta X^{\omega_2}).$$

By the same reasoning, we obtain $f_{\boldsymbol\omega}$ from an arbitrary polynomial f, by making the formal replacement

$$y^k\longmapsto X^{\omega_3-k\omega_2}Y^k,$$

and we obtain $g_{\boldsymbol\omega}$ from g by making the formal replacement

$$1\longmapsto X^{\omega_2}.$$

Suppose now that $\Delta_1>0$. Here, the blowing-up map is given by $(X,Y,Z)=(x,y,x^{\omega_3}z)$ and an element $(f,g)\in\mathrm{Pr}\mathcal{G}_x$ corresponds to a change of coordinates of the form

$$\tilde{z}=z+\xi y^{\Delta_2},$$

where $\xi=0$ if $\Delta_2\notin\mathbf{N}$. The corresponding change of coordinates in the (X,Y,Z) variables is given by

$$\widetilde{Z}=Z+\xi X^{\omega_3}Y^{\Delta_2}$$

and, therefore, it suffices to get $(f_{\boldsymbol{\omega}},g_{\boldsymbol{\omega}})=(\xi X^{\omega_3}Y^{\Delta_2},0)$. □

Remark 4.36. If $(f,g)\in\mathrm{Pr}\mathcal{G}_x$ is such that $g=0$, then the map $(f_{\boldsymbol{\omega}},g_{\boldsymbol{\omega}})\in\mathcal{G}_{\Delta,C}$ given by Lemma 4.35 is such that $g_{\boldsymbol{\omega}}=0$. In particular, for $g=0$, the map $(f_{\boldsymbol{\omega}},g_{\boldsymbol{\omega}})$ belongs to the subgroup $\mathcal{G}^2_{\Delta,C}$.

4.13. x-directional blowing-up (case $\boldsymbol{\Delta_1=0}$)

In this subsection, we shall study the x-directional blowing-up of a stable Newton data Ω in the case where $\Delta_1=0$.

Our goal is to prove that the main invariant inv strictly decreases at each nonelementary point $\tilde{p}\in\Phi^{-1}(p)\cap\mathrm{NElem}(\widetilde{\mathbf{M}})$ which lies in the domain of the x-directional blowing-up.

The following example shows that the height m_3 of the main vertex $\mathbf{m}$ can increase after an x-directional blowing-up. This is the main reason for introducing the concept of *virtual height* $\mathfrak{h}$ in §4.2.

Example 4.37. Consider the vector field $\chi=(y^2+xz^3)\,\partial/\partial y+z^3\,\partial/\partial z$. The associated Newton polyhedron is depicted in Figure 24 (left). The primary invariant is given by $(\mathfrak{h},m_2+1,m_3)=(2,1,2)$. The x-directional blowing-up with weight $\boldsymbol{\omega}=(1,2,1)$ results in the vector field

$$\widetilde{\chi}=(y^2+z^3)\frac{\partial}{\partial y}+z^3\frac{\partial}{\partial z}$$

(see Figure 24 (right)). Note that $\widetilde{m}_3=3>2=m_3$. However, the primary invariant associated with $\widetilde{\chi}$ is given by $(\tilde{\mathfrak{h}},\widetilde{m}_2+1,\widetilde{m}_3)=(2,0,3)$, which is lexicographically smaller than $(2,1,2)$.

Up to a preliminary transformation of type Bl^1_x (see the previous subsection), we may assume that the weight-vector $\boldsymbol{\omega}$ is such that $\omega_1=1$.

The following simple lemma will be the key to understanding the behavior of the virtual height under blowing-up and to prove Proposition 4.30.

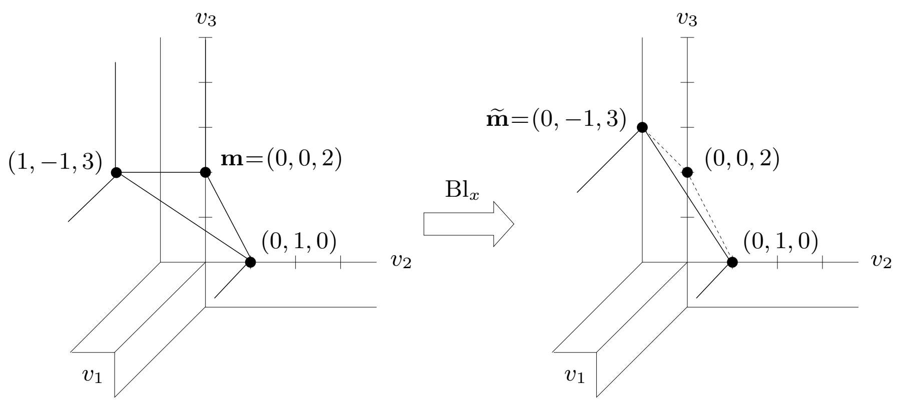


Figure 24. The height of the main vertex increases after an x-directional blowing-up.

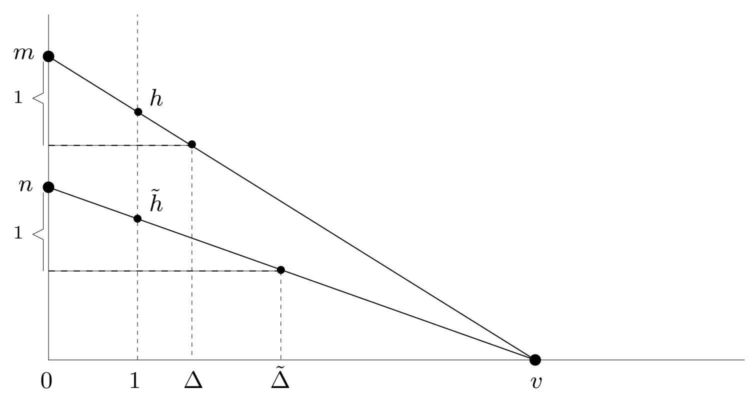


Figure 25. Illustration of Lemma 4.38.

LEMMA 4.38. *Let us consider three rational points in* $\mathbf{Q}^2$, *with coordinates* $(0,m)$, $(0,n)$ *and* $(v,0)$ *such that* $v\geqslant 2$ *and* $1\leqslant n<m$. *Let* $\Delta:=v/m$ *and* $\tilde{\Delta}:=v/n$ *be the slope of the lines* $\overline{\mathbf{m},\mathbf{v}}$ *and* $\overline{\mathbf{n},\mathbf{v}}$, *respectively. Consider the rational numbers*

$$h:=m-\frac{1}{\Delta} \quad \textit{and} \quad \tilde{h}:=n-\frac{1}{\tilde{\Delta}}.$$

Then, one necessarily has $\tilde{h}<h$. *Moreover, one the following situations occurs*:

(i) $\tilde{\Delta}<1$, *or*

(ii) $h\geqslant n-1$.

Proof. Figure 25 illustrates the statement of the lemma. The assertion that $\tilde{h}<h$ is obvious. Suppose now that $v/n=\tilde{\Delta}\geqslant 1$. Then, $\Delta\geqslant n/m$, and if we write $m=n+s$ (for some $s>0$), we get

$$h=m-\frac{1}{\Delta}\geqslant\frac{n^2+n(s-1)-s}{n}.$$

Therefore, the quantity $h-n+1\geqslant s(1-1/n)$ is always positive or zero. $\square$

LEMMA 4.39. *Suppose that* $\Delta_1=0$. *Suppose further that the x-directional blowing-up* $\mathrm{Bl}_x\Omega$ *is centered at a nonelementary point* $\tilde{p}\in\mathrm{NElem}(\widetilde{\mathbf{M}})$. *If the main vertex* $\overline{\mathbf{m}}$ *of* $\mathrm{Bl}_x\Omega$ *is such that* $\overline{\mathbf{m}}\neq\mathbf{m}$, *then one necessarily has* $\mathrm{inv}_1(\widetilde{\Omega})<_{\mathrm{lex}}\mathrm{inv}_1(\Omega)$, *where* $\widetilde{\Omega}=\mathrm{StBl}_x\Omega$ *is the stabilization of* $\mathrm{Bl}_x\Omega$.

Proof. First of all, let us suppose that the vertex $\overline{\mathbf{m}}=(0,\overline{m}_2,\overline{m}_3)$ is such that

$$(\overline{m}_2,\overline{m}_3)<_{\mathrm{lex}}(m_2,m_3).$$

Under this hypothesis, we split the discussion into three cases:

(a) $\overline{m}_2=m_2=0$ and $\overline{m}_3<m_3$;

(b) $\overline{m}_2=m_2=-1$ and $\overline{m}_3<m_3$;

(c) $\overline{m}_2=-1$ and $m_2=0$.

In case (a), it is obvious that $\overline{\mathbf{m}}$ is also the main vertex of $\widetilde{\Omega}$, because no regular-nilpotent transition can occur in the passage from $\mathrm{Bl}_x\Omega$ to $\widetilde{\Omega}$. Hence,

$$\mathrm{inv}_1(\widetilde{\Omega})=(\tilde{\mathfrak{h}},1,\widetilde{m}_3)<_{\mathrm{lex}}(\mathfrak{h},1,m_3)=\mathrm{inv}_1(\widetilde{\Omega}),$$

because $\tilde{\mathfrak{h}}=\overline{m}_3<m_3=\mathfrak{h}$.

To study cases (b) and (c), we consider the main vertex $\widetilde{\mathbf{m}}=(0,\widetilde{m}_2,\widetilde{m}_3)$ of $\widetilde{\Omega}$. It is obvious that either $\widetilde{\Omega}$ is in a regular configuration and $\widetilde{\mathbf{m}}=\overline{\mathbf{m}}$, or $\widetilde{\Omega}$ is in a nilpotent configuration and there occurs a regular-nilpotent transition in the passage from $\mathrm{Bl}_x\Omega$ to $\widetilde{\Omega}$. Notice that, in both cases, we have $\widetilde{m}_3\leqslant\overline{m}_3$.

To study case (b), we observe that the main edge $\mathfrak{e}$ of Ω has the form $\mathfrak{e}=\overline{\mathbf{m},\mathbf{v}}$ for some point $\mathbf{v}=(0,v_2,v_3)\in\mathrm{supp}(\Omega)$ such that $v_2\geqslant1$ and $v_3<m_3$. Two cases can occur:

(b.i) $\overline{m}_3\leqslant v_3$;

(b.ii) $\overline{m}_3>v_3$.

In case (b.i), we get

$$\mathrm{inv}_1(\widetilde{\Omega})=(\tilde{\mathfrak{h}},\widetilde{m}_2+1,\widetilde{m}_3)<_{\mathrm{lex}}(\mathfrak{h},m_2+1,m_3)=\mathrm{inv}_1(\widetilde{\Omega}),$$

because $\tilde{\mathfrak{h}}\leqslant\widetilde{m}_3\leqslant v_3<\mathfrak{h}$.

To treat case (b.ii), we define the numbers

$$m:=m_3-v_3,\quad n:=\overline{m}_3-v_3\quad\text{and}\quad v:=v_2+1.$$

By construction, we know that $1\leqslant n<m$ and $v\geqslant2$, and we can apply Lemma 4.38 to the points $(0,m)$, $(0,n)$ and $(v,0)$. If we denote the displacement vector of $\mathrm{Bl}_x\Omega$ by $\bar{\Delta}=(0,\bar{\Delta}_2)$, and the associated virtual height by $\bar{\mathfrak{h}}$, it follows that $\bar{\mathfrak{h}}\leqslant\mathfrak{h}$, and one of the following situations occurs:

(b.ii.1) $\bar{\Delta}_2<1$;

(b.ii.2) $\mathfrak{h}\geqslant\overline{m}_3$.

In case (b.ii.1), we know that $\mathrm{Bl}_x\Omega$ has a stable edge (because $\mathcal{G}^i_\Delta=\{(0,0)\}$). In particular, no regular-nilpotent transition can occur in the passage from $\mathrm{Bl}_x\Omega$ to $\widetilde{\Omega}$. Therefore,

$$\mathrm{inv}_1(\widetilde{\Omega})=(\bar{\mathfrak{h}},0,\overline{m}_3)<_{\mathrm{lex}}(\mathfrak{h},0,m_3)=\mathrm{inv}_1(\Omega).$$

In case (b.ii.2), if no regular-nilpotent transition occurs in the passage from $\mathrm{Bl}_x\Omega$ to $\widetilde{\Omega}$, we obtain the same conclusion by the estimate $\tilde{\mathfrak{h}}\leqslant\overline{m}_3\leqslant\mathfrak{h}$.

On the other hand, if such a regular-nilpotent transition occurs, then $\tilde{\mathfrak{h}}=\widetilde{m}_3<\overline{m}_3\leqslant\mathfrak{h}$ by the definition of nilpotent configurations.

Let us now study case (c). Here, keeping the notation as in the previous case, we consider the following possibilities:

(c.i) $\overline{m}_3\leqslant v_3$;

(c.ii) $\overline{m}_3>v_3$.

Case (c.i) is treated exactly as case (b.i). To study case (c.ii), it suffices to consider the points

$$m:=m_3+\frac{1}{\Delta_2}-v_3,\quad n:=\overline{m}_3-v_3\quad\text{and}\quad v:=v_2+1.$$

Since the vertex $\overline{m}$ has the form $\overline{m}=\pi_x(\mathbf{n})$, for some $\mathbf{n}\in\mathcal{F}\setminus\mathfrak{e}$, it follows that

$$0>\langle\boldsymbol{\omega},\mathbf{m}-\overline{\mathbf{m}}\rangle=\omega_2(-1)+\omega_3(\overline{m}_3-m_3),$$

and therefore (since $\Delta_2=\omega_3/\omega_2$), we have

$$n=\widetilde{m}_3-v_3<m_3+\frac{1}{\Delta_2}-v_3=m.$$

Using the same arguments as in case (b.ii), we conclude that $\mathrm{inv}_1(\widetilde{\Omega})<_{\mathrm{lex}}\mathrm{inv}_1(\Omega)$.

It remains to study the case where $(\overline{m}_2,\overline{m}_3)>_{\mathrm{lex}}(m_2,m_3)$. Here, one necessarily has the conditions

$$\overline{m}_2=0>-1=m_2\quad\text{and}\quad\overline{m}_3<\mathfrak{h},$$

where the second inequality follows from the fact that $\widetilde{\mathbf{m}}=\pi_x(\mathbf{n})$, for some point $\mathbf{n}\in\mathcal{F}$. Therefore, the Newton data $\mathrm{Bl}_x\Omega$ is already in a nilpotent configuration and $\overline{\mathbf{m}}$ is the main vertex of $\widetilde{\Omega}$. These conditions imply that $\tilde{\mathfrak{h}}=\overline{m}_3<\mathfrak{h}$. □

LEMMA 4.40. *Suppose that* $\Delta_1=0$. *Suppose further that the x-directional blowing-up* $\mathrm{Bl}_x\Omega$ *is centered at a nonelementary point* $\tilde{p}\in\mathrm{NElem}(\widetilde{\mathbf{M}})$. *If the main vertex of* $\mathrm{Bl}_x\Omega$ *coincides with that of* Ω, *then*

$$\mathrm{inv}(\widetilde{\Omega})<_{\mathrm{lex}}\mathrm{inv}(\Omega),$$

where $\widetilde{\Omega}=\mathrm{StBl}_x\Omega$ *is the stabilization of* $\mathrm{Bl}_x\Omega$.

Proof. We denote by

$$\bar{\Omega} := (f, 0, 1, 1, 1)\cdot \mathrm{Bl}_x\Omega$$

the analytic edge-preparation which is associated with $\mathrm{Bl}_x\Omega$ (see Lemma 4.23).

If there exists a regular-nilpotent transition in this preparation (see Lemma 4.9), then we are done. In fact, it is clear that the virtual height $\bar{\mathfrak{h}}$ associated with $\bar{\Omega}$ is at most equal to $\mathfrak{h}-1$, because the main vertex of $\bar{\Omega}$ is $\bar{\mathbf{m}}=(0,0,m_3-1)$ and $\bar{\mathfrak{h}}=m_3-1<m_3=\mathfrak{h}$ (a regular-nilpotent transition can only occur if $m_2=-1$ and the vertical displacement vector of $\mathrm{Bl}_x\Omega$ is equal to $\Delta=(0,1)$).

Therefore, let us assume that $\mathbf{m}$ is also the main vertex of $\bar{\Omega}$. Let $\bar{\Delta}=(\bar{\Delta}_1,\bar{\Delta}_2)$ be the main displacement vector which is associated with $\bar{\Omega}$. Then, by definition,

$$\mathrm{inv}_1(\widetilde{\Omega}) = (\bar{\mathfrak{h}}, m_2+1, m_3) \quad \text{and} \quad \mathrm{inv}_2(\widetilde{\Omega}) = (\#\iota_{\tilde{p}}-1, M\bar{\Delta}_1, M\max\{0,\bar{\Delta}_2\}).$$

Since $\#\iota_{\tilde{p}}=\#\iota_p=i$, it clearly suffices to prove the following claim.

Claim. $\bar{\Delta}_1=\Delta_1=0$ and $\bar{\Delta}_2<\Delta_2$.

To prove the claim, suppose by contradiction that either $\bar{\Delta}_1>0$ or $\bar{\Delta}_2\geqslant\Delta_2$. Then, if we write the Taylor series of the map $f(x,y)$ as

$$f(x,y) = \sum_{i+j\geqslant 1} f_{ij}x^i y^j,$$

it follows that the polynomial truncation $f_t:=\sum_{i=0}^b f_{i0}x^i$ (with $b:=\lfloor\Delta_2\rfloor$) is such that the Newton data

$$\Omega_t := (f_t, 0)\cdot\widetilde{\Omega}$$

has a displacement vector $\Delta_t \geqslant_{\mathrm{lex}} \Delta$. Since the map $(f_t,0)$ belongs to the x-directional projected group $\mathrm{Pr}\mathcal{G}_x$, it follows from Lemma 4.35 that there exists a unique map $(f_{\boldsymbol{\omega}},0)\in \mathcal{G}^i_{\Delta,C}$ such that

$$\Omega_t = \mathrm{Bl}_x(f_{\boldsymbol{\omega}},0)\cdot\Omega.$$

We conclude, by Lemma 4.31, that the Newton data $(f_{\boldsymbol{\omega}},0)\cdot\Omega$ does not belong to $\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$. This contradicts the hypothesis that Ω is a stable Newton data. □

PROPOSITION 4.41. *Suppose that* $\Delta_1=0$ *and that the x-directional blowing-up* $\mathrm{Bl}_x\Omega$ *is centered at a nonelementary point* $\tilde{p}\in\mathrm{NElem}(\widetilde{\mathbf{M}})$. *Then, the Newton data* $\widetilde{\Omega}=\mathrm{St}\,\mathrm{Bl}_x\Omega$ *is such that*

$$\mathrm{inv}(\widetilde{\Omega}) <_{\mathrm{lex}} \mathrm{inv}(\Omega).$$

Proof. This is an immediate consequence of Lemmas 4.39 and 4.40. □

4.14. Effect of translations in the x-directional blowing-up (case $\mathbf{\Delta_1{=}0}$)

In this subsection, we study the effect of the coordinate translations to the Newton data $\mathrm{Bl}_x\Omega$. As pointed out in Remark 4.2, the notions of stability and edge-stability have been introduced precisely to take these effects into account.

PROPOSITION 4.42. *Suppose that* $\Delta_1{=}0$ *and that the Newton data* $(\xi,\eta)\cdot\mathrm{Bl}_x\Omega$ *is centered at a nonelementary point* $\tilde{p}\in\mathrm{NElem}(\widetilde{\mathbf{M}})$, *where* $(\xi,\eta)\in\mathcal{G}_x^{\mathrm{tr}}$ *is an* x*-directional translation. Then*,

$$\mathrm{inv}(\widetilde{\Omega}_{\xi,\eta}) <_{\mathrm{lex}} \mathrm{inv}(\Omega),$$

where $\widetilde{\Omega}_{\xi,\eta}{=}\mathrm{St}(\xi,\eta)\cdot\mathrm{Bl}_x\Omega$ *is the stabilization of* $(\xi,\eta)\cdot\mathrm{Bl}_x\Omega$.

Proof. Defining $i{=}\#\iota_p$, we split the proof into two cases:

(a) $i{=}2$ and $\eta{\neq}0$;

(b) $i{=}1$ or $\eta{=}0$.

To treat case (a), we observe that the x-projected face $\widetilde{\mathcal{F}}{:=}\{\pi_x(\mathbf{v}){:}\mathbf{v}{\in}\mathcal{F}\}$ is equal to the intersection $\mathrm{supp}(\widetilde{\Omega})\cap(\{0\}\times\mathbf{Z}^2)$.

In particular, if we denote the main vertex of $\widetilde{\Omega}_{\xi,\eta}$ by $\widetilde{\mathbf{m}}$, it is immediate to see that

$$\widetilde{\mathbf{m}} \leqslant_{\mathrm{lex}} \overline{\mathbf{m}}$$

(where $\overline{\mathbf{m}}$ is the main vertex of $\mathrm{Bl}_x\Omega$). It follows that

$$\mathrm{inv}_1(\widetilde{\Omega}_{\xi,\eta}) = (\tilde{\mathfrak{h}}, \widetilde{m}_2{+}1, \widetilde{m}_3) \leqslant_{\mathrm{lex}} (\mathfrak{h}, m_2{+}1, m_3) = \mathrm{inv}_1(\Omega),$$

because $\tilde{\mathfrak{h}}{\leqslant}m_3{=}\mathfrak{h}$. On the other hand, if we have an equality of the primary multiplicity $\mathrm{inv}_1(\cdot)$, then

$$\mathrm{inv}_2(\widetilde{\Omega}_{\xi,\eta}) = (0, \lambda\bar{\Delta}_1, \lambda\max\{0,\overline{\Delta}_2\}) <_{\mathrm{lex}} (1, \lambda\Delta_1, \lambda\max\{0,\Delta_2\}) = \mathrm{inv}_2(\Omega)$$

(because the assumption $\eta{\neq}0$ implies that the translated Newton data $\widetilde{\Omega}_{\xi,\eta}$ is centered at a point $\tilde{p}$ such that $\#\iota_{\tilde{p}}{=}1{<}2{=}\#\iota_p$).

We now treat case (b). It follows from Lemma 4.35, that there exists a unique map $(f,g)\in\mathcal{G}^i_{\Delta,C}$ such that

$$\widetilde{\Omega}_{\xi,\eta} = \mathrm{St}\,\mathrm{Bl}_x(f,g)\cdot\Omega.$$

More explicitly, (f,g) is given by $(\xi x^{\omega_3}, \eta x^{\omega_2})$, where $\eta{=}0$ if $i{=}2$.

Since Ω is a stable Newton data, the Newton data $\Omega_{f,g}{:=}(f,g)\cdot\Omega$ is also stable. Moreover,

$$\mathrm{inv}(\Omega_{f,g}) = \mathrm{inv}(\Omega).$$

Thus, the result follows by applying Proposition 4.41 to $\Omega_{f,g}$, instead of Ω. □

4.15. x-directional blowing-up (case $\Delta_1 > 0$)

In this subsection, we keep the assumption that $\omega_1 = 1$. Recall that this condition can always be obtained, up to a preliminary transformation of type Bl_x^1.

LEMMA 4.43. *Suppose that $\Delta_1 > 0$. Suppose further that the x-directional blowing-up $\mathrm{Bl}_x\Omega$ is centered at a nonelementary point $\tilde{p} \in \mathrm{NElem}(\widetilde{\mathbf{M}})$. If $\Delta_2 \leqslant 0$, then*

$$\mathrm{inv}_1(\widetilde{\Omega}) <_{\mathrm{lex}} \mathrm{inv}_1(\Omega),$$

where $\widetilde{\Omega} = \mathrm{St}\,\mathrm{Bl}_x\Omega$ is the stabilization of $\mathrm{Bl}_x\Omega$.

Proof. Under the hypothesis of the lemma, we know that the main edge of Ω is given by $\mathfrak{e} = \overline{\mathbf{m}, \mathbf{v}}$, where $\mathbf{v} \in \mathrm{supp}(\Omega)$ is such that $v_2 \leqslant m_2$ and $v_3 < m_3$. Using Lemma 4.31, we conclude that the main vertex of $\mathrm{Bl}_x\Omega$ is given either by

$$\overline{\mathbf{m}} = (0, v_2, v_3)$$

(if $\mathrm{Bl}_x\Omega$ is in a regular configuration) or by

$$\overline{\mathbf{m}} = (0, 0, \overline{m}_3)$$

for some $\overline{m}_3 < v_3$ (if $\mathrm{Bl}_x\Omega$ is in a nilpotent configuration). Therefore, after applying the stabilization map St to $\mathrm{Bl}_x\Omega$, we get

$$\mathrm{inv}_1(\widetilde{\Omega}) = (\tilde{\mathfrak{h}}, \widetilde{m}_2 + 1, \widetilde{m}_3) <_{\mathrm{lex}} (\mathfrak{h}, m_2 + 1, m_3),$$

because $\tilde{\mathfrak{h}} \leqslant v_3 < m_3 = \mathfrak{h}$. □

LEMMA 4.44. *Suppose that $\Delta_1 > 0$. Suppose further that the x-directional blowing-up $\mathrm{Bl}_x\Omega$ is centered at a nonelementary point $\tilde{p} \in \mathrm{NElem}(\widetilde{\mathbf{M}})$. If $\Delta_2 > 0$, then the stabilization $\widetilde{\Omega} = \mathrm{St}\,\mathrm{Bl}_x\Omega$ of $\mathrm{Bl}_x\Omega$ is such that*

$$\mathrm{inv}(\widetilde{\Omega}) <_{\mathrm{lex}} \mathrm{inv}(\Omega).$$

Proof. Under the hypothesis $\Delta_2 > 0$, we consider separately the following cases:

(a) $\mathrm{Bl}_x\Omega$ is in a hidden nilpotent configuration;

(b) $\mathrm{Bl}_x\Omega$ is not in a hidden nilpotent configuration.

In case (a), let

$$\overline{\Omega} = (f, 0) \cdot \mathrm{Bl}_x\Omega$$

be the edge-preparation of $\mathrm{Bl}_x\Omega$. Then, the virtual height $\bar{\mathfrak{h}}$ associated with $\overline{\Omega}$ is strictly smaller than $\mathfrak{h} = m_3$.

Consider now case (b). The main vertex of both $\mathrm{Bl}_x\Omega$ and $\bar{\Omega}$ is $\mathbf{m}$. Moreover, the displacement vector Δ' of $\mathrm{Bl}_x\Omega$ is given by

$$\Delta'_1 = 0 \quad \text{and} \quad \Delta'_2 = \Delta_2.$$

The argument is now similar to the one used in the proof of Lemma 4.40. Let

$$\bar{\Omega} = (f,0)\cdot \mathrm{Bl}_x\Omega$$

be the edge-preparation of $\mathrm{Bl}_x\Omega$, and let $\bar{\Delta}=(\bar{\Delta}_1,\bar{\Delta}_2)$ be the displacement vector associated with $\bar{\Omega}$. We claim that

$$\bar{\Delta}_1 = 0 \quad \text{and} \quad \bar{\Delta}_2 = \Delta_2.$$

Indeed, suppose the contrary. Then, if we consider the polynomial truncation of f given by $f_t=\xi y^{\Delta_2}$ (with $\xi\in\mathbf{R}$ equal to zero if $\Delta_2\notin\mathbf{N}$), it follows that

$$\Omega_t := (f_t,0)\cdot \mathrm{Bl}_x\Omega$$

has a displacement vector $\Delta_t >_{\mathrm{lex}} \Delta$. Since $(f_t,0)$ belongs to the x-projected group $\mathrm{Pr}\mathcal{G}_x$, we can use Lemma 4.35 to conclude that that there exists a map $(f_{\boldsymbol{\omega}},0)\in\mathcal{G}^i_{\Delta,C}$ such that

$$(f_{\boldsymbol{\omega}},0)\cdot\Omega$$

has a vertical displacement vector which is (lexicographically) strictly greater than Δ. But this contradicts the hypothesis that Ω is stable. The claim is proved.

Using the claim, we easily conclude that

$$\mathrm{inv}_1(\widetilde{\Omega}) \leqslant_{\mathrm{lex}} \mathrm{inv}_1(\Omega) \quad \text{and} \quad \mathrm{inv}_2(\widetilde{\Omega}) <_{\mathrm{lex}} \mathrm{inv}_2(\Omega). \qquad \square$$

Proposition 4.45. *Suppose that* $\Delta_1>0$. *Suppose further that the* x*-directional blowing-up* $\mathrm{Bl}_x\Omega$ *is centered at a nonelementary point* $\tilde{p}\in\mathrm{NElem}(\widetilde{\mathbf{M}})$*. Then*,

$$\mathrm{inv}(\widetilde{\Omega}) <_{\mathrm{lex}} \mathrm{inv}(\Omega),$$

where $\widetilde{\Omega}=\mathrm{St}\mathrm{Bl}_x\Omega$ *is the stabilization of* $\mathrm{Bl}_x\Omega$.

Proof. It suffices to use Lemmas 4.43 and 4.44. □

4.16. Effect of translations in the x-directional blowing-up (case $\Delta_1>0$)

Let us now study the effect of the translations in the x-directional blowing-up chart for the case where $\Delta_1>0$.

PROPOSITION 4.46. *Suppose that* $\Delta_1>0$ *and that the Newton data* $(\xi,0)\cdot\mathrm{Bl}_x\Omega$ *is centered at a nonelementary point* $\tilde{p}\in\mathrm{NElem}(\widetilde{\mathbf{M}})$, *where* $(\xi,0)\in\mathcal{G}_x^{\mathrm{tr}}$ *is an* x*-directional translation. Then*,

$$\mathrm{inv}(\widetilde{\Omega}_{\xi,\eta})<_{\mathrm{lex}}\mathrm{inv}(\Omega),$$

where $\widetilde{\Omega}_{\xi,\eta}=\mathrm{St}(\xi,0)\cdot\mathrm{Bl}_x\Omega$ *is the stabilization of* $(\xi,\eta)\cdot\mathrm{Bl}_x\Omega$.

Proof. If $\xi=0$, this follows from Lemma 4.45.

Let us assume that $\xi\neq0$. We split the proof into three cases:

(a) $\Delta_2<0$;

(b) $\Delta_2>0$;

(c) $\Delta_2=0$.

In case (a), we know that $\mathbf{m}=(0,0,m_3)$. Moreover, the main vertex of $\mathrm{Bl}_x\Omega$ is given by $\overline{\mathbf{m}}=(0,-1,\overline{m}_3)$ for some $\overline{m}_3<m_3$. It follows that the Newton data $(\xi,0)\cdot\mathrm{Bl}_x\Omega$ is in a final situation and therefore it is centered at an elementary point $\tilde{p}\in\mathrm{Elem}(\widetilde{\mathbf{M}})$. This contradicts the hypothesis in the statement.

In case (b), we have $\overline{\mathbf{m}}=\mathbf{m}$. Let $\bar{\chi}$ be the vector field associated with $\mathrm{Bl}_x\Omega$. Then,

$$\bar{\chi}|_{\mathbf{m}}=y^{m_2}z^{m_3}\left(\alpha x\frac{\partial}{\partial x}+\beta y\frac{\partial}{\partial y}+\gamma z\frac{\partial}{\partial z}\right),\quad (\alpha,\beta,\gamma)\in\mathbf{R}^3\setminus\{(0,0,0)\}$$

(where $\alpha=\gamma=0$ if $m_2=-1$). Since

$$\pi_x(\mathcal{F})\cap(\{0\}\times\{m_2\}\times\mathbf{R})=\{\mathbf{m}\}$$

and $m_2\in\{-1,0\}$, it is clear that after the translation $\tilde{z}=z+\xi$ we get a Newton data $(\xi,0)\cdot\mathrm{Bl}_x\Omega$ which is in a final situation. Again, this contradicts the hypothesis in the statement.

It remains to study case (c). Here, we observe that the translation map $(\xi,0)$ belongs to the x-projected group $\mathrm{Pr}\mathcal{G}_x$. It follows from Lemma 4.35 that there exists a unique map $(f,g)\in\mathcal{G}^i_{\Delta,C}$ such that

$$\widetilde{\Omega}_\xi=\mathrm{St}\,\mathrm{Bl}_x(f,g)\cdot\Omega.$$

More explicitly, (f,g) is given by $(\xi x^{\omega_3},0)$.

Since Ω is a stable Newton data, the same holds for the Newton data $\Omega_{f,g}:=(f,g)\cdot\Omega$. Moreover,

$$\mathrm{inv}(\Omega_{f,g})=\mathrm{inv}(\Omega).$$

Thus, the result follows directly by applying Proposition 4.45 to $\Omega_{f,g}$, instead of Ω. $\square$

We are finally ready to give the proof of Proposition 4.30.

Prof of Proposition 4.30. In case $\Delta_1=0$, we apply Proposition 4.42. In case $\Delta_1>0$, we apply Proposition 4.46. □

4.17. *y*-directional blowing-up

Let $(\mathbf{M},\mathrm{Ax})$ be a controlled singularly foliated manifold and let $p\in\mathfrak{D}\cap A$ be a divisor point in $\mathrm{NElem}(\mathbf{M})$.

Let $\Omega\in\mathrm{New}^{\boldsymbol{i},\mathbf{m}}_{\Delta,C}$ be a stable Newton data, associated with some adapted chart at p (and some local generator χ of L). In this subsection, we assume that the corresponding weight-vector $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)$ is such that $\omega_2>0$.

The *y-directional translation group* is the group $\mathcal{G}^{\mathrm{tr}}_y:=\mathcal{G}^1_{(0,0),\infty}$. In other words, an element of $\mathcal{G}^{\mathrm{tr}}_y$ corresponds to a translation

$$\tilde{z}=z+\xi$$

for some constant $\xi\in\mathbf{R}$. We denote this element simply by $(\xi,0)$.

PROPOSITION 4.47. *Given a stable Newton data* Ω, *let* $\bar{\Omega}=\mathrm{Bl}_y\Omega$ *be its y-directional blowing-up. Then*, *for each* $(\xi,0)\in\mathcal{G}^{\mathrm{tr}}_y$, *either the translated Newton data*

$$\bar{\Omega}_\xi:=(\xi,0)\cdot\bar{\Omega}$$

is centered at an elementary point $\tilde{p}\in\mathrm{Elem}(\widetilde{\mathbf{M}})$ *or*

$$\mathrm{inv}(\widetilde{\Omega}_\xi)<_{\mathrm{lex}}\mathrm{inv}(\Omega),$$

where $\widetilde{\Omega}_\xi=\mathrm{St}\,\bar{\Omega}_\xi$ *is the stabilization of* $\bar{\Omega}_\xi$.

The proof of the proposition will be given at the end of §4.18. First of all, we state the following analogue of Lemma 4.31.

LEMMA 4.48. *Let* $\bar{\Omega}:=\mathrm{Bl}_y\Omega$ *be the y-directional blowing-up of* Ω. *Then*, *there exists a bijective correspondence*

$$\begin{aligned}\mathrm{supp}(\Omega)\cap\mathcal{F}&\longrightarrow\mathrm{supp}(\bar{\Omega})\cap\{0\}\times\mathbf{Z}^2,\\ \mathbf{v}=(v_1,v_2,v_3)&\longmapsto\pi_y(\mathbf{v})=(0,v_1,v_3),\end{aligned}$$

such that the corresponding Newton maps $\bar{\Theta}$ *and* Θ *satisfy* $\bar{\Theta}[\pi_y(\mathbf{v})]=IM_y\Theta[\mathbf{v}]$.

Proof. This is an immediate consequence of the definition of Bl_y. □

We remark that the Newton data $\mathrm{Bl}_y\Omega=((\bar{x},\bar{y},\bar{z}),\bar{\iota},\overline{\Theta})$ is always such that $\#\bar{\iota}=2$.

As in the discussion for the x-directional blowing-up, we can decompose the map Bl_y into two maps Bl_y^2 and Bl_y^1, which are respectively associated with the singular changes of coordinates

$$x=\bar{x}, \quad y=\varepsilon\bar{y}^{\omega_2} \quad \text{and} \quad z=\bar{z},$$

and

$$x=\bar{x}^{\omega_1}\bar{y}, \quad y=\bar{x} \quad \text{and} \quad z=\bar{x}^{\omega_3}\bar{z},$$

followed by a division by $\bar{x}^{\mu}$ (for $\varepsilon\in\{-1,1\}$). The first change of coordinates corresponds to a *ramification*, and the second change of coordinates can always be written as a composition of a finite sequence of homogeneous blowing-ups.

The following lemma is an analogue of Lemma 4.33.

LEMMA 4.49. *Suppose that Ω is edge-stable. Then $\overline{\Omega}=\mathrm{Bl}_y^1\Omega$ is also an edge-stable Newton data.*

Proof. The proof is very similar to the proof of Lemma 4.34. We omit the details for shortness. □

Using this lemma, we may assume that $\omega_2=1$ without loss of generality in our results.

LEMMA 4.50. *Suppose that $\omega_2=1$. Then, given a translation map $(\xi,0)\in\mathcal{G}_y^{\mathrm{tr}}$, there exists a unique map $(f_{\boldsymbol{\omega}},0)\in\mathcal{G}_\Delta$ which makes the following diagram commutative*:

$$\begin{array}{ccc} \mathrm{New} & \xrightarrow{(f_{\boldsymbol{\omega}},g_{\boldsymbol{\omega}})\cdot} & \mathrm{New} \\ {\scriptstyle \mathrm{Bl}_y}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Bl}_y} \\ \mathrm{New} & \xrightarrow{(f,g)\cdot} & \mathrm{New}. \end{array}$$

Proof. The proof is analogous to the proof of Lemma 4.35. Consider the change of coordinates $\tilde{z}=z+\xi$, and apply the map $(X,Y,Z)=(x^{\omega_1}y,\varepsilon x,x^{\omega_3}z)$ on both sides of the equality. Cancelling common powers of x, we get

$$\widetilde{Z}=Z+\bar{\xi}Y^{\omega_3} \quad \text{for } \bar{\xi}=\varepsilon^{\omega_3}\xi,$$

which corresponds to the map $(\bar{\xi}Y^{\omega_3},0)$ in the group $\mathcal{G}_\Delta$. □

4.18. Effect of translations in the y-directional blowing-up

Let us keep the notation as in the previous subsection. Recall that we may assume, without loss of generality, that $\omega_2=1$.

The proof of Proposition 4.47 will be given by considering separately the cases $\Delta_2>1$, $\Delta_2=1$ and $\Delta_2<1$.

Proof of Proposition 4.47 (case $\Delta_2>1$). Suppose initially that $\xi=0$. Write the main edge of Ω as $\mathfrak{e}=\overline{\mathbf{m},\mathbf{v}}$, where $\mathbf{v}=(0,v_2,v_3)\in\operatorname{supp}(\Omega)$ is such that $v_2>m_2$ and $v_3<m_3$.

By Lemma 4.48, we conclude that the main vertex of $\widetilde{\Omega}:=\widetilde{\Omega}_0$ is given by $\widetilde{\mathbf{m}}=(0,0,v_3)$. Therefore,

$$\operatorname{inv}_1(\widetilde{\Omega})=(\tilde{\mathfrak{h}},1,v_3)<_{\text{lex}}(\mathfrak{h},m_2+1,m_3)=\operatorname{inv}_1(\Omega),$$

because $\tilde{\mathfrak{h}}=v_3<m_3=\mathfrak{h}$ (the last equality follows from the assumption that $\Delta_2>1$).

Suppose now that $\xi\neq 0$. We claim that the main vertex of $\widetilde{\Omega}_\xi$ has the form $\overline{\mathbf{m}}=(0,0,\overline{m}_3)$ for some $\overline{m}_3\leqslant m_3-1$.

Indeed, if this is not the case then necessarily $\overline{\mathbf{m}}=\mathbf{m}$ (by Lemma 4.48). Using Lemma 4.50, we conclude that there exists a map $(f,0)\in\mathcal{G}_\Delta$ such that

$$(f,0)\cdot\Omega$$

has a vertical displacement vector which is (lexicographically) strictly greater than Δ. But this contradicts the hypothesis that Ω is stable (and, in particular, edge-stable). The claim is proved.

Using the claim, we conclude again that $\operatorname{inv}_1(\widetilde{\Omega}_\xi)<_{\text{lex}}\operatorname{inv}_1(\Omega)$. ☐

Let us now consider the case $\Delta_2=1$.

Proof of Proposition 4.47 (case $\Delta_2=1$). Let $\mathbf{m}=(0,m_2,m_3)$ be the main vertex of Ω and Δ be the vertical displacement vector. We split the proof into two cases:

(a) $m_2=-1$;

(b) $m_2=0$.

In case (a), the primary invariant is given by

$$\operatorname{inv}_1(\Omega)=(m_3,0,m_3).$$

We claim that $\widetilde{\Omega}_\xi$ has a main vertex $\overline{\mathbf{m}}=(0,0,\overline{m}_3)$ such that $\overline{m}_3\leqslant m_3-2$.

Indeed, if there exists $\xi\in\mathbf{R}$ such that $\widetilde{\Omega}_\xi$ has a main vertex with height $\overline{m}_3=m_3-1$, then it follows from the hypothesis that Ω should necessarily be in a hidden nilpotent configuration. Using Lemma 4.50, this conclusion contradicts the assumption that Ω is stable and $m_2=-1$.

As a consequence of the claim, $\operatorname{inv}_1(\widetilde{\Omega}_\xi)<_{\text{lex}}\operatorname{inv}_1(\Omega)$, because $\tilde{\mathfrak{h}}=\widetilde{m}_3\leqslant m_3-2<\mathfrak{h}$.

Case (b) can be treated as in the proof of the case $\Delta_2>1$. ☐

To conclude the proof of Proposition 4.47, we treat the case $\Delta_2<1$.

Proof of Proposition 4.47 (case $\Delta_2<1$). We consider separately the following cases:

(a) $m_2=0$;

(b) $m_2=-1$.

In case (a), we can use exactly the same argument as in the proof of the case $\Delta_2>1$ to conclude that the main vertex $\overline{\mathbf{m}}$ of $\widetilde{\Omega}_\xi$ is such that $\overline{m}_3 \leqslant m_3-1$. Therefore,

$$\mathrm{inv}_1(\widetilde{\Omega}) = (\bar{\mathfrak{h}}, 1, \overline{m}_3) <_{\mathrm{lex}} (\mathfrak{h}, 0, m_3) = \mathrm{inv}_1(\Omega),$$

because $\bar{\mathfrak{h}} = \overline{m}_3 < m_3 = \mathfrak{h}$.

Let us treat case (b). Suppose initially that $\xi=0$. Write the main edge of Ω as $\mathfrak{e} = \overline{\mathbf{m}, \mathbf{v}}$, where $\mathbf{v}=(0, v_2, v_3)$ is such that $v_2>1$ (the strict inequality follows from the fact that Ω is not in a nilpotent configuration). Therefore $v_3 \leqslant \lfloor m_3-1/\Delta_2 \rfloor$. By Lemma 4.48, we conclude that the main vertex of $\widetilde{\Omega} := \widetilde{\Omega}_0$ is given by $\widetilde{\mathbf{m}}=(0,0,v_3)$. Therefore

$$\mathrm{inv}_1(\widetilde{\Omega}) = (\tilde{\mathfrak{h}}, 1, v_3) <_{\mathrm{lex}} (\mathfrak{h}, 0, m_3) = \mathrm{inv}_1(\Omega),$$

because, by the definition of virtual height, $\tilde{\mathfrak{h}} = v_3 \leqslant \lfloor m_3 - 1/\Delta_2 \rfloor < \lfloor m_3 - 1/\Delta_2 + 1 \rfloor = \mathfrak{h}$.

Suppose now that $\xi \neq 0$. Let χ be the vector field which is associated with Ω. Then, its restriction to the main edge $\mathfrak{e}$ can be written as

$$\chi|_{\mathfrak{e}} = F(y,z)x\frac{\partial}{\partial x} + G(y,z)\frac{\partial}{\partial y} + H(y,z)\frac{\partial}{\partial z},$$

where F, G and H are (ω_2, ω_3)-quasihomogeneous polynomials of degree μ, $\mu+\omega_2$ and $\mu+\omega_3$, respectively. The hypothesis $m_2=-1$ implies that $G(0,z)=\beta z^{m_3}$ for some nonzero constant $\beta \in \mathbf{R}$.

Using Lemma 4.48, we see that the vector field $\widetilde{\chi}$ which is associated with $\mathrm{Bl}_y \Omega$ (before the translation by ξ) is such that its restriction to $\pi_y(\mathfrak{e})$ has the form

$$\frac{1}{\omega_2}G(1,z)x\frac{\partial}{\partial x} + \left(F(1,z)y - \frac{\omega_1}{\omega_2}G(1,z)y \right)\frac{\partial}{\partial y} + \left(H(1,z) - \frac{\omega_3}{\omega_2}G(1,z)z \right)\frac{\partial}{\partial z}. \tag{29}$$

Let us consider the polynomial

$$\tilde{g}(z) := \frac{1}{\omega_2}G(1,z)$$

and denote by $\bar{\mathfrak{h}}$ the virtual height associated with $\widetilde{\Omega}_\xi$.

It follows that ξ is a root of $\tilde{g}(z)$ of multiplicity $\bar{\mathfrak{h}}$. On the other hand, Corollary B.3 (Appendix B) implies that

$$\mu_\xi(\tilde{g}) \leqslant \left\lfloor m_3 - \frac{1}{\Delta_2} \right\rfloor < \left\lfloor m_3 - \frac{1}{\Delta_2} + 1 \right\rfloor = \mathfrak{h}.$$

This concludes the proof of Proposition 4.47. □

4.19. The z-directional blowing-up

Let $(\mathbf{M},\mathrm{Ax})$ be a controlled singularly foliated manifold and let $p\in\mathfrak{D}\cap A$ be a divisor point in $\mathrm{NElem}(\mathbf{M})$.

Let $\Omega\in\mathrm{New}^{i,\mathbf{m}}_{\Delta,C}$ be a stable Newton data at p. From our constructions, it is clear that the associated weight-vector $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)$ is always such that $\omega_3>0$.

LEMMA 4.51. *Let $\overline{\Omega}:=\mathrm{Bl}_z\Omega$ be the z-directional blowing-up of Ω. Then, there exists a bijective correspondence*

$$\mathrm{supp}(\Omega)\cap\mathcal{F}\longrightarrow\mathrm{supp}(\overline{\Omega})\cap\{0\}\times\mathbf{Z}^2,$$
$$\mathbf{v}=(v_1,v_2,v_3)\longmapsto\pi_z(\mathbf{v})=(0,v_1,v_2),$$

such that the corresponding Newton maps $\overline{\Theta}$ and Θ satisfy $\overline{\Theta}[\pi_z(\mathbf{v})]=JM_z\Theta[\mathbf{v}]$.

Proof. This is an immediate consequence of the definition of Bl_z. □

PROPOSITION 4.52. *Given a stable Newton data Ω, its z-directional blowing-up $\overline{\Omega}=\mathrm{Bl}_z\Omega$ is always centered at an elementary point $\tilde{p}\in\mathrm{Elem}(\widetilde{\mathbf{M}})$.*

Proof. Using Proposition 4.3, it is sufficient to prove that $\mathrm{Bl}_z\Omega$ is in a final situation.

If we write the main vertex of Ω as $\mathbf{m}=(0,m_2,m_3)$, it follows from Lemma 4.51 that $\widetilde{\Omega}$ contains the point

$$\widetilde{\mathbf{m}}:=\pi_z(\mathbf{m})=(0,0,m_2)$$

in its support. It is clear that this point is necessarily the new main vertex of $\widetilde{\Omega}$. Moreover, since $m_2\in\{-1,0\}$, the Newton data $\widetilde{\Omega}$ is in a final situation. □

4.20. Proof of the local resolution of singularities

Proof of Theorem 4.29. Consider the local blowing-up $\Phi:\widetilde{\mathbf{M}}\to\mathbf{M}\cap U$ defined in the statement of the theorem, and write

$$\widetilde{\mathbf{M}}=(\widetilde{M},\widetilde{\Upsilon},\widetilde{\mathfrak{D}},\tilde{L}).$$

Let V^x, V^y and V^z denote the domain of the x-, y- and z-directional charts, respectively.

First of all, we define an open subset $\tilde{A}\subset\widetilde{M}$ and an analytic line field $\tilde{\mathfrak{z}}$ on $\tilde{A}$ by

$$\tilde{A}=\Phi^{-1}(A)\cap(V^x\cup V^y)\quad\text{and}\quad\tilde{\mathfrak{z}}=\Phi_*(\mathfrak{z})|_{\tilde{A}},$$

where $\Phi_*(\mathfrak{z})$ denotes the pull-back of $\mathfrak{z}$. We make the following observations:

(i) Proposition 4.52 implies that $\tilde{A}$ is an open neighborhood of $\mathrm{NElem}(\widetilde{\mathbf{M}})$;

(ii) on the domain $V^x\cup V^y$, the pull-back foliation $\Phi_*(\mathfrak{z})$ is everywhere regular. Hence $\mathrm{Ze}(\tilde{\mathfrak{z}})=\varnothing$.

It follows that the pair $\widetilde{\mathrm{Ax}}=(\tilde{A},\tilde{\mathfrak{z}})$ satisfies all the conditions of Definition 2.14. Hence, $\widetilde{\mathrm{Ax}}$ is an axis for $\widetilde{\mathbf{M}}$.

Now, let $\tilde{p}\in\Phi^{-1}(p)$ be a point belonging to $\mathrm{NElem}(\widetilde{\mathbf{M}})$. Then, either $\tilde{p}$ lies in the domain V^x or $\tilde{p}$ lies in $V^y\setminus V^x$.

Firstly, suppose that $\tilde{p}\in V^x$ and let $(\bar{x},\bar{y},\bar{z})$ be the global coordinates of the x-directional chart (given in §4.9). It follows that there exists a unique pair of constants $(\xi,\eta)\in\mathbf{R}^2$ such that the coordinates

$$(\bar{x},\bar{y}-\eta,\bar{z}-\xi)$$

define an adapted local chart at $\tilde{p}$. The stabilization of this chart corresponds to the stabilization of the Newton data $(\xi,\eta)\cdot\mathrm{Bl}_x\Omega$ (where Ω is the Newton data centered at p). Therefore, it follows from Proposition 4.30 that

$$\mathrm{inv}(\widetilde{\mathbf{M}},\widetilde{\mathrm{Ax}},\tilde{p}\,) <_{\mathrm{lex}} \mathrm{inv}(\mathbf{M},\mathrm{Ax},p). \tag{30}$$

This proves the theorem in the case where $\tilde{p}\in V^x$.

Suppose now that $\tilde{p}\in V^y\setminus V^x$ and let $(\bar{x},\bar{y},\bar{z})$ be the global coordinates of the y-directional chart (given in §4.9). Then, there exists a unique constant $\xi\in\mathbf{R}$ such that

$$(\bar{x},\bar{y},\bar{z}-\xi)$$

defines an adapted local chart at $\tilde{p}$. It suffices now to apply Proposition 4.30 to conclude that (30) also holds. This proves the theorem. □

5. Global theory

5.1. Upper semicontinuity of the virtual height at NElem∩$\mathfrak{D}$

In this subsection, our goal is to prove the upper semicontinuity of the virtual height. In other words, we will prove that each point $p\in\mathrm{NElem}(\mathbf{M})\cap\mathfrak{D}$ has an open neighborhood $V\subset M$ such that

$$\mathfrak{h}(\mathbf{M},\mathrm{Ax},q)<\mathfrak{h}(\mathbf{M},\mathrm{Ax},p)$$

for each point $q\in\mathrm{NElem}(\mathbf{M})\cap\mathfrak{D}\cap V$. For simplicity, denote the set of nonelementary points simply by NElem, and let

$$\mathfrak{h}\colon \mathrm{NElem}\cap\mathfrak{D}\longrightarrow\mathbf{N}$$

be the virtual height function. The *stratum of virtual height* h at $\mathfrak{D}$ is the subset

$$S_h\cap\mathfrak{D}=\{p\in\mathrm{NElem}(\mathbf{M})\cap\mathfrak{D}:\mathfrak{h}(p)=h\}.$$

To state the next result, we introduce the following notion. Let $D\subset\mathfrak{D}$ be an irreducible component of the divisor and let $p\in\mathrm{NElem}\cap\mathfrak{D}$ be a point in D. We shall say that a local chart $(U,(x,y,z))$ at p is *D-adapted* if

(i) $\mathfrak{z}$ is locally generated by $\partial/\partial z$;

(ii) $D=\{x=0\}$;

(iii) $\mathfrak{D}\cap U\subset\{xy=0\}$.

We further say that the chart $(U,(x,y,z))$ is *D-stable* if the corresponding Newton data $\Omega=((x,y,z),\iota,\Theta)$ is stable (for some choice of local generator for the line field).

Remark 5.1. If the point p belongs to the intersection $D\cap D'$ of two divisors, then in a D-adapted chart $(U,(x,y,z))$ we necessarily have $D=\{x=0\}$ and $D'=\{y=0\}$.

PROPOSITION 5.2. *Given an irreducible component of the divisor $D\subset\mathfrak{D}$ and a point $p\in S_h\cap D$, there exists an open neighborhood $V\subset M$ of p such that $\mathfrak{h}(q)\leqslant h$ for each point $q\in V\cap D\cap\mathrm{NElem}(\mathbf{M})$. Moreover, with a fixed D-stable local chart $(U,(x,y,z))$ at p,*

(i) *if $\Delta_1(D)>0$, then we locally have $S_h\cap D=\{x=z=0\}$;*

(ii) *if $\Delta_1(D)=0$, then we locally have $S_h\cap D=\{p\}$;*

where $\Delta(D)=(\Delta_1(D),\Delta_2(D))$ is the vertical displacement vector of the corresponding Newton data Ω.

Proof. First of all, we consider the case where $\Delta_1(D)>0$. We will show that there exists an open neighborhood of the origin $U\subset\mathbf{R}^2$ such that for each $(\xi,\eta)\in U$, the translation map

$$\tilde{y}=y+\eta \quad\text{and}\quad \tilde{z}=z+\xi \tag{31}$$

is such that one of the following two situations occurs:

(i.1) if $\xi\neq 0$, then the translated Newton data $\widetilde{\Omega}=(\xi,\eta)\cdot\Omega$ is in final situation;

(i.2) if $\xi=0$, then the virtual height at the translated point $\tilde{p}$ (i.e. the point which is obtained from p by the local translation (31)) is equal to h.

Item (i.1) is easy. Indeed, let $\mathbf{m}=(0,m_2,m_3)$ be the main vertex of Ω. Then, it is immediate that there exists a constant $C>0$ and a neighborhood of the origin $U\subset\mathbf{R}^2$ such that for each $(\xi,\eta)\in U$, the translated data $\widetilde{\Omega}$ evaluated at the point $\widetilde{\mathbf{m}}=(0,m_2,0)$ is such that

$$\|\widetilde{\Omega}(\widetilde{\mathbf{m}})\|\geqslant C|\xi|^{m_3+1}$$

(where $\|\cdot\|$ denotes the Euclidean norm). Since $m_2\in\{-1,0\}$, we see that $\widetilde{\Omega}$ is in final situation if $\xi\neq 0$.

Let us prove item (i.2). If $\xi=0$, there exist constants $C>0$ and $\delta>0$ such that for each translation $(0,\eta)$, with $|\eta|<\delta$, we have

$$\|\widetilde{\Omega}(\mathbf{m})\|\geqslant C.$$

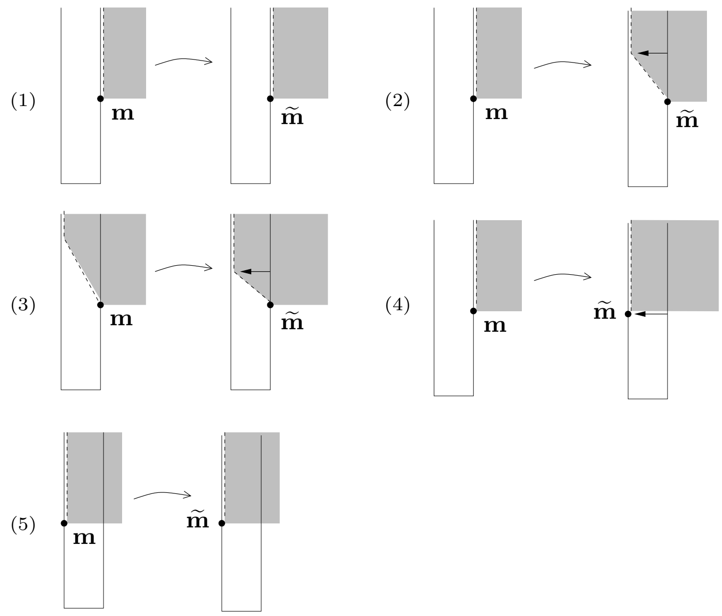


Figure 26. The effect of a translation.

Looking at the restriction of Ω to the set $\{0\}\times\mathbf{Z}^2$, the translation causes the five possible movements shown in Figure 26 (we denote the main vertex of $\widetilde{\Omega}$ by $\widetilde{\mathbf{m}}$). In each case, it is immediate that $\mathfrak{h}(\tilde{p})=h$.

We proceed now with the proof of the proposition in the case where $\Delta_1(D)=0$. We will show that there exists an open neighborhood of the origin $U\subset\mathbf{R}^2$ such that for each $(\xi,\eta)\in U$, the translation map

$$\tilde{y}=y+\eta \quad \text{and} \quad \tilde{z}=z+\xi$$

is such that one of the following two situations occurs:

(ii.1) if $\xi\neq 0$ and $\eta=0$, then the translated Newton data $\widetilde{\Omega}=(\xi,\eta)\cdot\Omega$ is in final situation;

(ii.2) if $\eta\neq 0$, then the virtual height of the translated point $\tilde{p}$ is strictly smaller than h.

The proof of (ii.1) is analogous to the proof of (i.1). In order to prove (ii.2), we consider the following *blowing-up* in the parameters (ξ,η):

$$\begin{aligned} \phi\colon \mathbf{R}^+\times\mathbf{S}^1 &\longrightarrow \mathbf{R}^2, \\ (r,\theta) &\longmapsto (\eta,\xi)=(r\cos\theta, r^s\sin\theta), \end{aligned}$$

where $s:=\Delta_2(D)$. We claim that there exists a neighborhood $\overline{U}\subset\mathbf{R}^+\times\mathbf{S}^1$ of the set $\{r=0\}$ such that the corresponding neighborhood of the origin $U:=\phi(\overline{U})$ satisfies the conditions stated above.

From (ii.1), we know that $\widetilde{\Omega}$ is in final situation for $\theta=\pi/2$ and r sufficiently small. Therefore, since this is an open condition, there exists an open neighborhood $\overline{V}\subset\mathbf{R}^+\times\mathbf{S}^1$ of the point $(r,\theta)=(0,\pi/2)$ such that $\widetilde{\Omega}$ is in final situation for each translation by (ξ,η) in $\phi(\overline{V})$.

To complete the study, it suffices to study the collection of all translations by (ξ,η) which are contained in the image of the directional blowing-up

$$\eta=\bar{\eta} \quad \text{and} \quad \xi=\bar{\eta}^s\bar{\xi},$$

with $\bar{\eta}$ varying in $\mathbf{R}$ and $\bar{\xi}$ belonging to some compact subset $K\subset\mathbf{R}$.

For this, we fix some stable local chart $(V,(x,y,z))$ at p and consider the local blowing-up for $(\mathbf{M},\mathrm{Ax})$:

$$\Phi\colon\widetilde{\mathbf{M}}\longrightarrow\mathbf{M}\cap V$$

(see §4.8). Under this blowing-up, the above collection of translations can be studied in the domain of the y-directional chart, where the blowing-up can be written as

$$x=\bar{y}^{\omega_1}\bar{x}, \quad y=\bar{y}^{\omega_2} \quad \text{and} \quad z=\bar{y}^{\omega_3}\bar{z}$$

(with $\omega_2/\omega_3=1/s$). Fixed $\bar{\xi}\in K$, let $\bar{p}$ denote the point on the exceptional divisor $\widetilde{D}=\Phi^{-1}(Y_p)$ which is obtained by the y-directional blowing-up followed by the vertical translation

$$\tilde{z}=\bar{z}+\bar{\xi}.$$

It follows from the proof of Proposition 4.47 that $\mathfrak{h}(\bar{p})<\mathfrak{h}(p)=h$. Therefore, using the compactness of K, it suffices to prove the following claim.

Claim. Let $(\overline{U},(\bar{x},\bar{y},\bar{z}))$ be a stable local chart at $\bar{p}$. Then, there exists a constant $\delta>0$ such that for each translation

$$\tilde{y}=\bar{y}+\bar{\eta}, \tag{32}$$

with $|\bar{\eta}|<\delta$, the corresponding translated point $\tilde{p}$ is such that $\mathfrak{h}(\tilde{p})\leqslant\mathfrak{h}(\bar{p})$.

The proof of the claim is similar to the proof of (ii.1). Let $\bar{\Omega}$ be the Newton data at $\bar{p}$, and $\bar{\Delta}=(\bar{\Delta}_1,\bar{\Delta}_2)$ be the corresponding vertical displacement. Two cases can occur:

(ii.2.a) $\bar{\Delta}_1>0$;

(ii.2.b) $\bar{\Delta}_1=0$.

In case (ii.2.a), item (i) treated above implies that $\mathfrak{h}(\tilde{p})=\mathfrak{h}(\bar{p})$.

In case (ii.2.b), let $\mathbf{m}=(0,0,m_3)$ be the main vertex associated with $\bar{\Omega}$, and let $\mathfrak{e}=\overline{\mathbf{m},\mathbf{v}}$ be the corresponding main edge. Then, if we write $\mathbf{v}=(v_1,v_2,v_3)$ (with $v_3<m_3$), there exist constants $C>0$ and $\delta>0$ such that

$$\|\widetilde{\Omega}(\tilde{\mathbf{v}})\| \geqslant C|\bar{\eta}|^{v_2+1}, \quad \text{with } \tilde{\mathbf{v}}=(0,0,v_3),$$

where $\widetilde{\Omega}$ is the Newton data obtained by the translation (32). We easily conclude that $\mathfrak{h}(\tilde{p})\leqslant\mathfrak{h}(\bar{p})$. □

5.2. Upper semicontinuity of the invariant at NElem∩$\mathfrak{D}$

Using the results of the previous subsection, let us prove the upper semicontinuity of the function

$$\mathrm{inv}: \mathrm{NElem}\cap\mathfrak{D} \longrightarrow \mathbf{N}^6,$$

where $\mathrm{inv}(p)$ is a shorter notation for $\mathrm{inv}(\mathbf{M},\mathrm{Ax},p)$.

We recall that the invariant $\mathrm{inv}(p)=(\mathrm{inv}_1(p),\mathrm{inv}_2(p))$ is given by

$$\mathrm{inv}_1=(\mathfrak{h},m_2+1,m_3) \quad \text{and} \quad \mathrm{inv}_2=(\#\iota_p-1,\lambda\Delta_1,\lambda\max\{\Delta_2,0\}),$$

where these quantities are computed using some stable local chart $(U,(x,y,z))$ for $(\mathbf{M},\mathrm{Ax})$ at p. The following remark will be useful in the sequel.

Remark 5.3. The definition of inv implies the following facts:

(1) if $p\in S_h\cap\mathfrak{D}$ is such that $\#\iota_p=2$, then $\mathrm{inv}(p)=(h,1,h,1,*,*)$;

(2) if $p\in S_h\cap\mathfrak{D}$ is such that $\#\iota_p=1$, then either

$$\mathrm{inv}(p)=(h,0,m_3,0,*,*) \quad \text{or} \quad \mathrm{inv}(p)=(h,1,h,0,*,*);$$

where the $*$'s denote some arbitrary natural numbers.

LEMMA 5.4. *Let $p\in S_h\cap\mathfrak{D}$ be a point such that $\#\iota_p=1$. Assume that the displacement vector $\Delta=(\Delta_1,\Delta_2)$ satisfies*

$$\Delta_1>0.$$

Then, there exists a neighborhood $V\subset M$ of p such that for each point $q\in(S_h\cap\mathfrak{D})\cap V$, the corresponding displacement vector $\tilde{\Delta}=(\tilde{\Delta}_1,\tilde{\Delta}_2)$ satisfies $\tilde{\Delta}_1=\Delta_1$.

Proof. Let us fix a stable local chart $(U,(x,y,z))$ for $(\mathbf{M},\mathrm{Ax})$ at p. Then, the local blowing-up center is given by $Y_p=\{x=z=0\}$. Let $\Phi:\widetilde{\mathbf{M}}\to\mathbf{M}\cap U$ be the local blowing-up with center Y_p and weight-vector $\boldsymbol{\omega}_p=(q_1,0,p_1)$ (where $\Delta_1=p_1/q_1$ is the irreducible rational representation of Δ_1).

Suppose, by contradiction, that there exists a sequence of real numbers $\{\eta_k\}$, with $\eta_k \to 0$, such that the corresponding sequence of Newton data $\widetilde{\Omega}_k$ which are obtained by the translations $\tilde{y}_k = y + \eta_k$ have a displacement vector $\Delta^k = (\Delta_1^k, \Delta_2^k)$ such that

$$\Delta_1^k > \Delta_1.$$

Let $\{q_k\}$, with $q_k \to p$, denote the sequence of points in Y_p which are obtained by these translations.

Using Lemmas 4.31 and 4.35, we see that, for each k, the set $\Phi^{-1}(q_k)$ contains at least one nonelementary point $\tilde{q}_k$ such that

$$\mathfrak{h}(\tilde{q}_k) = \mathfrak{h}(q_k) = h. \tag{33}$$

In fact, we can choose this point as the origin in the x-directional chart of the blowing-up.

On the other hand, the proofs of Lemmas 4.43 and 4.44 imply that each nonelementary point $\tilde{p}$ in $\Phi^{-1}(p)$ satisfies one of the following conditions:

(a) $\mathfrak{h}(\tilde{p}) < h$;

(b) $\mathfrak{h}(\tilde{p}) = h$ and $\tilde{\Delta}_1 = 0$;

where $\tilde{\Delta} = (\tilde{\Delta}_1, \tilde{\Delta}_2)$ is the vertical displacement vector of the Newton data at $\tilde{p}$ (for some fixed stable local chart).

Using item (ii) of Proposition 5.2 and the compactness of $\Phi^{-1}(p)$, we conclude that there exists some neighborhood $\widetilde{V} \subset \widetilde{M}$ of $\Phi^{-1}(p)$ such that each nonelementary point $\tilde{q} \in \widetilde{V} \setminus \Phi^{-1}(p)$ satisfies $\mathfrak{h}(\tilde{q}) < h$. This contradicts (33). □

PROPOSITION 5.5. *The function* $\mathrm{inv}\colon \mathrm{NElem} \cap \mathfrak{D} \to \mathbf{N}^6$ *is upper semicontinuous* (*for the lexicographical ordering on* $\mathbf{N}^6$).

Proof. Given a point $p \in \mathrm{NElem} \cap \mathfrak{D}$, we have to prove that there exists a neighborhood $V \subset M$ of p such that for each point $q \in \mathrm{NElem} \cap \mathfrak{D} \cap V$,

$$\mathrm{inv}(q) \leqslant_{\mathrm{lex}} \mathrm{inv}(p).$$

The upper semicontinuity of the initial segment of the local invariant, namely

$$(\mathfrak{h}, m_2 + 1, m_3, \#\iota_p - 1),$$

is obvious by Remark 5.3 and Proposition 5.2.

Let us fix $p \in S_h \cap \mathfrak{D}$. We claim that there exists a neighborhood $V \subset M$ of p such that for each point $q \in (\mathrm{NElem} \cap \mathfrak{D}) \cap V$, we have

$$\mathrm{inv}_1(q) = \mathrm{inv}_1(p) \text{ and } \#\iota_q = \#\iota_p \quad \Longrightarrow \quad \mathrm{inv}_2(q) \leqslant_{\mathrm{lex}} \mathrm{inv}_2(p).$$

Indeed, if $\mathrm{inv}_1(q)=\mathrm{inv}_1(p)$ and $\#\iota_q=\#\iota_p$, then it follows from items (i) and (ii) of Proposition 5.2 and from Remark 5.3 that

$$\#\iota_p = 1 \quad \text{and} \quad \Delta_1 > 0.$$

Therefore, using Lemma 5.4, we conclude (up to restricting V to some smaller neighborhood of p) that $\Delta_1^q=\Delta_1$ for each point $q\in(\mathrm{NElem}\cap\mathfrak{D})\cap V$. Moreover, for each fixed stable local chart $(U,(x,y,z))$ at p, it is clear that the adapted local chart at q which is obtained by the translation

$$\tilde{x}=x, \quad \tilde{y}=y+\eta \quad \text{and} \quad \tilde{z}=z$$

(for some appropriately chosen constant $\eta\in\mathbf{R}$) is also stable. Therefore, we obviously have (up to a new restriction of V to some smaller neighborhood of p) that $\Delta_2^q\leqslant\Delta_2^p$. This concludes the proof. □

5.3. Points in NElem$\setminus\mathfrak{D}$ and generic Newton polygon

A point $p\in\mathrm{NElem}\setminus\mathfrak{D}$ will be called *smooth* if the germ of analytic sets NElem_p is locally a smooth 1-dimensional analytic curve.

We shall say that an adapted local chart $(U,(x,y,z))$ for $(\mathbf{M},\mathrm{Ax})$ at p is *smoothly adapted* if

$$\mathrm{NElem} = \{y=z=0\}.$$

It follows from Proposition 3.1 that the transition map between two smoothly adapted local charts $(U,(x,y,z))$ and $(U',(x',y',z'))$ has the form

$$x'=f(y)+xu(x,y), \quad y'=yv(x,y) \quad \text{and} \quad z'=yh(x,y)+zw(x,y,z), \tag{34}$$

where f, h, u, v and w are analytic functions such that $f(0)=0$ and u, v and w are units.

Let Ω be the Newton data for $(\mathbf{M},\mathrm{Ax})$ at the smooth point p, relative to some smoothly adapted local chart $(U,(x,y,z))$. The *generic Newton map* associated with Ω is the map $\Theta_G\colon\mathbf{Z}^2\to\{0,1\}$ given by

$$\Theta_G(\mathbf{v}) = \begin{cases} 0, & \text{if } (\mathbf{Z}\times\{\mathbf{v}\})\cap\mathrm{supp}(\Omega)=\varnothing, \\ 1, & \text{if } (\mathbf{Z}\times\{\mathbf{v}\})\cap\mathrm{supp}(\Omega)\neq\varnothing \end{cases}$$

(see Figure 27). The *generic Newton polygon* associated with Ω is the convex polygon in $\mathbf{R}^2$ given by $\mathcal{N}_G(\Omega)=\mathrm{supp}(\Theta_G)+\mathbf{R}_+^2$. The *generic higher vertex* of Ω is the minimal point $\mathbf{p}_G\in\mathcal{N}_G$ in the lexicographical ordering.

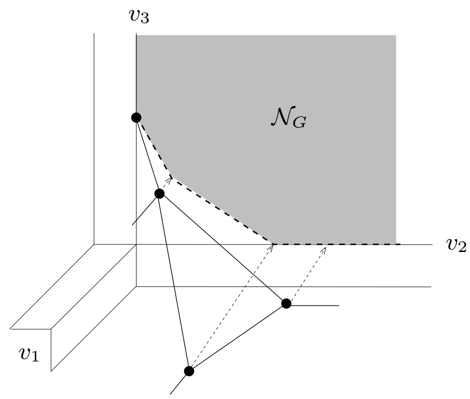


Figure 27. The generic Newton polygon.

Remark 5.6. The generic Newton polygon can be equivalently defined as

$$\mathcal{N}_G = \pi(\mathcal{N}),$$

where $\pi: \mathbf{R}^3 \to \mathbf{R}^2$ is the linear projection $\pi(v_1, v_2, v_3) = (v_2, v_3)$ and $\mathcal{N} = \mathcal{N}(\Omega)$ is the Newton polyhedron of Ω.

The triple $\Omega_G = ((x, y, z), \iota_p, \Theta_G)$ will be called the *generic Newton data* at p.

The *generic edge* associated with $\mathbf{p}_G$ is the unique edge $\mathfrak{e}(\mathbf{p}_G) \subset \mathcal{N}_G$ which intersects the horizontal line $\left\{(v_1, v_2) \in \mathbf{R}^2 : v_2 = p_2 - \frac{1}{2}\right\}$ (where we write $\mathbf{p}_G = (p_1, p_2)$), with the convention that $\mathfrak{e}(\mathbf{p}_G) = \varnothing$, if the intersection is empty.

We shall say that Ω_G is in a *nilpotent configuration* if the following conditions are satisfied:

(i) $\mathbf{p}_G = (-1, p_2)$ for some integer $p_2 \in \mathbf{Z}$;

(ii) the edge $\mathfrak{e}(\mathbf{p}_G)$ has the form $\overline{\mathbf{p}_G, \mathbf{n}}$, for some vertex $\mathbf{n} = (0, n_2)$ with $n_2 \in \mathbf{Z}$.

If one of these conditions fails, we shall say that Ω_G is in a *regular configuration.*

The *generic main vertex* is a vertex $\mathbf{m}_G \in \mathcal{N}_G$ which is chosen as follows:

(i) if Ω_G is in a regular configuration, then $\mathbf{m}_G := \mathbf{p}_G$;

(ii) if Ω_G is in a nilpotent configuration, then $\mathbf{m}_G := \mathbf{n}$ (where $\mathfrak{e}(\mathbf{p}_G) = \overline{\mathbf{p}_G, \mathbf{n}}$).

Let us write $\mathbf{m}_G = (m_1, m_2)$. The *generic main edge* is the edge $\mathfrak{e}_G \subset \mathcal{N}_G$ which intersects the horizontal line $\left\{(v_1, v_2) \in \mathbf{R}^2 : v_2 = m_2 - \frac{1}{2}\right\}$, with the convention that $\mathfrak{e}_G = \varnothing$, if this intersection is empty.

Note that we can write the generic main edge as

$$\mathfrak{e}_G = \mathbf{m}_G + t(\Delta, -1),$$

where t belongs to a real interval of the form $[0, L]$ (for some $L > 0$) and $\Delta \in \mathbf{Q}_{>0}$ is a positive rational number. In this setting, we shall shortly say that the generic Newton data Ω_G belongs to the class $\mathrm{New}_{\Delta}^{\mathbf{m}_G}$.

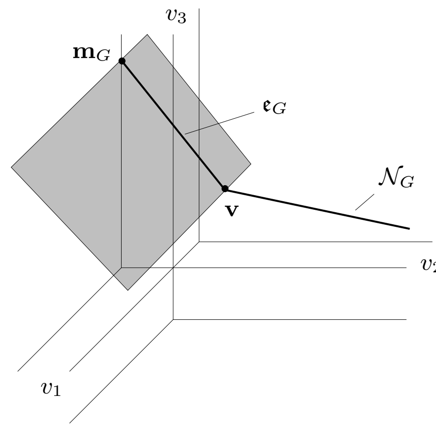


Figure 28. Newton polyhedron at a generic point.

The point $p \in \mathrm{NElem} \setminus \mathfrak{D}$ will be called *generic* with respect to an adapted local chart $(U,(x,y,z))$ if the following conditions hold:

(i) p is a smooth point;

(ii) $(U,(x,y,z))$ is a smoothly adapted local chart at p;

(iii) all the vertices of the corresponding Newton polyhedron $\mathcal{N}(\Omega)$ belong to the region $\{-1,0\} \times \mathbf{Z}^2$.

Remark 5.7. Suppose that $p \in \mathrm{NElem} \setminus \mathfrak{D}$ is generic with respect to an adapted local chart $(U,(x,y,z))$. Then, the generic main edge $\mathfrak{e}_G$ defines a face of the Newton polygon $\mathcal{N} = \mathcal{N}(\Omega)$. More precisely, the Minkowski sum

$$\mathcal{F} := \mathfrak{e}_G + \{t \cdot (1,0,0) : t \in \mathbf{R}_+\}$$

is a face of $\mathcal{N}$.

5.4. Generic edge-stability and equireducible points

Let $p \in \mathrm{NElem} \setminus \mathfrak{D}$ be a smooth point and let $(U,(x,y,z))$ be a smoothly adapted local chart for $(\mathbf{M}, \mathrm{Ax})$ at p. We denote respectively by Ω and Ω_G the associated Newton data and generic Newton data (for some choice of local generator for the line field).

Given a rational number $\delta \in \mathbf{Q}_{\geqslant 0}$, the group of $\mathcal{G}_\delta$-maps is the group of all analytic changes of coordinates of the form

$$\tilde{z} = z + g(y),$$

where g is given by $g(y) = \xi y^\delta$ (for some constant $\xi \in \mathbf{R}$) if $\delta \in \mathbf{N}$, and $g \equiv 0$ otherwise.

The group $\mathcal{G}_\delta$ acts naturally on the class of Newton data via the coordinate change $(x,y,z)\mapsto(x,y,\tilde{z})$. Given a map $g\in\mathcal{G}_\delta$, we denote this action on Ω simply by $g\cdot\Omega$.

If Ω_G belongs to the class $\mathrm{New}_\Delta^{\mathbf{m}_G}$, then we say that Ω is *generic edge-stable* if

$$(g\cdot\Omega)_G\in\mathrm{New}_\Delta^{\mathbf{m}_G}$$

for all maps $g\in\mathcal{G}_\Delta$ with $g(0)=0$. In other words, the generic Newton data associated with $g\cdot\Omega$ lies in the class $\mathrm{New}_\Delta^{\mathbf{m}_G}$ for all $g\in\mathcal{G}_\Delta$. The local chart $(U,(x,y,z))$ will be called *generic edge-stable* if Ω is generic edge-stable.

Lemma 5.8. *Suppose that the smoothly adapted local chart $(U,(x,y,z))$ is* not *generic edge-stable. Then, there exists a unique map $g\in\mathcal{G}_\Delta$ such that the transformed generic Newton data $(g\cdot\Omega)_G$ does not belongs to $\mathrm{New}_\Delta^{\mathbf{m}_G}$.*

Proof. The result can be proved by straightforward modifications of the proof of Lemma 4.13. □

Now, we are ready to give the main definition of this subsection. We shall say that a smooth point $p\in\mathrm{NElem}\setminus\mathfrak{D}$ is *equireducible* if there exists a smoothly adapted local chart $(U,(x,y,z))$ for $(\mathbf{M},\mathrm{Ax})$ at p such that

(i) p is generic with respect to $(U,(x,y,z))$;

(ii) the corresponding Newton data Ω is generic edge-stable.

In this case, $(U,(x,y,z))$ will be called an *equireduction chart* for $(\mathbf{M},\mathrm{Ax})$ at p.

Lemma 5.9. *Let $(U,(x,y,z))$ and $(U',(x',y',z'))$ be two equireduction charts at an equireducible point p. Then, the transition map (see (34)) has necessarily the form*

$$x'=f(y)+xu(x,y),\quad y'=yv(x,y)\quad\text{and}\quad z'=yh(x,y)+zw(x,y,z),$$

for some analytic functions f, h, u, v and w such that $f(0)=0$ and u, v and w are units. Moreover, the support of the function $H(x,y)=yh(x,y)$ satisfies the following property

$$\mathrm{supp}(H)\subset\{(v_1,v_2)\in\mathbf{N}^2:v_2>\Delta\}.$$

Proof. This is a direct corollary of Lemma 5.8. □

As a consequence of the second part of Lemma 5.9, the Newton data Ω' associated with the chart $(U',(x',y',z'))$ is such that

$$\Omega_G\in\mathrm{New}_\Delta^{\mathbf{m}_G}\iff(\Omega')_G\in\mathrm{New}_\Delta^{\mathbf{m}_G}.$$

Let us now characterize the generic Newton data centered at points in $\mathrm{NElem}\setminus\mathfrak{D}$.

First of all, we introduce the following notion. A generic Newton data Ω_G is *in a final situation* if one of the following conditions holds:

(i) the generic main vertex $\mathbf{m}_G=(m_1,m_2)$ is such that $m_2\in\{-1,0\}$;

(ii) the main edge is given by $\mathfrak{e}_G=\overline{\mathbf{m}_G,\mathbf{v}}$, where $\mathbf{m}_G=(-1,1)$ and $\mathbf{v}=(1,-1)$.

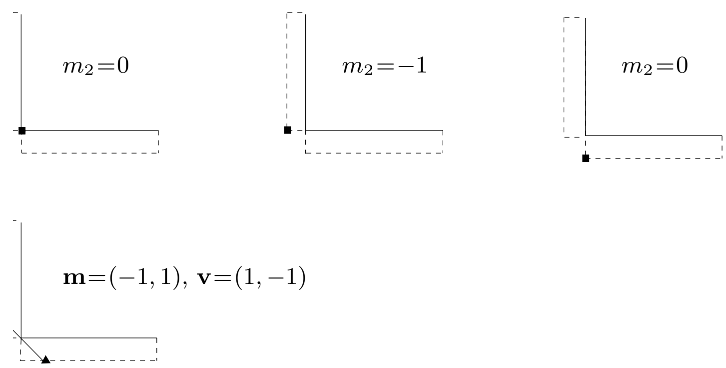


Figure 29. The final situations for the generic Newton data.

As a consequence of the definition of equireducible point, we get the following result.

PROPOSITION 5.10. *Let* $p \in \mathrm{NElem} \setminus \mathfrak{D}$ *be an equireducible point and let* $(U,(x,y,z))$ *be an equireduction chart at* p. *Then, the associated generic Newton data* Ω_G *is not in a final situation.*

Proof. This is analogous to the proof of Proposition 4.3. □

5.5. Local blowing-up at equireducible points

Let $p \in \mathrm{NElem}(\mathbf{M}) \setminus \mathfrak{D}$ be an equireducible point. Let Ω be the Newton data for $(\mathbf{M}, \mathrm{Ax})$ at p, with respect to some equireduction chart $(U,(x,y,z))$.

The *generic virtual height* for $(\mathbf{M}, \mathrm{Ax})$ at p is defined as

$$\mathfrak{h}_G(\mathbf{M}, \mathrm{Ax}, p) := \begin{cases} \lfloor m_2 + 1 - 1/\Delta \rfloor, & \text{if } m_1 = -1, \\ m_2, & \text{if } m_1 = 0, \end{cases}$$

where $\mathbf{m}_G = (m_1, m_2)$ is the main vertex of the generic Newton polygon of $\mathcal{N}_G$.

The *local blowing-up center* associated with $(\mathbf{M}, \mathrm{Ax})$ at p is the submanifold

$$Y_p = \{y = z = 0\}.$$

Assume that the generic Newton data Ω_G belongs to the class $\mathrm{New}_{\Delta}^{\mathbf{m}_G}$. The *weight-vector* associated with $(\mathbf{M}, \mathrm{Ax})$ at p is given by

$$\boldsymbol{\omega} = (0, q, p),$$

where $\Delta = p/q$ is the irreducible rational representation of Δ.

Remark 5.11. It follows from Lemma 5.9 that $\mathfrak{h}_G(\mathbf{M}, \mathrm{Ax}, p)$, Y_p and ω are independent of the choice of the equireduction chart $(U,(x,y,z))$.

The *local blowing-up* for $(\mathbf{M}, \mathrm{Ax})$ at p is the $\boldsymbol{\omega}$-weighted blowing-up

$$\Phi: \widetilde{\mathbf{M}} \longrightarrow \mathbf{M} \cap U$$

with center on Y_p, with respect to the trivialization given by $(U,(x,y,z))$.

Remark 5.12. Lemma 5.9 implies that the transition map between two equireduction charts always preserves the $\boldsymbol{\omega}$-quasihomogeneous structure on $\mathbf{R}^3$.

The following theorem is a version of the local resolution of singularities for equireducible points.

THEOREM 5.13. *Let* $(\mathbf{M}, \mathrm{Ax})$ *be a controlled singularly foliated manifold and let* $p \in A \setminus \mathfrak{D}$ *be an equireducible point in* $\mathrm{NElem}(\mathbf{M})$. *Consider the* local blowing-up *for* $(\mathbf{M}, \mathrm{Ax})$ *at* p,

$$\Phi: \widetilde{\mathbf{M}} \longrightarrow \mathbf{M} \cap U,$$

with respect to some equireduction chart $(U,(x,y,z))$. *Then, there exists an axis* $\widetilde{\mathrm{Ax}} = (\tilde{A}, \tilde{\mathfrak{z}})$ *for* $\widetilde{\mathbf{M}}$ *such that each point* $\tilde{p} \in \Phi^{-1}(p) \cap \tilde{A}$ *belonging to* $\mathrm{NElem}(\widetilde{\mathbf{M}})$ *is such that*

$$\mathfrak{h}(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}}, \tilde{p}) < \mathfrak{h}_G(\mathbf{M}, \mathrm{Ax}, p).$$

Proof. This is analogous to the proof of Theorem 4.29, using now the definition of equireducible points. □

The *local invariant* for $(\mathbf{M}, \mathrm{Ax})$ at an equireducible point $p \in \mathrm{NElem} \setminus \mathfrak{D}$ is the vector of natural numbers

$$\mathrm{inv}(\mathbf{M}, \mathrm{Ax}, p) = (\mathfrak{h}_G(\mathbf{M}, \mathrm{Ax}, p), 0, 0, 0, 0, 0) \in \mathbf{N}^6.$$

5.6. Distinguished vertex blowing-up

In this subsection, we describe a procedure which will be used to treat the points $p \in \mathrm{NElem} \setminus \mathfrak{D}$ which are not equireducible. The basic idea is to *include* these points in the divisor $\mathfrak{D}$ by an appropriately chosen weighted blowing-up.

Let $(\mathbf{M}, \mathrm{Ax})$ be a controlled singularly foliated manifold. We fix a point $p \in \mathrm{NElem}$, a local generator χ for the line field L and a local generator Z for the line field $\mathfrak{z}$ which defines the axis Ax.

The *primitive height* for $(\mathbf{M}, \mathrm{Ax})$ at p is the minimal integer $h = H(\mathbf{M}, \mathrm{Ax}, p)$ such that the vector field

$$\chi^h := (\mathcal{L}_Z)^h(\chi)$$

is nonzero at p. Here, $(\mathcal{L}_Z)^h$ is the h-fold composition of the Lie bracket operator $\mathcal{L}_Z(\cdot) = [Z, \cdot]$. By convention, we set $H(\mathbf{M}, \mathrm{Ax}, p) = \infty$ if $\chi^h(p) = 0$ for all $h \in \mathbf{N}$.

LEMMA 5.14. *For $p \in \mathrm{NElem} \setminus \mathfrak{D}$, the primitive height $H(\mathbf{M}, \mathrm{Ax}, p)$ is a well-defined natural number. Moreover, it is independent of the choice of the local generators χ and Z.*

Proof. Let us prove that $H(\mathbf{M}, \mathrm{Ax}, p)$ is finite. For this, we fix an adapted local chart $(U, (x, y, z))$ at p and write

$$\chi = F(x, y, z) \frac{\partial}{\partial x} + G(x, y, z) \frac{\partial}{\partial y} + H(x, y, z) \frac{\partial}{\partial z}$$

for some analytic germs F, G and H. We can also choose $Z = \partial/\partial z$. Therefore,

$$\chi^h = \frac{\partial^h F}{\partial z^h}(x, y, z) \frac{\partial}{\partial x} + \frac{\partial^h G}{\partial z^h}(x, y, z) \frac{\partial}{\partial y} + \frac{\partial^h H}{\partial z^h}(x, y, z) \frac{\partial}{\partial z}.$$

If the collection of vector fields $\{\chi^h\}$ vanishes at the origin for all $h \in \mathbf{N}$, then the germs F, G and H necessarily belong to the ideal $(x, y)\mathcal{O}_p$. This contradicts the fact that Ax is an axis for $\mathbf{M}$ (see Definition 2.14).

We now prove that the primitive height is independent of the choice of χ and Z. For this, it suffices to observe that, if we write $\chi' = U\chi$ and $Z' = VZ$, for some units U and V, then

$$[Z', \chi'] = [VZ, U\chi] = UV[Z, \chi] + VZ(U)\chi + U\chi(V)Z.$$

Proceeding by induction, we conclude that $(\mathcal{L}_{Z'})^h(\chi')$ vanishes at p if and only if $(\mathcal{L}_Z)^h(\chi)$ vanishes at p. □

An adapted local chart $(U, (x, y, z))$ at a point $p \in \mathrm{NElem} \setminus \mathfrak{D}$ will be called *strongly adapted* if the associated Newton data Ω has a polyhedron with a vertex of the form $\mathbf{d} = (-1, 0, d_3)$, where

$$d_3 = H(\mathbf{M}, \mathrm{Ax}, p).$$

The vertex $\mathbf{d}$ will be called *distinguished vertex*.

The following lemma shows that we can always construct a strongly adapted local chart.

LEMMA 5.15. *Given an adapted local chart $(U, (x, y, z))$ at $p \in \mathrm{NElem} \setminus \mathfrak{D}$, there exists a linear change of coordinates of the form*

$$\tilde{x} = x, \quad \tilde{y} = y + \xi x \quad \text{and} \quad \tilde{z} = z \quad (\text{for some constant } \xi \in \mathbf{R}),$$

such that the resulting local chart $(U, (\tilde{x}, \tilde{y}, \tilde{z}))$ is strongly adapted.

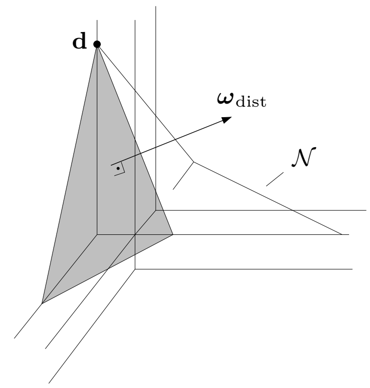


Figure 30. Distinguished vertex and $\boldsymbol{\omega}_{\rm dist}$.

Proof. Indeed, since $H=H(\mathbf{M},\mathrm{Ax},p)$ is finite, the Newton data Ω associated with the chart $(U,(x,y,z))$ has a Newton polyhedron with at least one vertex of the form $(0,-1,H)$ or $(-1,0,H)$.

In the latter case, we are done. In the former case, it is immediate that a change of coordinates as described in the statement leads to the desired situation. □

Let us fix a strongly adapted local chart $(U,(x,y,z))$ at $p\in\mathrm{NElem}\setminus\mathfrak{D}$. Let Ω be the corresponding Newton data for $(\mathbf{M},\mathrm{Ax})$ and let $\mathcal{N}(\Omega)$ be its Newton polyhedron.

The *distinguished weight-vector* for $(\mathbf{M},\mathrm{Ax})$ at p (with respect to the fixed chart $(U,(x,y,z))$) is the weight-vector $\boldsymbol{\omega}_{\rm dist}\in\mathbf{N}^3_{>0}$ of minimal norm, for which there exists an integer $\mu\in\mathbf{Z}$ such that

$$\mathcal{N}\cap\{\mathbf{v}\in\mathbf{R}^3:\langle\boldsymbol{\omega}_{\rm dist},\mathbf{v}\rangle=\mu\}=\{\mathbf{d}\},$$

where $\mathbf{d}=(-1,0,H(\mathbf{M},\mathrm{Ax},p))$ is the distinguished vertex. In other words, there exists an integer μ such that the plane $\{\mathbf{v}:\langle\boldsymbol{\omega}_{\rm dist},\mathbf{v}\rangle=\mu\}$ intersects $\mathcal{N}$ at the single point $\mathbf{d}$.

The *distinguished vertex blowing-up* of $(\mathbf{M},\mathrm{Ax})$ at p (with respect to the chart $(U,(x,y,z))$) is the $\boldsymbol{\omega}_{\rm dist}$-weighted blowing-up

$$\Phi:\widetilde{\mathbf{M}}\longrightarrow\mathbf{M}$$

with center at p, relative to the local trivialization given by $(U,(x,y,z))$.

Proposition 5.16. *Let $\Phi:\widetilde{\mathbf{M}}\to\mathbf{M}$ be as above. Then, there exists an axis $\widetilde{\mathrm{Ax}}$ for $\widetilde{\mathbf{M}}$ such that*

$$\mathfrak{h}(\widetilde{\mathbf{M}},\widetilde{\mathrm{Ax}},\tilde{p})\leqslant H(\mathbf{M},\mathrm{Ax},p)$$

for each point $\tilde{p}\in\mathrm{NElem}(\widetilde{\mathbf{M}})\cap\Phi^{-1}(p)$.

Proof. Let Ω be the Newton data associated with the local chart $(U,(x,y,z))$. We consider separately points lying in the domain of the z-, x- and y-directional charts of the blowing-up, and use the computations made in the previous subsection.

In the z-directional chart, Lemma 4.51 implies that $\mathrm{Bl}_z\Omega$ is in a final situation.

In the x-directional chart, it follows from Lemma 4.31 that the distinguished vertex $\mathbf{d}=(0,-1,H(\mathbf{M},\mathrm{Ax},p))$ becomes the higher vertex of $\mathrm{Bl}_x\Omega$. As a consequence,

$$\mathfrak{h}(\widetilde{\mathbf{M}},\widetilde{\mathrm{Ax}},\tilde{p})\leqslant H(\mathbf{M},\mathrm{Ax},p)$$

for each nonelementary point $\tilde{p}\in\Phi^{-1}(p)$ which lies in the domain of the x-directional chart.

In the y-directional chart, it follows from Lemma 4.48 that the distinguished vertex $\mathbf{d}=(0,-1,H(\mathbf{M},\mathrm{Ax},p))$ is mapped to the point $\tilde{\mathbf{d}}=(0,0,H(\mathbf{M},\mathrm{Ax},p))$ which belongs to the support of $\mathrm{Bl}_y\Omega$. Moreover,

$$\mathrm{supp}(\mathrm{Bl}_y\Omega)\cap(\{0\}\times\mathbf{Z}^2)=\{\tilde{\mathbf{d}}\}.$$

Therefore, we conclude that $\mathfrak{h}(\widetilde{\mathbf{M}},\widetilde{\mathrm{Ax}},\tilde{p})\leqslant H(\mathbf{M},\mathrm{Ax},p)$ for each point $\tilde{p}\in\Phi^{-1}(p)$ which lies in the domain of the y-directional chart.

To finish the proof, we can define an axis $\widetilde{\mathrm{Ax}}$ for $\widetilde{\mathbf{M}}$ exactly as in the proof of Theorem 4.29. □

5.7. Nonequireducible points are discrete

Let us now prove that the set of nonequireducible points in $\mathrm{NElem}\setminus\mathfrak{D}$ is finite on each compact subset of the ambient space.

The first lemma is an easy result of analytic geometry.

LEMMA 5.17. *Given an arbitrary point $p\in\mathrm{NElem}$, there exists an open neighborhood $U\subset M$ of p such that each point $q\in(\mathrm{NElem}\setminus\mathfrak{D})\cap(U\setminus\{p\})$ is smooth.*

Proof. This is obvious, since the set of nonsmooth points is a Zariski closed subset of the analytic set NElem. □

LEMMA 5.18. *Let $p\in\mathrm{NElem}\setminus\mathfrak{D}$ be an equireducible point. Then, there exists an open neighborhood $V\subset M$ of p such that each point $q\in\mathrm{NElem}\cap V$ is also an equireducible point. Moreover, if $(U,(x,y,z))$ is an equireduction chart at p, then the translated coordinates*

$$\tilde{x}=x+\varrho,\quad \tilde{y}=y\quad \textit{and}\quad \tilde{z}=z$$

are equireduction coordinates at q (for some appropriately chosen constant $\varrho\in\mathbf{R}$).

Proof. We have to prove that the generic Newton data associated with the translated coordinates $(\tilde{x},\tilde{y},\tilde{z})=(x+\varrho,y,z)$ is edge-stable, for all $|\varrho|$ sufficiently small.

Suppose, by contradiction, that this is not the case. Then, for each $\varepsilon>0$, there exists a constant $\varrho\in\mathbf{R}$ with $|\varrho|<\varepsilon$ such that the corresponding translated point q (with the coordinates $(\tilde{x},\tilde{y},\tilde{z})$) satisfies the following condition: there exists a map in G_Δ of the form

$$\bar{z}=\tilde{z}+\xi\tilde{y}^{\Delta},$$

such that the transformed generic Newton data $\bar{\Omega}_G$ (at the point q) belongs to the class $\mathrm{New}_{\bar{\Delta}}^{\mathbf{m}_G}$ for some $\bar{\Delta}>\Delta$.

Applying the local blowing-up $\Phi\colon\widetilde{\mathbf{M}}\to\mathbf{M}\cap U$ for $(\mathbf{M},\mathrm{Ax})$ at p, we can choose a point $\bar{q}\in\Phi^{-1}(q)$ (for instance, the origin in the y-directional chart of the blowing-up) such that its virtual height satisfies

$$\mathfrak{h}(\bar{q})\geqslant\mathfrak{h}_G(q)\geqslant\mathfrak{h}_G(p).$$

But this contradicts Theorem 5.13 and Proposition 5.2. □

As an immediate consequence, we get the following result.

COROLLARY 5.19. (i) *The set of equireducible points*, *denoted by* Eq, *is an open subset of* $\mathrm{NElem}\setminus\mathfrak{D}$ (*for the topology induced by the topology of* M).

(ii) *Given two equireducible points p and q in the same connected component of* Eq, *the corresponding generic Newton data necessarily belong to the same class* $\mathrm{New}_{\Delta}^{\mathbf{m}_G}$.

Finally, we can state the main result of this subsection.

PROPOSITION 5.20. *The set of nonequireducible points in* $\mathrm{NElem}\setminus\mathfrak{D}$ *is finite on each compact subset* $K\subset M$.

Proof. By the compactness of $\mathrm{NElem}\cap K$, we just need to prove the following claim.

Claim. For each point $p\in\mathrm{NElem}\cap K$, there exists an open neighborhood $U\subset M$ of p such that each point in the set $(\mathrm{NElem}\setminus\mathfrak{D})\cap(U\setminus\{p\})$ is equireducible.

In order to prove the claim, we consider separately the following cases:

(1) $p\in\mathrm{NElem}\setminus\mathfrak{D}$ is an equireducible point;

(2) $p\in\mathrm{NElem}\cap\mathfrak{D}$;

(3) $p\in\mathrm{NElem}\setminus\mathfrak{D}$ is nonequireducible.

In case (1), the claim is a direct consequence of Corollary 5.19.

In case (2), it suffices to prove that the result holds for each irreducible branch of the (possibly singular) germ of analytic sets NElem_p. Let us fix one such branch, which we denote by γ. Let Y_p be the local blowing-up center for $(\mathbf{M}, \mathrm{Ax})$ at p. We consider separately the following two cases:

(2.a) $\gamma = Y_p$;

(2.b) $\gamma \neq Y_p$.

In case (2.a), if we fix an arbitrary stable chart $(U, (x, y, z))$ at p, then necessarily

$$\gamma = \{y = z = 0\}.$$

Using the same reasoning as in the proof of Lemma 5.18, we conclude that, for each sufficiently small constant $\varrho \in \mathbf{R}$, the translated coordinates $(x+\varrho, y, z)$ are equireduction coordinates. Therefore, each point of γ which is sufficiently near p is equireducible.

In case (2.b), we consider the local blowing-up $\Phi: \widetilde{\mathbf{M}} \to \mathbf{M} \cap U$ for $(\mathbf{M}, \mathrm{Ax})$ at p. The strict transform of γ accumulates at some point

$$\tilde{p} \in \mathrm{NElem}(\widetilde{\mathbf{M}}) \cap \Phi^{-1}(p).$$

We can now repeat the analysis on this point $\tilde{p}$. If we fall in case (2.b), we make another local blowing-up and proceed inductively.

By Theorem 4.29, we necessarily fall in case (2.a) after a finite number of such steps.

Finally, in case (3), we argue as follows. Let us fix some strongly adapted local chart $(U, (x, y, z))$ at p and let $\Phi: \widetilde{\mathbf{M}} \to \mathbf{M} \cap U$ be a distinguished local blowing-up for $(\mathbf{M}, \mathrm{Ax})$ at p. Then, looking at the strict transform of NElem and using the compactness of $\Phi^{-1}(p)$, the result immediately follows from case (2). □

We shall say that a controlled singularly foliated manifold $(\mathbf{M}, \mathrm{Ax})$ is *equireducible outside the divisor* if each point in $\mathrm{NElem}(\mathbf{M}) \setminus \mathfrak{D}$ is equireducible.

Lemma 5.21. *Let* $(\mathbf{M}, \mathrm{Ax})$ *be a controlled singularly foliated manifold and* $U \subset M$ *be a relatively compact subset. Let* $\{p_1, ..., p_k\} \subset \mathrm{NElem} \setminus \mathfrak{D}$ *be the distinct nonequireducible points of* $(\mathbf{M}, \mathrm{Ax})$ *on* $U \setminus \mathfrak{D}$. *Then, there exist a blowing-up*

$$\Phi: \widetilde{\mathbf{M}} \longrightarrow \mathbf{M}$$

with center on p_1, *and an axis* $\widetilde{\mathrm{Ax}}$ *for* $\widetilde{\mathbf{M}}$, *such that the points*

$$\{\Phi^{-1}(p_2), ..., \Phi^{-1}(p_k)\} \subset \mathrm{NElem}(\widetilde{\mathbf{M}}) \setminus \widetilde{\mathfrak{D}}$$

are the only nonequireducible points for $(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}})$ *on the relatively compact subset*

$$\Phi^{-1}(U) \setminus \widetilde{\mathfrak{D}}.$$

Proof. We fix a strongly adapted local chart $(U,(x,y,z))$ at the point p_1 and let

$$\Phi\colon \widetilde{\mathbf{M}} \longrightarrow \mathbf{M}$$

be a distinguished vertex blowing-up at p_1, as defined in §5.6. The result immediately follows from Proposition 5.16. □

COROLLARY 5.22. *Let* $(\mathbf{M},\mathrm{Ax})$ *be a controlled singularly foliated manifold and let* $U\subset M$ *be a relatively compact subset. Then, there exist a finite sequence of blowing-ups*

$$\mathbf{M}=\mathbf{M}_0 \xleftarrow{\Phi_1} \mathbf{M}_1 \xleftarrow{\Phi_2} \dots \xleftarrow{\Phi_k} \mathbf{M}_k,$$

and an axis Ax_k *for* $\mathbf{M}_k$, *such that* $(\mathbf{M}_k,\mathrm{Ax}_k)$ *is equireducible outside the divisor, when restricted to* $(\Phi_k\circ\dots\circ\Phi_1)^{-1}(U)$.

Let $(\mathbf{M},\mathrm{Ax})$ be a singularly foliated manifold which is equireducible outside the divisor. Then, each connected component Y of $\mathrm{NElem}\setminus\mathfrak{D}$ is a smooth 1-dimensional analytic curve. In this case, we define the *generic virtual height* for $(\mathbf{M},\mathrm{Ax})$ along Y as the natural number

$$\mathfrak{h}(\mathbf{M},\mathrm{Ax},Y):=\mathfrak{h}_G(\mathbf{M},\mathrm{Ax},p),$$

where p is an arbitrary point on Y. Using Corollary 5.19, one concludes that $\mathfrak{h}(\mathbf{M},\mathrm{Ax},Y)$ is independent of the choice of the particular point $p\in Y$.

5.8. Extending the invariant to $\mathrm{NElem}\setminus\mathfrak{D}$

In this subsection, let us assume that $(\mathbf{M},\mathrm{Ax})$ is equireducible outside the divisor. In particular, the virtual height function $\mathfrak{h}(\mathbf{M},\mathrm{Ax},\cdot)\colon \mathrm{NElem}\cap\mathfrak{D}\to\mathbf{N}$ can be extended to the whole set NElem by setting

$$\mathfrak{h}(\mathbf{M},\mathrm{Ax},\cdot):=\mathfrak{h}_G(\mathbf{M},\mathrm{Ax},\cdot) \quad \text{on } \mathrm{NElem}\setminus\mathfrak{D}.$$

We denote this function shortly by $\mathfrak{h}(p)$.

The *stratum of virtual height* h is the subset

$$S_h=\{p\in\mathrm{NElem}:\mathfrak{h}(p)=h\}.$$

LEMMA 5.23. *Given a connected equireducible curve* $Y\subset\mathrm{NElem}\setminus\mathfrak{D}$, *let* $\overline{Y}\subset\mathrm{NElem}$ *be the smallest closed analytic subset which contains* Y. *Then, for each point* $p\in\overline{Y}\cap\mathfrak{D}$, *we have* $\mathfrak{h}(Y)\leqslant\mathfrak{h}(p)$.

Proof. Let $(U,(x,y,z))$ be a stable adapted chart at p. Firstly, suppose that the point p is such that $Y_p=\overline{Y}\cap U$ (i.e. $\overline{Y}$ locally coincides with the local blowing-up at p). Then, it follows from the argument used in the proof of Lemma 5.4 that $\mathfrak{h}(Y)=\mathfrak{h}(p)$.

Suppose now that $Y_p\neq\overline{Y}\cap U$ and assume, by contradiction, that

$$\mathfrak{h}(Y)>\mathfrak{h}(p).$$

We make the local blowing-up $\Phi:\widetilde{\mathbf{M}}\to\mathbf{M}\cap U$ for $(\mathbf{M},\mathrm{Ax})$ at p, and look at the strict transform Y' of the curve Y. The closure of this curve $\overline{Y}'$ necessarily intersects the exceptional divisor $\widetilde{D}=\Phi^{-1}(Y_p)$ in at least one nonelementary point $\tilde{p}$. Moreover,

$$\mathfrak{h}(Y')=\mathfrak{h}(Y)>\mathfrak{h}(p)\geqslant\mathfrak{h}(\tilde{p}),$$

as a consequence of Theorem 4.29.

Let us now set $p:=\tilde{p}$ and $Y:=Y'$, and iterate the process. Theorem 4.29 implies that after some finite number of iterations, we fall into a situation where $\widetilde{D}$ has no nonelementary points. This is a contradiction. □

PROPOSITION 5.24. *The function* $\mathfrak{h}:\mathrm{NElem}\to\mathbf{N}$ *is upper semicontinuous.*

Proof. This is an immediate consequence of Proposition 5.2 and Lemma 5.23. □

The Newton invariant $\mathrm{inv}(\mathbf{M},\mathrm{Ax},p)$ can also be defined globally on NElem. We denote it shortly by $\mathrm{inv}(p)$ and remark that the following relations hold:

(1) if $p\in S_h\setminus\mathfrak{D}$, then $\mathrm{inv}(p)=(h,0,0,0,0,0)$;

(2) if $p\in S_h\cap\mathfrak{D}$ is such that $\#\iota_p=2$, then $\mathrm{inv}(p)=(h,1,h,1,*,*)$;

(3) if $p\in S_h\cap\mathfrak{D}$ is such that $\#\iota_p=1$, then either

$$\mathrm{inv}(p)=(h,0,m_3,0,*,*)\quad\text{or}\quad\mathrm{inv}(p)=(h,1,h,0,*,*)$$

for some $m_3\geqslant h$ (where the $*$'s denote some arbitrary natural numbers).

As a consequence of this, combined with Propositions 5.5 and 5.24, we conclude the following result.

PROPOSITION 5.25. *The function* $\mathrm{inv}:\mathrm{NElem}\to\mathbf{N}^6$ *is upper semicontinuous* (*for the lexicographical ordering on* $\mathbf{N}^6$).

5.9. Extended center, bad points and bad trees

A controlled singularly foliated manifold $(\mathbf{M},\mathrm{Ax})$ will be called a *restriction* if it is given by the restriction of a controlled singularly foliated manifold $(\mathbf{M}',\mathrm{Ax}')$ to some relatively compact open subset U of the ambient space M'.

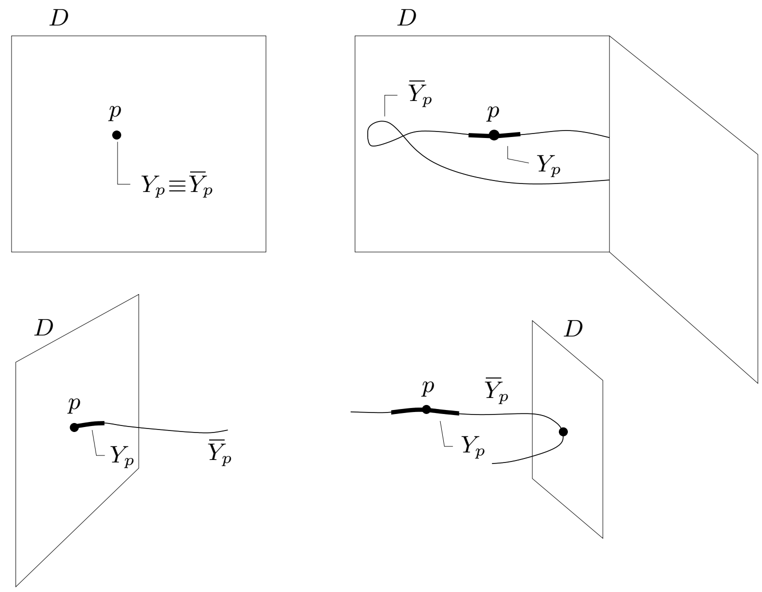


Figure 31. The extended centers.

In this subsection, we shall suppose that $(\mathbf{M}, \mathrm{Ax})$ is a restriction and, moreover, that it is equireducible outside the divisor. In particular, this implies (by the upper semicontinuity of the height function) that

$$\mathfrak{h}_{\max} := \sup\{\mathfrak{h}(p) : p \in \mathrm{NElem}\}$$

is a finite natural number and that the divisor list $\Upsilon \in \mathbf{L}$ has finite length.

The set $S_{\mathfrak{h}_{\max}} = \{p \in \mathrm{NElem} : \mathfrak{h}(p) = \mathfrak{h}_{\max}\}$ will be called the *stratum of maximal height.*

Lemma 5.26. *The stratum of maximal height $S_{\mathfrak{h}_{\max}}$ is a closed analytic subset of* NElem. *Moreover*, $S_{\mathfrak{h}_{\max}} \cap \mathfrak{D}$ *is a union of isolated points and closed analytic curves which have normal crossings with the divisor.*

Proof. The set $S_{\mathfrak{h}_{\max}}$ is closed by the upper semicontinuity of the function $\mathfrak{h}$. Moreover, it follows from conditions (i) and (ii) of Proposition 5.2 that the set $S_{\mathfrak{h}_{\max}} \cap \mathfrak{D}$ is locally smooth at each point $p \in S_{\mathfrak{h}_{\max}} \cap \mathfrak{D}$. □

The *extended center* associated with a point $p \in \mathrm{NElem}$ is the smallest closed analytic subset $\overline{Y}_p \subset \mathrm{NElem}$ which coincides with the local blowing-up center Y_p in a neighborhood of p.

Remark 5.27. For instance, if $Y_p = \{p\}$, then $\overline{Y}_p = \{p\}$. On the other hand, if Y_p is contained in some irreducible divisor component $D \subset \mathfrak{D}$, then $\overline{Y}_p$ is entirely contained in D.

We say that $\overline{Y}_p$ is a *divisorial center* if $\overline{Y}_p \subset \mathfrak{D}$.

LEMMA 5.28. *Let $p \in S_{\mathfrak{h}_{\max}}$ be such that the extended center $\overline{Y}_p$ is divisorial. Then, $\overline{Y}_p$ is either an isolated point or a smooth analytic curve which has normal crossings with the divisor $\mathfrak{D}$.*

Proof. This is an obvious consequence of Lemma 5.26, since the extended center $\overline{Y}_p$ is an irreducible closed analytic subset of $S_{\mathfrak{h}_{\max}} \cap \mathfrak{D}$. □

We say that the extended center $\overline{Y}_p$ is *permissible* at a point $q \in \overline{Y}_p$ if $\overline{Y}_q \equiv \overline{Y}_p$. In other words, $\overline{Y}_p$ is permissible at q if the the local blowing-up center for $(\mathbf{M}, \mathrm{Ax})$ at q locally coincides with $\overline{Y}_p$. A point $q \in \overline{Y}_p$ is called a *bad point* if $\overline{Y}_p$ is not permissible at q. We denote by $\mathrm{Bad}(p)$ the set of all bad points in $\overline{Y}_p$. We shall say that the extended center $\overline{Y}_p$ is *globally permissible* if $\mathrm{Bad}(p) = \varnothing$.

PROPOSITION 5.29. *Fix a point $p \in S_{\mathfrak{h}_{\max}}$.*

(i) *If $Y_p = \{p\}$, then $\overline{Y}_p$ is globally permissible.*

(ii) *Suppose that $\overline{Y}_p$ is a smooth curve contained in some divisor component $D_i \subset \mathfrak{D}$. Then, each point of* $\mathrm{Bad}(p)$ *is contained in the intersection $D_i \cap D_j$ for some index $j > i$.*

(iii) *If $p \in S_{\mathfrak{h}_{\max}} \setminus \mathfrak{D}$ is an equireducible point, then* $\mathrm{Bad}(p)$ *is a subset of $\overline{Y}_p \cap \mathfrak{D}$.*

Proof. Fact (i) is trivial. To prove fact (ii), notice that for each point $q \in \overline{Y}_p$, the following three situations can appear:

(a) $\iota_q = [i]$;

(b) $\iota_q = [i, k]$;

(c) $\iota_q = [j, i]$;

for some indices $k < i < j$. In cases (a) and (b), it is clear that the extended center $\overline{Y}_p$ is permissible at q, because $\Delta_1^q = \Delta_1^p > 0$ (by Lemma 5.4). Therefore, a bad point of $\overline{Y}_p$ necessarily lies in the intersection of D_i with some divisor D_j of larger index.

Fact (iii) is a direct consequence of the assumption that $(\mathbf{M}, \mathrm{Ax})$ is equireducible outside the divisor $\mathfrak{D}$. □

COROLLARY 5.30. *For each point $p \in S_{\mathfrak{h}_{\max}} \cap D_i$, the following properties hold*:

(1) *if $\#\iota_p = 2$, then the set* $\mathrm{Bad}(p)$ *has at most one point*;

(2) *if $\#\iota_p = 1$ and $\overline{Y}_p \subset D_i$, then the set* $\mathrm{Bad}(p)$ *has at most two points.*

In both cases each point $q \in \mathrm{Bad}(p)$ is such that $\iota_q = [i, j]$ for some index $j > i$.

Proof. This is a direct consequence of Proposition 5.29 and the description of the set $S_{\mathfrak{h}_{\max}} \cap \mathfrak{D}$ given by Lemma 5.26. □

LEMMA 5.31. *Let $p \in S_{\mathfrak{h}_{\max}} \setminus \mathfrak{D}$ be an equireducible point. Then, for each point*

$$q \in \mathrm{Bad}(p),$$

the associated local blowing-up center Y_q is such that

$$\overline{Y}_q \subset \mathfrak{D},$$

i.e. $\overline{Y}_q$ *is necessarily a divisorial center.*

Proof. Indeed, suppose by contradiction that Y_q is not a divisorial center and $\overline{Y}_q \neq \overline{Y}_p$.

We fix a stable local chart $(U,(x,y,z))$ at q and let

$$\Phi: \widetilde{\mathbf{M}} \longrightarrow \mathbf{M} \cap U$$

be the local blowing-up (with center Y_q) of $(\mathbf{M}, \mathrm{Ax})$ at q. It follows from Propositions 4.47 and 4.52 that each point $\tilde{p} \in \Phi^{-1}(q)$ is such that either the Newton data is in a final situation or $\mathfrak{h}(\tilde{p}) < \mathfrak{h}(q) = \mathfrak{h}_{\max}$.

On the other hand, the strict transform of $\overline{Y}_p$ under Φ contains at least one point of $\Phi^{-1}(q)$. But this contradicts the fact that $\overline{Y}_p \subset S_{\mathfrak{h}_{\max}}$. □

A *bad chain* is a (possibly infinite) sequence of points $\{p_n\}_{n \geqslant 0}$ which is contained in $S_{\mathfrak{h}_{\max}}$ and is such that

$$p_{n+1} \in \mathrm{Bad}(p_n) \quad \text{for } n \geqslant 0.$$

We shall say that a finite bad chain $\{p_0, ..., p_l\}$ is *complete* if $\mathrm{Bad}(p_l) = \varnothing$. The number l will be called the *length* of the complete bad chain. A complete bad chain of length 2 is illustrated in Figure 32.

Remark 5.32. It follows from Lemma 5.31 and Corollary 5.30 that for a bad chain $\{p_n\}_{n \geqslant 0}$, we always have $\#\iota_{p_1} \geqslant 1$ and $\iota_{p_n} = 2$ for all $n \geqslant 2$.

Lemma 5.33. *Each bad chain has a finite number of points.*

Proof. By Remark 5.32, each bad chain $\{p_n\}_{n \geqslant 0}$ is such that $\#\iota_{p_n} = 2$ for all $n \geqslant 2$. Moreover, if we write $\iota_{p_n} = [j_n, i_n]$ then

$$i_n < j_n = i_{n+1} < j_{n+1} = i_{n+2} < j_{n+2} = ...,$$

and therefore the indices $\{i_n\}_{n \geqslant 0} \subset \Upsilon$ form a strictly increasing sequence (where Υ is the list of divisor indices). Since we supposed that $(\mathbf{M}, \mathrm{Ax})$ is a restriction, the list Υ is necessarily finite. The lemma is proved. □

Given a point $p \in S_{\mathfrak{h}_{\max}}$, the set of bad chains *starting at* p is the set $\mathcal{B}(p)$ of all complete bad chains $\{p_n\}_{n \geqslant 0}$ such that $p_0 = p$.

Remark 5.34. It follows from Lemma 5.33 that the set $\mathcal{B}(p)$ has only a finite number of elements.

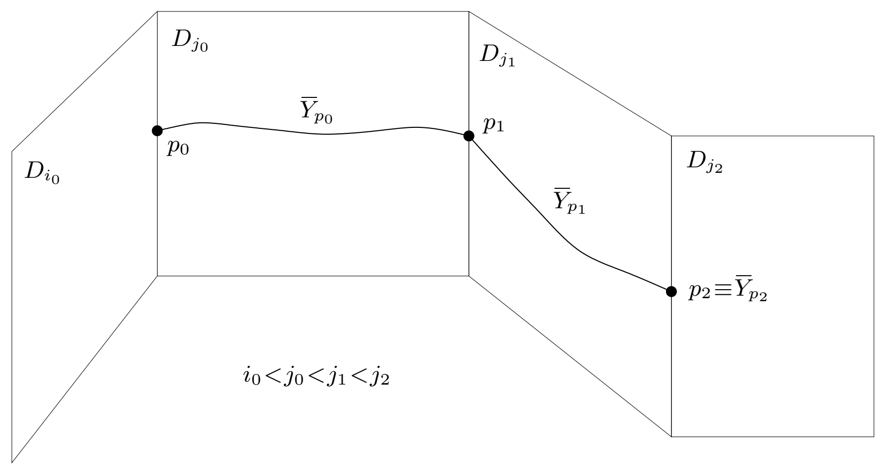


Figure 32. The bad chain $\{p_0, p_1, p_2\}$ (here $\mathrm{Bad}(p_2)=\varnothing$).

More generally, given a finite set of points $P\subset S_{\mathfrak{h}_{\max}}$, we define the *P-bad chain* as the union

$$\mathcal{B}(P):=\bigcup_{p\in P}\mathcal{B}(p)$$

of all bad chains starting at points in P. Associated to $\mathcal{B}(P)$, let us consider a directed graph $T=(V,E)$ defined as follows:

(i) the set of vertices V corresponds to the set of points of all bad chains starting at points in P (for simplicity, we identify each element of V with the corresponding point in the bad chain);

(ii) the directed edge $q\rightarrow r$ belongs to the set of edges E if there exists a bad chain $\{p_n\}_{n=0}^{l}$ in $\mathcal{B}(P)$ such that

$$p_i=q \quad \text{and} \quad p_{i+1}=r$$

for some $0\leqslant i\leqslant l-1$.

LEMMA 5.35. *The graph $T=(V,E)$ is a directed tree.*

Proof. We need to prove that T has no cycles. Let us suppose, by contradiction, that there exists a cycle in T,

$$q_0\longrightarrow q_1\longrightarrow \dots \longrightarrow q_r\longrightarrow q_{r+1}=q_0,$$

where $q_{n+1}\in\mathrm{Bad}(q_n)$ for each $0\leqslant n\leqslant r$. Let us write $\iota_{q_n}=[i_n]$ (if $\#\iota_{q_n}=1$) and $\iota_{q_n}=[j_n,i_n]$ (if $\#\iota_{q_n}=2$).

First of all, suppose that the extended center $\overline{Y}_{q_0}$ is divisorial (i.e. contained in $\mathfrak{D}$). Then, it follows from Corollary 5.30 that the sequence $i_1<i_2<\dots<i_n$ is strictly increasing. Thus no cycle may appear.

Suppose now that $\overline{Y}_{q_0}$ is not divisorial. Then, Lemma 5.31 implies that $\overline{Y}_{q_n}$ is divisorial, for all $n\geqslant 1$. This contradicts the fact that $q_{n+1}=q_0$. □

Definition 5.36. The directed tree $T=(V,E)$ defined above will be called the *bad tree* associated with P. We shall denote it by $\mathrm{Tr}\mathcal{B}(P)$.

From now on, we adopt the usual nomenclature for trees. Thus, a *branch* is any succession of points and directed edges:

$$p_0 \longrightarrow p_1 \longrightarrow \ldots \longrightarrow p_k.$$

In this case, the number k is called the *length* of the branch. A point $q\in V$ is called a *descendant* of a point p if there exists a branch of positive length as above such that $p_0=p$ and $p_k=q$. A point q is called a *terminal* if it has no descendants in the tree.

Remark 5.37. For each terminal point $q\in\mathrm{Tr}\mathcal{B}(P)$, the extended center $\overline{Y}_q$ is globally permissible (because $\mathrm{Bad}(q)=\varnothing$).

The *maximal length* of a bad tree is the length $L(\mathrm{Tr}\mathcal{B}(P))\in\mathbf{N}$ of the longest branch of $\mathrm{Tr}\mathcal{B}(P)$.

Let $F\subset\mathcal{B}(P)$ be the set of all terminal points which lie in branches of maximal length (i.e. those branches of $\mathrm{Tr}\mathcal{B}(P)$ which have length $L(\mathrm{Tr}\mathcal{B}(P))$). We define the *maximal final invariant* of $\mathrm{Tr}\mathcal{B}(P)$ as

$$\mathrm{inv}(\mathrm{Tr}\mathcal{B}(P)) := \max{}_{\mathrm{lex}}\{\mathrm{inv}(q) : q\in F\},$$

where the maximum is taken in the lexicographical ordering in $\mathbf{N}^6$. The *maximal final locus* is the finite set of points

$$\mathrm{Loc}(\mathrm{Tr}\mathcal{B}(P)) := \{q\in F : \mathrm{inv}(q)=\mathrm{inv}(\mathrm{Tr}\mathcal{B}(P))\}.$$

Finally, we define the *multiplicity* of the bad tree $\mathrm{Tr}\mathcal{B}(P)$ as the vector

$$\mathrm{Mult}(\mathrm{Tr}\mathcal{B}(P)) := (L(\mathrm{Tr}\mathcal{B}(P)), \mathrm{inv}(\mathrm{Tr}\mathcal{B}(P)), \#\mathrm{Loc}(\mathrm{Tr}\mathcal{B}(P)))\in\mathbf{N}^8, \tag{35}$$

where $\#\mathrm{Loc}(\mathrm{Tr}\mathcal{B}(P))$ is the cardinality of the set $\mathrm{Loc}(\mathrm{Tr}\mathcal{B}(P))$.

5.10. Maximal invariant locus and global multiplicity

In this subsection, we continue to assume that $(\mathbf{M},\mathrm{Ax})$ is a a controlled singularly foliated manifold which is a restriction (see §5.9) and equireducible outside the divisor.

Therefore, the maximum of the invariant $\mathrm{inv}(p)$,

$$\mathrm{inv}_{\max}(\mathbf{M},\mathrm{Ax}) := \sup{}_{\mathrm{lex}}\{\mathrm{inv}(p) : p \in \mathrm{NElem}\},$$

is a finite vector in $\mathbf{N}^6$. If $(\mathbf{M},\mathrm{Ax})$ is clear from the context, we denote this number simply by $\mathrm{inv}_{\max}$. The subset

$$S_{\mathrm{inv}_{\max}} := \{p \in \mathrm{NElem} : \mathrm{inv}(p) = \mathrm{inv}_{\max}\} \subset S_{\mathfrak{h}_{\max}}$$

will be called the *maximal invariant stratum* of $(\mathbf{M},\mathrm{Ax})$.

Consider the subsets $\mathfrak{D}_i := \{p \in \mathrm{NElem} : \#\iota_p = i\}$ for $i=0,1,2$. We establish the following definition: we say that $S_{\mathrm{inv}_{\max}}$ is of

$$\begin{cases} 2\text{-}\textit{boundary type}, & \text{if } S_{\mathrm{inv}_{\max}} \cap \mathfrak{D}_2 \neq \varnothing, \\ 1\text{-}\textit{boundary type}, & \text{if } S_{\mathrm{inv}_{\max}} \cap \mathfrak{D}_2 = \varnothing \text{ and } S_{\mathrm{inv}_{\max}} \cap \mathfrak{D}_1 \neq \varnothing, \\ 0\text{-}\textit{boundary type}, & \text{if } S_{\mathrm{inv}_{\max}} \cap (\mathfrak{D}_1 \cup \mathfrak{D}_2) = \varnothing \text{ and } S_{\mathrm{inv}_{\max}} \cap \mathfrak{D}_0 \neq \varnothing. \end{cases}$$

Using this classification, the following result establishes some properties of $S_{\mathrm{inv}_{\max}}$.

LEMMA 5.38. *The maximal invariant stratum has the following properties*:

(i) *if* $S_{\mathrm{inv}_{\max}}$ *is of* 2-*boundary type*, *then* $S_{\mathrm{inv}_{\max}} \subset \mathfrak{D}_2$;

(ii) *if* $S_{\mathrm{inv}_{\max}}$ *is of* 1-*boundary type*, *then* $S_{\mathrm{inv}_{\max}} \subset \mathfrak{D}_1$;

(iii) *if* $S_{\mathrm{inv}_{\max}}$ *is of* 0-*boundary type*, *then* $S_{\mathrm{inv}_{\max}} = S_{\mathfrak{h}_{\max}} \subset \mathfrak{D}_0$.

Proof. The result is a direct consequence of the definition of inv and Remark 5.3. □

In the next lemmas, we give a more detailed description of $S_{\mathfrak{h}_{\max}}$.

LEMMA 5.39. *A* 2-*boundary type* $S_{\mathrm{inv}_{\max}}$ *is the union of a finite number of distinct points* $\{p_1, ..., p_m\}$.

Proof. This follows immediately from the description of the set $S_{\mathrm{inv}_{\max}} \cap \mathfrak{D}$ which is given in Lemma 5.26. □

LEMMA 5.40. *A* 1-*boundary type* $S_{\mathrm{inv}_{\max}}$ *is a finite union of distinct closed analytic sets*

$$Y_1 \cup ... \cup Y_r \cup \{p_1, ..., p_m\} \cup \{q_1, ..., q_n\},$$

for some natural numbers $r, m, n \in \mathbf{N}$, *such that the following conditions hold*:

(i) $Y_1, ..., Y_r$ *are globally permissible* 1-*dimensional extended centers contained in* $\mathfrak{D}_1$;

(ii) *each* p_i *is an isolated point of* $S_{\mathrm{inv}_{\max}} \cap \mathfrak{D}_1$ *such that* $\overline{Y}_{p_i}$ *is a globally permissible extended center contained in* $\mathfrak{D}_1$;

(iii) *each* q_j *is an isolated point of* $S_{\mathrm{inv}_{\max}} \cap \mathfrak{D}_1$ *such that the extended center* $\overline{Y}_{q_j}$ *is not divisorial.*

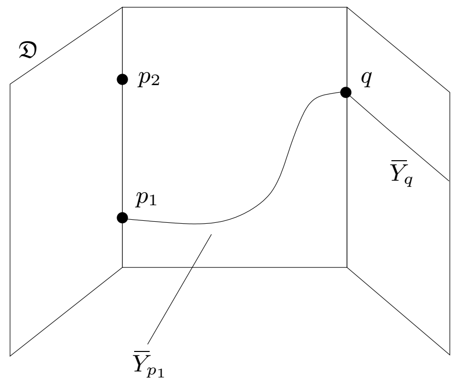


Figure 33. The 2-boundary maximal invariant stratum. Here, $\mathrm{Bad}(p_1)=\{q\}$.

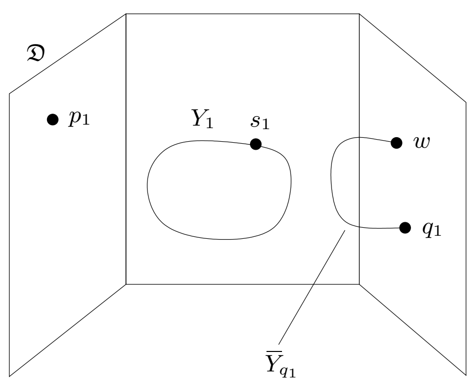


Figure 34. The 1-boundary maximal invariant stratum. Here, $\mathrm{Bad}(q_1)=\{w\}$.

Proof. This follows immediately from the description of the set $S_{\mathfrak{h}_{\max}}\cap\mathfrak{D}$ which is given in Lemma 5.26, and the assumption that $S_{\mathrm{inv}_{\max}}\cap\mathfrak{D}_2=\varnothing$. □

LEMMA 5.41. *A* 0*-boundary type* $S_{\mathrm{inv}_{\max}}$ *is a finite union* $Y_1\cup...\cup Y_r$ *of distinct globally permissible* 1*-dimensional extended centers which are contained in* $\mathfrak{D}_0$.

Proof. It follows immediately from the assumptions that $S_{\mathrm{inv}_{\max}}\cap(\mathfrak{D}_2\cup\mathfrak{D}_1)=\varnothing$ and that $(\mathbf{M},\mathrm{Ax})$ is equireducible outside the divisor. □

Based on the above description of $S_{\mathrm{inv}_{\max}}$, we state the following definition.

Definition 5.42. A *maximal point locus* of $(\mathbf{M},\mathrm{Ax})$ is a finite collection of distinct points $P_{\max}\subset S_{\mathrm{inv}_{\max}}$, which is obtained as follows:

(i) if $S_{\mathrm{inv}_{\max}}$ is a 2-boundary maximal invariant stratum, then

$$P_{\max}:=\{p_1,...,p_m\}=S_{\mathrm{inv}_{\max}}, \qquad (36)$$

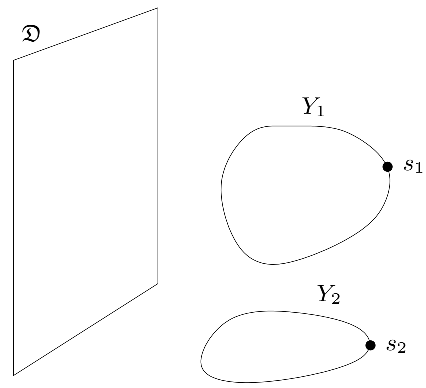


Figure 35. The 0-boundary maximal invariant stratum.

where $\{p_1, ..., p_m\}$ is given by Lemma 5.39 (see Figure 33);

(ii) if $S_{\text{inv}_{\max}}$ is a 1-boundary maximal invariant stratum, then

$$P_{\max} = \{s_1, ..., s_r, p_1, ..., p_m, q_1, ..., q_n\}, \tag{37}$$

where each s_i is an arbitrary point contained in the curve Y_i, and the set

$$\{Y_1, ..., Y_r, p_1, ..., p_m, q_1, ..., q_n\}$$

is given by Lemma 5.40 (see Figure 34);

(iii) if $S_{\text{inv}_{\max}}$ is a 0-boundary maximal invariant stratum, then

$$P_{\max} := \{s_1, ..., s_r\}, \tag{38}$$

where each s_i is an arbitrary point contained in the curve Y_i, and the set $\{Y_1, ..., Y_r\}$ is given by Lemma 5.41 (see Figure 35).

The *global multiplicity* associated with $(\mathbf{M}, \text{Ax})$ is the vector

$$\text{Mult}(\mathbf{M}, \text{Ax}) := (\text{inv}_{\max}(\mathbf{M}, \text{Ax}), \text{Mult}(\text{Tr}\mathcal{B}(P_{\max}))),$$

where $P_{\max}$ is a maximal point locus of $(\mathbf{M}, \text{Ax})$ and $\text{Mult}(\text{Tr}\mathcal{B}(P_{\max}))$ is the multiplicity of the bad tree $\text{Tr}\mathcal{B}(P_{\max})$ (see definition (35)).

Remark 5.43. It is obvious that the value of $\text{Mult}(\mathbf{M}, \text{Ax})$ is independent of the choice of the points $s_i \in Y_i$ $(i=1, ..., r)$ which is made in (37) and (38).

The globally permissible extended center $Y = \overline{Y}_q$ which is associated with a terminal point $q \in \text{Loc}(\text{Tr}\mathcal{B}(P_{\max}))$ will be called a *blowing-up center* for $(\mathbf{M}, \text{Ax})$.

PROPOSITION 5.44. *Let* $Y \subset M$ *be a blowing-up center for* $(\mathbf{M}, \mathrm{Ax})$. *Then, there exists a weight-vector* $\boldsymbol{\omega} \in \mathbf{N}^3$ *such that the following properties hold*:

(i) *for every point* $p \in Y \setminus \mathfrak{D}$ *and every equireduction chart* $(U_p, (x_p, y_p, z_p))$ *for* $(\mathbf{M}, \mathrm{Ax})$ *at* p,

$$\boldsymbol{\omega} = \omega_p \quad \textit{and} \quad Y \cap U_p = Y_p$$

(*i.e.* ω *is the local weight-vector at* p *and* $Y \cap U_p$ *is the local blowing-up center*);

(ii) *for every point* $p \in Y \cap \mathfrak{D}$ *and every stable adapted chart* $(U_p, (x_p, y_p, z_p))$ *for* $(\mathbf{M}, \mathrm{Ax})$ *at* p,

$$\boldsymbol{\omega} = \omega_p \quad \textit{and} \quad Y \cap U_p = Y_p.$$

Moreover, the collection of charts $\{(U_p, (x_p, y_p, z_p))\}_{p \in A}$ *as defined above is an* $\boldsymbol{\omega}$-*weighted trivialization atlas for* $Y \subset M$.

Proof. Properties (i) and (ii) follow from the fact that Y is a globally permissible center.

In order to prove the last statement, we need to prove that the transition between two charts in the above trivialization, say $(U_p, (x_p, y_p, z_p))$ and $(U_q, (x_q, y_q, z_q))$, preserves the $\boldsymbol{\omega}$-quasihomogeneous structure on $\mathbf{R}^3$.

If Y is a single point, it suffices to apply Proposition 4.28. If Y is a smooth curve, then we can locally write

$$Y = \{x_p = z_p = 0\} \quad \text{or} \quad Y = \{y_p = z_p = 0\}.$$

In these cases we claim that, for each point $q \in Y$ which is sufficiently close to p, the respective translated chart,

$$(\tilde{x}_q, \tilde{y}_q, \tilde{z}_q) = (x_p, y_p - \varrho, z_p) \quad \text{or} \quad (\tilde{x}_q, \tilde{y}_q, \tilde{z}_q) = (x_p - \varrho, y_p, z_p),$$

is a stable local chart (or an equireduction chart) for $(\mathbf{M}, \mathrm{Ax})$ at q (for some conveniently chosen constant $\varrho \in \mathbf{R}$).

Indeed, this claim can be proved by easy modifications of the proofs of Lemmas 5.4 and 5.18. Using the claim, combined with Proposition 4.28 and Remark 5.12, we conclude that $\{(U_p, (x_p, y_p, z_p))\}_{p \in A}$ is a trivialization of Y which preserves the $\boldsymbol{\omega}$-quasihomogeneous structure on $\mathbf{R}^3$. □

It follows from Proposition 2.12 that we can define the $\boldsymbol{\omega}$-weighted blowing-up of $\mathbf{M}$ with center on Y:

$$\Phi: \widetilde{\mathbf{M}} \longrightarrow \mathbf{M} \tag{39}$$

(with respect to the trivialization given by Proposition 5.44). The transformed singularly foliated manifold $\widetilde{\mathbf{M}}$ is defined according to §2.6.

The above map will be called a *good blowing-up* for $(\mathbf{M}, \mathrm{Ax})$.

5.11. Global reduction of singularities

To state our next result, we recall that a controlled singularly foliated manifold $(\mathbf{M}, \mathrm{Ax})$ is called a *restriction* if it is defined by a restriction of a controlled singularly foliated manifold $(\mathbf{M}', \mathrm{Ax}')$ to some relatively compact open subset of the ambient space.

THEOREM 5.45. *Let* $(\mathbf{M}, \mathrm{Ax})$ *be a controlled singularly foliated manifold which is a restriction and is equireducible outside the divisor. Let*

$$\Phi: \widetilde{\mathbf{M}} \longrightarrow \mathbf{M}$$

be a good blowing-up for $(\mathbf{M}, \mathrm{Ax})$. *Then, either* $\widetilde{\mathbf{M}}$ *is an elementary singularly foliated manifold, or there exists an axis* $\widetilde{\mathrm{Ax}}=(\tilde{A}, \tilde{\mathfrak{z}})$ *for* $\widetilde{\mathbf{M}}$ *such that*:

(i) *the singularly foliated manifold* $(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}})$ *is a restriction*;

(ii) $(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}})$ *is equireducible outside the divisor*;

(iii) $\mathrm{Mult}(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}}) <_{\mathrm{lex}} \mathrm{Mult}(\mathbf{M}, \mathrm{Ax})$.

Proof. Assume that the set $\mathrm{NElem}(\widetilde{\mathbf{M}})$ of nonelementary points of $\widetilde{\mathbf{M}}$ is nonempty.

Let us denote by Y be the blowing-up center and recall that Φ locally coincides with the local blowing-up for $(\mathbf{M}, \mathrm{Ax})$ at each point $p\in Y$. From this, we conclude from Theorems 4.29 and 5.13 that there exists an axis $\widetilde{\mathrm{Ax}}=(\tilde{A}, \tilde{\mathfrak{z}})$ for $\mathbf{M}$ (obtained by analytic gluing) such that $(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}})$ is a controlled singularly foliated manifold such that:

(i) $(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}})$ is a restriction;

(ii) $(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}})$ is equireducible outside the divisor.

Moreover, by Theorem 4.29, we conclude that

$$\mathrm{inv}_{\max}(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}}) \leqslant_{\mathrm{lex}} \mathrm{inv}_{\max}(\mathbf{M}, \mathrm{Ax}).$$

If the inequality is strict, we are done. Otherwise, let us choose a maximal point locus $\widetilde{P}_{\max}$ for $(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}})$. Using again Theorems 4.29 and 5.13, we can write

$$\widetilde{P}_{\max} = \Phi^{-1}(P_{\max} \backslash Y)$$

for some maximal point locus $P_{\max}$ of $\mathbf{M}$. Indeed, we have

$$\mathrm{inv}(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}}, \tilde{q}) <_{\mathrm{lex}} \mathrm{inv}_{\max}(\mathbf{M}, \mathrm{Ax})$$

for each point $\tilde{q}\in\Phi^{-1}(Y)$, and therefore $S_{\mathrm{inv}_{\max}} \cap \Phi^{-1}(Y) = \varnothing$.

For simplicity, let us write the respective multiplicities of the bad trees $\mathrm{Tr}\mathcal{B}(P_{\max})$ and $\mathrm{Tr}\mathcal{B}(\widetilde{P}_{\max})$ simply as

$$\mathrm{Mult}(\mathrm{Tr}\mathcal{B}(P_{\max})) = (L, I, \#) \quad \text{and} \quad \mathrm{Mult}(\mathrm{Tr}\mathcal{B}(\widetilde{P}_{\max})) = (\tilde{L}, \tilde{I}, \tilde{\#}).$$

Then, we need to prove that

$$(L, I, \#) <_{\mathrm{lex}} (\tilde{L}, \tilde{I}, \tilde{\#}).$$

First of all, we claim that $L \leqslant \tilde{L}$. Indeed, it suffices to study how each branch

$$p_0 \longrightarrow p_1 \longrightarrow ... \longrightarrow p_l \quad (p_0 \in P_{\max}) \tag{40}$$

of the bad tree $\mathrm{Tr}\mathcal{B}(P_{\max})$ is transformed by the blowing-up.

If $p_i \notin Y$ for all $i=0, ..., l$, then this branch is mapped isomorphically to a branch of length l of the new bad tree $\mathrm{Tr}\mathcal{B}(\widetilde{P}_{\max})$.

Now, suppose that there exists an index $0 \leqslant i \leqslant l$ such that

$$\{p_0, ..., p_{i-1}\} \cap Y = \varnothing \quad \text{and} \quad p_i \in Y.$$

If $i=0$, it is immediate to see that the branch is completely destroyed. So, we suppose that $i \geqslant 1$. It follows from Propositions 4.30, 4.47 and 4.52 that each nonelementary point $\tilde{q} \in \Phi^{-1}(p_i)$ satisfies one of the following conditions:

(1) $\mathfrak{h}(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}}, \tilde{q}) < \mathfrak{h}(\mathbf{M}, \mathrm{Ax}, p_i)$;

(2) $\mathfrak{h}(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}}, \tilde{q}) = \mathfrak{h}(\mathbf{M}, \mathrm{Ax}, p_i)$ and $\Delta_1^{\tilde{q}} = 0$,

where $\Delta^{\tilde{q}} = (\Delta_1^{\tilde{q}}, \Delta_2^{\tilde{q}})$ is the vertical displacement vector associated with $\tilde{q}$.

As a consequence, the points lying in case (2) are isolated points of $\widetilde{S}_{\mathfrak{h}\max} \cap \widetilde{D}$ (where $\widetilde{D} := \Phi^{-1}(Y)$ is the exceptional divisor of the blowing-up and $\widetilde{S}_{\mathfrak{h}\max}$ is the stratum of maximal height for $(\widetilde{\mathbf{M}}, \widetilde{\mathrm{Ax}})$).

Now, observe that the strict transform of the extended center $\overline{Y}_{p_{i-1}}$ intersects the set $\Phi^{-1}(p_i)$ in a unique point $\tilde{q} \in S_{\mathfrak{h}\max}$, which necessarily lies in case (2). This immediately implies that $\mathrm{Bad}(\tilde{q}) = \varnothing$, and therefore the branch (40) is mapped to a unique branch in $\mathrm{Tr}\mathcal{B}(\widetilde{P}_{\max})$, which has one of the following forms

$$\tilde{p}_0 \longrightarrow \tilde{p}_1 \longrightarrow ... \longrightarrow \tilde{p}_{i-1} \longrightarrow \tilde{q}, \quad \text{or} \quad \tilde{p}_0 \longrightarrow \tilde{p}_1 \longrightarrow ... \longrightarrow \tilde{p}_{i-1},$$

where $\tilde{p}_j = \Phi^{-1}(p_j)$ for $j=0, ..., i-1$. In both cases, it is clear that the new branch has length at most equal to the length of the original branch. We have proved that $\tilde{L} \leqslant L$.

Let us suppose that $\tilde{L} = L$. Then, since the blowing-up creates no new branches of maximal length L, it follows immediately from the theorem of local resolution of singularities (Theorem 4.29) that

$$\tilde{I} \leqslant_{\mathrm{lex}} I.$$

It remains to prove that the conditions $\tilde{L} = L$ and $\tilde{I} = I$ imply that $\tilde{\#} < \#$. To see this, it suffices to remark that the blowing-up satisfies the following properties:

(1) the blowing-up Φ creates no new branches of length L;

(2) the center Y contains at least one terminal point of a branch which has length exactly equal to L.

Applying again Theorems 4.29 and 5.13, we immediately conclude that $\tilde{\#}<\#$. This completes the proof of the theorem. □

5.12. Proof of the main theorem

We are now ready to prove the main theorem of this work.

Proof of Theorem 1.1. Let $\mathbf{M}=(M,\varnothing,\varnothing,L_\chi)$ be the singularly foliated manifold associated with χ and let Ax be an axis for $\mathbf{M}$, defined as in Proposition 2.16. Given a relatively compact subset $U\subset M$, we denote by $(\mathbf{M}',\mathrm{Ax}')$ the restriction of $(\mathbf{M},\mathrm{Ax})$ to U.

Using Lemma 5.21, we know that there exists a finite sequence of blowing-ups

$$(\mathbf{M}',\mathrm{Ax}')=(\mathbf{M}_0,\mathrm{Ax}_0)\longrightarrow(\mathbf{M}_1,\mathrm{Ax}_1)\longrightarrow\ldots\longrightarrow(\mathbf{M}_k,\mathrm{Ax}_k),$$

such that the resulting singularly foliated manifold $(\mathbf{M}_k,\mathrm{Ax}_k)$ is equireducible outside the divisor.

To finish the proof, it suffices to consider the controlled singularly foliated manifold $(\mathbf{M}_k,\mathrm{Ax}_k)$ and apply Theorem 5.45 repeatedly. □

Appendix A. Faithful flatness of $\mathbf{C}[[x,y,z]]$

In the proof of the stabilization of adapted charts, we need the following simple consequence of the fact that $\mathbf{C}\{x,y,z\}$ is a unique factorization domain and that its completion $\mathbf{C}[[x,y,z]]$ is faithfully flat (see e.g. [Ma, §4.C and §24.A]).

LEMMA A.1. *Let $I\subset\mathbf{C}\{x,y,z\}$ be a nonzero radical ideal and let*

$$I'=J'_1\cap\ldots\cap J'_k$$

be the irreducible primary decomposition of the ideal $I'=I\mathbf{C}[[x,y,z]]$ in the ring of formal series $\mathbf{C}[[x,y,z]]$. Then, each J'_i, $i=1,\ldots,k$, can be written as $J'_i=J_i\mathbf{C}[[x,y,z]]$ for some prime ideal $J_i\subset\mathbf{C}\{x,y,z\}$.

COROLLARY A.2. *Let $H=(H_1,\ldots,H_r)\in\mathbf{R}\{x,y,z\}^r$ be a nonzero germ of analytic maps. Suppose that we can write the factorization*

$$\begin{pmatrix}H_1\\H_2\\\vdots\\H_r\end{pmatrix}=(z-f(x,y))\begin{pmatrix}S_1\\S_2\\\vdots\\S_r\end{pmatrix},$$

where $f\in\mathbf{R}[[x,y]]$ and $S_1,\ldots,S_r\in\mathbf{R}[[x,y,z]]$. Then, necessarily $f\in\mathbf{R}\{x,y\}$ is an analytic germ.

Proof. Indeed, the hypothesis implies that the ideal $I=\mathrm{rad}(H_1,...,H_r)$ is contained in the principal ideal $J=(z-f(x,y))\mathbf{C}[[x,y,z]]$. In particular, J is a member of the irreducible primary decomposition of I in $\mathbf{C}[[x,y,z]]$. Therefore, it suffices to apply Lemma A.1 to conclude that f is necessarily an analytic germ. □

Given a nonzero natural number $a\in\mathbf{N}$, consider now the ideal

$$\hat{I}_a=(x^a)\mathbf{R}[[x,y,z]]\subset\mathbf{R}[[x,y,z]].$$

The elements of the quotient ring $\widehat{R}_a=\mathbf{R}[[x,y,z]]/I_a$ are uniquely represented by the polynomials in $\mathbf{R}[[y,z]][x]$ whose degree in the variable x is at most $a-1$. We let R_a denote the image of $\mathbf{R}\{x,y,z\}$ under the quotient map.

Corollary A.3. *Let $([H_1],...,[H_r])\in R_a^r$ be a nonzero germ. Suppose that we can write the factorization (in $\widehat{R}_a$)*

$$\begin{pmatrix}[H_1]\\ [H_2]\\ \vdots\\ [H_r]\end{pmatrix}=(z-[f(x,y)])\begin{pmatrix}[S_1]\\ [S_2]\\ \vdots\\ [S_r]\end{pmatrix},$$

where $[f]\in\widehat{R}_a$ and $[S_1],...,[S_r]\in\widehat{R}_a$. Then, the germ $[f]$ necessarily lies in R_a.

Proof. It suffices to use Corollary A.2. □

Appendix B. Virtual height

Let us start with an elementary version of the Descartes' lemma.

Lemma B.1. *Let $Q(z)$ be a polynomial in $\mathbf{C}[z]$ with m nonzero monomials. Then, the multiplicity of Q at a point $\xi\neq0$ is at most $m-1$.*

Proof. Given a polynomial $Q\in\mathbf{C}[z]$, let $\mu(Q)$ be the multiplicity of Q at the origin (i.e. the greatest natural number k such that z^k divides $Q(z)$). We consider the sequence of polynomials $Q_0(z)$, $Q_1(z)$, ... which is inductively defined as follows:

$$\begin{cases}Q_0=z^{-\mu(Q)}Q,\\ Q_{i+1}=z^{-\mu(Q_i')}Q_i' & \text{for } i\geqslant0,\end{cases}$$

(where $'$ means d/dz). By induction, we can easily prove that Q has multiplicity k at some point $\xi\neq0$ if and only if

$$Q_0(\xi)=...=Q_{k-1}(\xi)=0.$$

However, it follows from the hypothesis and the above construction that Q_{m-1} is necessarily a nonzero constant. Therefore, the maximum multiplicity of Q at a point $\xi\neq 0$ is at most $m-1$. □

For the rest of this appendix, we shall adopt the following notation. Let $P(x_1,...,x_n)$ be an n-variable polynomial whose support is contained in the straight line

$$r: t\in\mathbf{R}^+\longmapsto \mathbf{p}+t(\boldsymbol{\Delta},-1),$$

for some $\mathbf{p}=(p_1,...,p_n)\in\mathbf{N}^n$ and some vector $\boldsymbol{\Delta}\in\mathbf{Q}^{n-1}_{\geqslant 0}$ of the form

$$\boldsymbol{\Delta}=\left(\frac{a_1}{b_1},...,\frac{a_{n-1}}{b_{n-1}}\right),\quad \text{with } a_i\in\mathbf{N},\ b_i\in\mathbf{N}_*\text{ and }\gcd(a_i,b_i)=1.$$

Let c be the least common multiple of $b_1,...,b_{n-1}$ and $Q(z)$ be the 1-variable polynomial

$$Q(z)=P(1,...,1,z)=*\,z^{p_n}+...,$$

where $*$ denotes some nonzero coefficient.

Proposition B.2. *The multiplicity $\mu_\xi(Q)$ of the polynomial $Q(z)$ at a point $\xi\neq 0$ is at most equal to $\lfloor p_n/c\rfloor$.*

Proof. Let $\mathbf{p}^1,...,\mathbf{p}^k$ denote the points of intersection of the straight line $r(t)$ with the lattice $\mathbf{N}^n$, ordered according to the last coordinate (so that $\mathbf{p}^k=\mathbf{p}$). Lemma B.1 implies that the multiplicity $\mu_\xi(Q)$ is at most equal to k. The result now follows immediately by noticing that each point $\mathbf{p}^s$ is necessarily given by $\mathbf{p}^s=\mathbf{p}+(k-s)c(\boldsymbol{\Delta},-1)$. □

Corollary B.3. *Suppose that $p_n\geqslant c+1$ and $1\leqslant a_i<b_i$ for some $i\in\{1,...,n-1\}$. Then,*

$$\mu_\xi(Q)\leqslant p_n-\frac{b_i}{a_i},$$

for all $\xi\neq 0$.

Proof. Let us prove that the condition $p_n-b_i/a_i\geqslant\lfloor p_n/c\rfloor$ is satisfied. Since $b_i\leqslant c$, it is clearly satisfied if

$$p_n\geqslant\frac{c^2}{c-1}. \tag{41}$$

Now we use the facts that $p_n\geqslant c+1$ and $c\geqslant b_i\geqslant 2$. It follows that inequality (41) is immediately satisfied when $p_n>c+1$. For $p_n=c+1$, we compute

$$\left\lfloor\frac{p_n}{c}\right\rfloor=\left\lfloor 1+\frac{1}{c}\right\rfloor=1=p_n-c\leqslant p_n-\frac{b_i}{a_i}.$$

This concludes the proof. □

Appendix C. Comments on final models

In this appendix, we shall indicate some possible refinements of our main theorem. First of all, we introduce the notion of *strongly elementary vector field*.

We use the following notation: given a matrix $A\in\mathrm{Mat}(n,\mathbf{R})$ and a formal map $R=(R_1,...,R_n)\in\mathbf{R}[[\mathbf{x}]]^n$, the symbol

$$[A\mathbf{x}+R]\frac{\partial}{\partial\mathbf{x}}$$

denotes the formal vector field

$$\sum_{i=1}^{n}[\langle A_i,\mathbf{x}\rangle+R_i]\frac{\partial}{\partial x_i},$$

where A_i is the ith row of the matrix A.

Let $\iota\subset[n,...,1]$ be a sublist of indices and

$$\mathfrak{D}=\bigcup_{i\in\iota}\{x_i=0\}$$

be the corresponding divisor of coordinate hyperplanes in $\mathbf{R}^n$.

We say that a formal n-dimensional vector field η is $\mathfrak{D}$-*preserving* if it can be written in the form

$$\eta=\sum_{i\in\iota}a_ix_i\frac{\partial}{\partial x_j}+\sum_{j\in[n,...,1]\setminus\iota}a_j\frac{\partial}{\partial x_j},$$

where $a_1,...,a_n\in\mathbf{R}[[\mathbf{x}]]$ are formal series.

A formal n-dimensional vector field η is called a $\mathfrak{D}$-*final model* if η is $\mathfrak{D}$-preserving and has one of the following expressions:

(1) (nonsingular vector field)

$$\eta=(\lambda+r(\mathbf{x}))\frac{\partial}{\partial x_1},$$

for some nonzero constant $\lambda\in\mathbf{R}^*$ and a germ $r\in\mathbf{R}[[\mathbf{x}]]$ with $r(0)=0$;

(2) (singular vector field) there exists a decomposition of $\mathbf{R}^n$ into a cartesian product

$$\mathbf{x}=(\mathbf{x}_+,\mathbf{x}_-,\mathbf{x}_\mathbf{I},\mathbf{x}_0)\in\mathbf{R}^{n_+}\times\mathbf{R}^{n_-}\times\mathbf{R}^{n_\mathbf{I}}\times\mathbf{R}^{n_0},$$

with $n_++n_-+n_\mathbf{I}\geqslant1$, such that η can be written as

$$\eta=[J_+\mathbf{x}_++R_+(\mathbf{x})]\frac{\partial}{\partial\mathbf{x}_+}+[J_-\mathbf{x}_-+R_-(\mathbf{x})]\frac{\partial}{\partial\mathbf{x}_-}+[J_\mathbf{I}\mathbf{x}_\mathbf{I}+R_\mathbf{I}(\mathbf{x})]\frac{\partial}{\partial\mathbf{x}_\mathbf{I}}+R_0(\mathbf{x})\frac{\partial}{\partial\mathbf{x}_0},$$

and the following conditions hold:

(i) $(J_+, J_-, J_{\mathbf{I}}) \in \mathrm{Mat}(n_+, \mathbf{R}) \times \mathrm{Mat}(n_-, \mathbf{R}) \times \mathrm{Mat}(n_{\mathbf{I}}, \mathbf{R})$ are matrices whose eigenvalues are all nonzero and have strictly positive real part, strictly negative real part and zero real part, respectively;

(ii) $R_* \in \mathbf{R}[[\mathbf{x}]]^{n_*}$ is a formal germ such that $R_*(0) = DR_*(0) = 0$ (for $* \in \{+, -, \mathbf{I}, 0\}$); moreover,

$$R_+|_{\{\mathbf{x}_+=0\}} = R_-|_{\{\mathbf{x}_-=0\}} = R_{\mathbf{I}}|_{\{\mathbf{x}_{\mathbf{I}}=0\}} = 0,$$

and

$$R_0|_{\{\mathbf{x}_-=\mathbf{x}_0=0\}} = R_0|_{\{\mathbf{x}_+=\mathbf{x}_0=0\}} = R_0|_{\{\mathbf{x}_{\mathbf{I}}=\mathbf{x}_0=0\}} = 0;$$

as a consequence, the eigenspaces W_+, W_-, $W_{\mathbf{I}}$ and W_0 which correspond to J_+, J_-, $J_{\mathbf{I}}$ and the zero matrix, respectively, are (formal) invariant manifolds for η;

(iii) the zero set $Z = \{\eta = 0\} \subset W_0$ has normal crossings (i.e. it is given by a finite union of intersections of coordinate hyperplanes);

(iv) the restricted vector field $\eta_0 = \eta|_{W_0}$ has the form

$$\eta_0 = \mu \mathbf{x}_0^{\boldsymbol{\alpha}} U(\mathbf{x}) \tilde{\eta}, \quad \text{with } \mathbf{x}_0^{\boldsymbol{\alpha}} = \prod_{i=0}^{n_0} \mathbf{x}_{0,i}^{\alpha_i},$$

where $\mu \in \mathbf{R}$ is a real constant, $\boldsymbol{\alpha} \in \mathbf{N}^{n_0}$ is a vector of natural numbers, $U \in \mathbf{R}[[\mathbf{x}]]$ is a unit and $\tilde{\eta}$ is an n_0-dimensional $(\mathfrak{D} \cap W_0)$-final model.

In other words, condition (iv) requires that the *restriction* of η to the manifold W_0 is given (up to multiplication by a unit) by a monomial times a vector field η_0 which is a final model on a space of *strictly lower dimension*.

An analytic vector field χ defined on M is $\mathfrak{D}$*-strongly elementary* at a point $p \in M$ if there exists a $\mathfrak{D}$-adapted *formal coordinate system* $\mathbf{x} = (x_1, ..., x_n)$ at p such that χ, written in these coordinates, is a $\mathfrak{D}$-final model.

Remark C.1. We cannot replace the words *formal coordinate system* by *analytic coordinate system* in the above definition. It would be too restrictive. For instance, it would imply that the local center manifolds are necessarily analytic.

A singularly foliated manifold $\mathbf{M} = (M, \Upsilon, \mathfrak{D}, L)$ will be called *strongly elementary* if for each point $p \in M$, the line field L is locally generated by a vector field χ_p which is $\mathfrak{D}$-strongly elementary.

Conjecture. Let χ be a reduced analytic vector field defined on a real-analytic manifold M without boundary. Then, for each relatively compact set $U \subset M$, there exists a finite sequence of weighted blowing-ups

$$(U, \varnothing, \varnothing, L_\chi|_U) =: \mathbf{M}_0 \xleftarrow{\Phi_1} \mathbf{M}_1 \xleftarrow{\Phi_2} \dots \xleftarrow{\Phi_n} \mathbf{M}_n \tag{42}$$

such that the resulting singularly foliated manifold $\mathbf{M}_n$ is strongly elementary.

Let us see a few examples of final models in dimensions 1, 2 and 3.

Example C.2. For $n=1$, the complete list of final models is the following.

- Nonsingular case:

$$\eta=(\lambda+r(x))\frac{\partial}{\partial x},$$

where $r(0)=0$ and $\lambda\in\mathbf{R}^*$.

- Singular case:

$$\eta=(\lambda x+xr(x))\frac{\partial}{\partial x},$$

where $r(0)=0$ and $\lambda\in\mathbf{R}^*$.

(Note that the former case only occurs if $\mathfrak{D}=\varnothing$.)

Example C.3. For $n=2$ and $\mathfrak{D}=\varnothing$, the complete list of final models is the following.

- Nonsingular case:

$$\eta=(\lambda+r(\mathbf{x}))\frac{\partial}{\partial x_1},$$

where $r(0)=0$ and $\lambda\in\mathbf{R}^*$.

- Singular case with $n_+=1$ and $n_-=1$:

$$\eta=(\lambda_1x_1+x_1r_1(\mathbf{x}))\frac{\partial}{\partial x_1}+(-\lambda_2x_2+x_2r_2(\mathbf{x}))\frac{\partial}{\partial x_2},$$

where $\lambda_1,\lambda_2\in\mathbf{R}_{>0}$ and $r_i(0)=0$ for $i=1,2$.

- Singular case with $n_\pm=2$:

$$\eta=(\pm\lambda_1x_1+R_1(\mathbf{x}))\frac{\partial}{\partial x_1}+(\pm\lambda_2x_2+R_2(\mathbf{x}))\frac{\partial}{\partial x_2},$$

where $\lambda_1,\lambda_2\in\mathbf{R}_{>0}$ and $R_i(0)=DR_i(0)=0$ for $i=1,2$.

- Singular case with $n_\pm=1$ and $n_0=1$:

$$\eta=(\pm\lambda_1x_1+x_1r_1(\mathbf{x}))\frac{\partial}{\partial x_1}+\mu x_2^\alpha U(\mathbf{x})(\lambda_2+r_2(\mathbf{x}))\frac{\partial}{\partial x_2},$$

where $\lambda_1\in\mathbf{R}_{>0}$, $\lambda_2\in\mathbf{R}^*$, $\mu\in\mathbf{R}$, $\alpha\geqslant2$, U is a unit and $r_i(0)=0$ for $i=1,2$.

- Singular case with $n_\pm=n_0=0$ and $n_{\mathbf{I}}=2$:

$$\eta=(\lambda x_2+R_1(\mathbf{x}))\frac{\partial}{\partial x_1}+(-\lambda x_1+R_2(\mathbf{x}))\frac{\partial}{\partial x_2},$$

where $\lambda\in\mathbf{R}_{>0}$ and $R_i(0)=DR_i(0)=0$ for $i=1,2$.

Example C.4. For $n=3$, $\mathfrak{D}=\varnothing$, $n_0=1$ and $n_+=n_-=1$, the final model is given by

$$\eta=(\lambda_1x_1+x_1r_1(\mathbf{x}))\frac{\partial}{\partial x_1}+(-\lambda_2x_2+x_2r_2(\mathbf{x}))\frac{\partial}{\partial x_2}+\mu x_3^\alpha U(\mathbf{x})(\lambda_3+r_3(\mathbf{x}))\frac{\partial}{\partial x_3},$$

where $\mu\in\mathbf{R}$, $\lambda_1,\lambda_2\in\mathbf{R}_{>0}$, $\lambda_3\in\mathbf{R}^*$, $\alpha\geqslant2$, U is a unit and $r_i(0)=0$ for $i=1,2,3$.

Example C.5. For $n=3$, $\mathfrak{D}=\varnothing$, $n_{\pm}=0$, $n_0=1$ and $n_{\mathbf{I}}=2$, the final model is given by

$$\eta=(\lambda x_2+x_1r_1(\mathbf{x})+x_2s_1(\mathbf{x}))\frac{\partial}{\partial x_1}+(-\lambda x_1+x_1r_2(\mathbf{x})+x_2s_2(\mathbf{x}))\frac{\partial}{\partial x_2} +\mu x_3^{\alpha}U(\mathbf{x})(\lambda_3+r_3(\mathbf{x}))\frac{\partial}{\partial x_3},$$

where $\mu\in\mathbf{R}$, $\lambda\in\mathbf{R}_{>0}$, $\lambda_3\in\mathbf{R}^*$, $\alpha\geqslant 2$, U is a unit and $r_i(0)=s_i(0)=0$ for $i=1,2,3$.

Example C.6. For $n=3$, $\mathfrak{D}=\varnothing$, $n_0=2$ and $n_+=1$, the final model is given by

$$\eta=(\lambda_1x_1+x_1r_1(\mathbf{x}))\frac{\partial}{\partial x_1}+\mu x_2^{\alpha}x_3^{\beta}U(\mathbf{x})\left(\sum_{i=2}^{3}(f_i(x_2,x_3)+x_1r_i(\mathbf{x}))\frac{\partial}{\partial x_i}\right),$$

where $\mu\in\mathbf{R}$, $\lambda_1\in\mathbf{R}_{>0}$, $\alpha+\beta\geqslant 1$, U is a unit and the vector field obtained by restriction to the center manifold $W_0=\{x_1=0\}$, namely

$$\tilde{\eta}=f_2(x_2,x_3)\frac{\partial}{\partial x_2}+f_3(x_2,x_3)\frac{\partial}{\partial x_3},$$

has one of the forms given in Example C.3.

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