

ON THE BOUNDARY BEHAVIOUR OF ELLIPTIC MODULAR FUNCTIONS.

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I.

Introduction.

1. Let $\tau = x + iy$ be a complex variable, and $q = e^{i\pi\tau}$; let us write, following the notation of TANNERY and MOLK¹,

$$\mathcal{P}_2(\tau) = 2q^{\frac{1}{4}} + 2q^{\frac{9}{4}} + 2q^{\frac{25}{4}} + \dots$$

$$\mathcal{P}_3(\tau) = 1 + 2q + 2q^4 + 2q^9 + \dots$$

$$\mathcal{P}_4(\tau) = 1 - 2q + 2q^4 - 2q^9 + \dots$$

These series are convergent when $y > 0$ and represent functions which are analytic in the half-plane $y > 0$ and which cannot be continued across the x -axis. The behaviour of these functions, when τ tends to a real number ξ by moving along the straight line $x = \xi$, has many interesting features. When ξ is rational, the behaviour is fairly simple and can be obtained either directly², or by effecting on τ a suitable linear transformation

¹ TANNERY and MOLK, *Éléments de la théorie des Fonctions Elliptiques*, Vol. II (1896), p. 257. We shall refer to this book (Vol. II) as F.E.

² Cf. HARDY, On the representation of a number as the sum of any number of squares, and in particular of five, *Transactions of the American Mathematical Society*, Vol. XXI (1920) pp. 255—284 (p. 259). Though the direct method gives the result in many cases without necessitating an appeal to the transformation theory, the latter has the advantage of being applicable to all elliptic modular functions, including those for which a direct method is not available. See HARDY and RAMANUJAN, Asymptotic formulae in combinatory analysis, *Proceedings of the London Mathematical Society*, (Ser. 2) Vol. 17 (1918) pp. 75—115 (pp. 93, 94).

$$(1) \quad T = \frac{c + d\tau}{a + b\tau},$$

where a, b, c, d are integers such that $ad - bc = 1$. If $\xi = \frac{P}{Q}$, where P, Q are integers prime to each other, the latter method consists in taking $a = P, b = -Q$ and c, d to be any integers such that $Pd + Qc = 1$. It is then easily seen from (1) that, when τ traces the line $x = \xi$ in the half-plane $y > 0$, $T = X + iY$ traces the line $X = -\frac{d}{Q}$ in the half-plane $Y > 0$; and that on these lines as $y \rightarrow +0$, $Y \rightarrow +\infty$. Now the transformation theory of the Theta functions¹ enables us to express (say) $\mathcal{P}_3(\circ|\tau)$ in terms of one of the functions $\mathcal{P}_2(\circ|T), \mathcal{P}_3(\circ|T), \mathcal{P}_4(\circ|T)$; and in order to examine the behaviour of $\mathcal{P}_3(\circ|\tau)$ as τ tends² to ξ , it is only necessary to study the behaviour of one or other of the functions $\mathcal{P}_2(\circ|T), \mathcal{P}_3(\circ|T), \mathcal{P}_4(\circ|T)$ as $Y \rightarrow +\infty$ along a fixed line parallel to the Y -axis. This latter investigation is very simple and leads to the required result without any difficulty.

2. When ξ is irrational the behaviour of the Theta functions is complicated, and there appear to be no results so simple as those that exist when ξ is rational. HARDY and LITTLEWOOD³ have obtained some interesting results when ξ is irrational. They have proved in this case that

$$\mathcal{P}_3(\circ|\tau) = o\left(\frac{1}{\sqrt{y}}\right), \quad \mathcal{P}_3(\circ|\tau) \neq o\left(\frac{1}{\sqrt{y}}\right).$$

They have further proved that, if ξ is such that, when expressed as a simple continued fraction, the partial quotients form a bounded sequence, then two positive constants K_1, K_2 exist such that

$$\frac{K_1}{4\sqrt{y}} < |\mathcal{P}_3(\circ|\tau)| < \frac{K_2}{4\sqrt{y}}.$$

In particular these inequalities hold when ξ is a quadratic surd.

3. The object of this paper is to investigate more fully the case when ξ is a quadratic surd. It is shown here that the periodicity of the continued frac-

¹ F.E. p. 262.

² To avoid constant repetition we shall understand that throughout this paper the path along which τ tends to ξ is the straight line $x = \xi$.

³ HARDY and LITTLEWOOD, Some problems of Diophantine approximation (II), *Acta Mathematica*, Vol. 37 (1914), pp. 193-238 (pp. 226-230).

tion for ξ is reflected also in the behaviour of the Theta functions, and enables us to obtain formulae which involve actual limits, when we break up the range of variation of y in a manner to be explained presently.

We shall confine ourselves in the following to the function $\mathcal{P}_3(\circ|\tau)$. A reference to the formulae given at the end of TANNERY and MOLK's book¹ will show that the considerations developed in this paper are applicable, with suitable modifications, to the other Theta functions and also to the modular functions $h(\tau)$, $J(\tau)$, $\varphi(\tau)$ etc.

Let ξ be a quadratic surd², and let

$$\begin{aligned} \xi &= b_1 + \frac{1}{|b_2} + \dots + \frac{1}{|b_m} + \frac{1}{|a_1} + \frac{1}{|a_2} + \dots + \frac{1}{|a_r} + \frac{1}{|a_1} + \frac{1}{|a_2} + \dots + \frac{1}{|a_r} + \dots \\ &= [b_1, b_2, \dots, b_m, \overline{a_1, a_2, \dots, a_r}],^3 \end{aligned}$$

there being r partial quotients in a period of the continued fraction. Let $\frac{p_n}{q_n}$ be the n th convergent of the continued fraction. As y is to tend to zero, we may suppose that it is restricted to values satisfying

$$0 < y < 1 = \frac{1}{q_1^2}.$$

Let \mathcal{A}_n denote the interval

$$\frac{1}{q_{n+1}^2} \leq y < \frac{1}{q_n^2}, \quad (n = 1, 2, \dots)$$

so that the range of variation of y may be considered to be made up of the sequence of intervals

$$\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n, \dots$$

Now we divide this sequence of intervals into a finite number of sub-sequences

¹ F.E. pp. 262, 266, 267.

² We may suppose that ξ is positive; this does not imply any loss of generality, since the function $\mathcal{P}_3(\circ|\tau)$ has the period 2.

³ We shall use this notation for periodic continued fractions. We shall also denote the finite continued fraction

$$c_1 + \frac{1}{|c_2} + \dots + \frac{1}{|c_n}$$

by $[c_1, c_2, \dots, c_n]$ and the infinite continued fraction

$$c_1 + \frac{1}{|c_2} + \dots + \frac{1}{|c_n} + \dots$$

by $[c_1, c_2, \dots, c_n, \dots]$.

in a periodic manner; it is shown that there exists a positive integer H (which is a multiple of r) such that, if y is confined to any one of the following H sub-sequences

$$(2) \quad \left\{ \begin{array}{l} \Delta_1, \Delta_{1+H}, \Delta_{1+2H}, \dots \Delta_{1+\nu H}, \dots \\ \Delta_2, \Delta_{2+H}, \Delta_{2+2H}, \dots \Delta_{2+\nu H}, \dots \\ \Delta_3, \Delta_{3+H}, \Delta_{3+2H}, \dots \Delta_{3+\nu H}, \dots \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ \Delta_H, \Delta_{2H}, \Delta_{3H}, \dots \Delta_{(\nu+1)H}, \dots \end{array} \right.$$

the behaviour of $\mathcal{P}_3(\circ|\tau)$ is fairly simple, and can be obtained by making a combined use of some properties of periodic continued fractions and of the formulae of the transformation theory of the Theta functions. To enable us to write down asymptotic formulae, we introduce a continuous parameter σ which can take any value in the interval $0 \leq \sigma < 1$; and we consider the behaviour of $\mathcal{P}_3(\circ|\tau)$ as y takes the values

$$\frac{\sigma}{q_n^2} + \frac{1-\sigma}{q_{n+1}^2};$$

here σ is kept fixed, and n is allowed to tend to infinity through integers, which have all the same residue R to modulus H , R being in the interval $1 \leq R \leq H$. In other words, y tends to zero through a sequence of values, one in each of the intervals

$$\Delta_R, \Delta_{R+H}, \Delta_{R+2H}, \dots,$$

each value of y dividing the interval in which it lies in the constant ratio $1-\sigma:\sigma$.

Under these circumstances it is proved in this paper that

$$\sqrt[4]{y} \mathcal{P}_3(\circ|\tau)$$

(where $\sqrt[4]{y}$ denotes the real positive fourth root) tends to a finite limit. This limit depends on R and σ ; and explicit formulae showing the nature of this dependence are also obtained in the course of the paper. As the sub-sequences (2) cover the whole range of variation of y , and as R can take any one of the values $1, 2, \dots, H$ and σ can take any value in $0 \leq \sigma < 1$, it is clear that the proof

of the existence of the above-mentioned limits, and the method of their evaluation amount to a complete description (to the first order of approximation) of the asymptotic behaviour of $\mathcal{P}_g(\sigma|\tau)$ as $\tau \rightarrow \xi$ along the straight line $x = \xi$.

II.

Geometrical Preliminaries.

4. Let ξ be a quadratic surd and let

$$\xi = [b_1, b_2, \dots, b_m, \overline{a_1, a_2, \dots, a_r}],$$

there being r partial quotients in a period of the continued fraction. Let $\frac{p_n}{q_n}$ be the n th convergent of the continued fraction, and let η_n stand for $(-1)^n$. Further let g be equal to r or $2r$ according as r is even or odd¹; so that g is always an even number and is divisible by r . Let q denote an integer in the interval $1 \leq q \leq g$.

Let S_n denote the linear transformation

$$(3) \quad \tau = \frac{p_{n-1} + \eta_n p_n T}{q_{n-1} + \eta_n q_n T}, \quad (n = 2, 3, \dots)$$

or its equivalent

$$(4) \quad T = \eta_{n+1} \frac{p_{n-1} - q_{n-1} \tau}{p_n - q_n \tau}.$$

We consider in this section the effect of making the transformations S_n on the intervals \mathcal{A}_n and on the points

$$\tau = \tau_n(\sigma) = \xi + i y_n(\sigma),$$

where

$$y_n(\sigma) = \frac{\sigma}{q_n^2} + \frac{1 - \sigma}{q_{n+1}^2},$$

the suffix n in S_n , \mathcal{A}_n , $\tau_n(\sigma)$ being the same for any particular transformation.²

¹ In other words, g is the least common multiple of 2 and r .

² The transformations S_n and the intervals \mathcal{A}_n are suggested by the work of HARDY and LITTLEWOOD, loc. cit. 226, 229. We have not defined the transformation S_1 which is to be applied to points in \mathcal{A}_1 . The omission is unimportant; we may, if we wish to preserve formal completeness, define S_1 to be (say) the identical transformation $\tau = T$.

5. We shall require, to start with, a lemma on periodic continued fractions.¹

Lemma 1. *Let $\xi, \frac{p_n}{q_n}, g, \varrho$ be as defined above; let ϱ be fixed and let n tend to infinity by assuming all values congruent to $\varrho \pmod{g}$. Then $\frac{q_{n-1}}{q_n}$ and $p_n q_n - \xi q_n^2$ tend to finite limits.*

Let us write the finite sequence

$$\begin{aligned} & a_1, a_2, \dots, a_r, a_1, a_2, \dots, a_r \\ \text{also in the form} & d_1, d_2, \dots, d_r, d_{r+1}, d_{r+2}, \dots, d_{2r}, \\ \text{so that} & \\ (5) \quad & \begin{cases} d_1 = a_1, d_2 = a_2, \dots, d_r = a_r \\ d_{r+1} = a_1, d_{r+2} = a_2, \dots, d_{2r} = a_r \end{cases} \end{aligned}$$

For $1 \leq t \leq r$ we define²

$$\begin{aligned} \theta_t &= [0, \overline{d_{t+r}, d_{t+r-1}, \dots, d_{t+1}}], \\ \varphi_t &= [\overline{d_{t+1}, d_{t+2}, \dots, d_{t+r}}]. \end{aligned}$$

We shall also write for convenience the continued fraction for ξ in the form

$$\begin{aligned} & [c_1, c_2, \dots, c_n, \dots], \\ \text{so that} & \\ & c_1 = b_1, c_2 = b_2, \dots, c_m = b_m, \\ & c_{m+\nu} = a_s \end{aligned}$$

when $\nu \equiv s \pmod{r}$, s , of course, being one of the numbers $1, 2, \dots, r$.

Now by a known result we have³

$$(6) \quad \frac{q_{n-1}}{q_n} = [0, c_n, c_{n-1}, \dots, c_2].$$

¹ The results of Lemma 1 were given in a somewhat different form in my paper, Some Diophantine approximations connected with quadratic surds, *Journal of the Indian Mathematical Society*, Vol. XIV (1922), pp. 161–166.

² When $r=1$ there is only one θ and one φ ,

$$\theta = [0, a_1, a_1, a_1, \dots], \quad \varphi = [a_1, a_1, \dots].$$

³ See, for example, CHRYSTAL, *Algebra*, Vol. II (1922), p. 433.

Since $n \rightarrow \infty$ through values which differ by multiples of r , it is clear that, when $n > m$, c_n has always a constant value, say a_t . Remembering that $a_t = d_{t+r}$, it is easily seen from (6) that

$$\frac{q_{n-1}}{q_n} \rightarrow [0, \overline{d_{t+r}, d_{t+r-1}, \dots, d_{t+1}}] = \theta_t,$$

which is the first result of the lemma.

To prove the second result, we write

$$f_{n+1} = [c_{n+1}, c_{n+2}, \dots].$$

Then

$$\xi = \frac{f_{n+1} p_n + p_{n-1}}{f_{n+1} q_n + q_{n-1}},$$

and so, remembering that

$$p_n q_{n-1} - p_{n-1} q_n = \eta_n$$

we have

$$\frac{p_n}{q_n} - \xi = \frac{p_n}{q_n} - \frac{f_{n+1} p_n + p_{n-1}}{f_{n+1} q_n + q_{n-1}} = \frac{\eta_n}{q_n (f_{n+1} q_n + q_{n-1})},$$

$$p_n q_n - \xi q_n^2 = \frac{\eta_n q_n}{f_{n+1} q_n + q_{n-1}},$$

$$(7) \quad \frac{1}{p_n q_n - \xi q_n^2} = \eta_n f_{n+1} + \eta_n \frac{q_{n-1}}{q_n}.$$

Since n has the same residue to modulus g , which is an even number, it follows that n has the same parity, and so η_n has the constant value η_ϱ . Also by the remark made above, when $n > m$, c_n has a constant value a_t ; so that, c_{n+1} has the constant value d_{t+1} . Therefore f_{n+1} has the constant value φ_t . And so using the first result of the lemma we see from (7) that

$$\frac{1}{p_n q_n - \xi q_n^2} \rightarrow \eta_\varrho (\varphi_t + \theta_t).$$

This limit cannot be zero, since $\theta_t > 0$, $\varphi_t > 1$. Hence $p_n q_n - \xi q_n^2$ tends to a finite limit.

The limits of $\frac{q_{n-1}}{q_n}$ and $p_n q_n - \xi q_n^2$ depend on ϱ and we shall denote them respectively by L_ϱ and A_ϱ , ($\varrho = 1, 2, \dots, g$). We shall also have by definition

$$A_{g+1} = A_1, \quad A_0 = A_g,$$

$$L_{g+1} = L_1, \quad L_0 = L_g.$$

6. Let us now return to the relation (3) between τ and T and consider the curve traced by $T = X + iY$ when $\tau = x + iy$ describes the part of the line $x = \xi$ which lies between $y = \frac{1}{q_n^2}$ and $y = \frac{1}{q_{n+1}^2}$. We shall regard the τ -plane and the T -plane as coincident, so that the x - and X -axes coincide, and the y - and Y -axes coincide (in position as well as in direction). There will be no confusion in doing so, as τ is confined to the fixed line $x = \xi$ and T traces circular arcs whose positions we will now describe. Writing $\tau = \xi + iy$, $T = X + iY$ in (3), and equating the real parts, we see after an easy calculation that, as τ describes the semi-infinite line $x = \xi$, $y > 0$, T describes the semi-circle, whose equation is

$$(8) \quad \begin{cases} (X^2 + Y^2)(p_n q_n - \xi q_n^2) + \eta_n X(p_n q_{n-1} + p_{n-1} q_n - 2\xi q_{n-1} q_n) \\ + (p_{n-1} q_{n-1} - \xi q_{n-1}^2) = 0, \end{cases}$$

and which lies in the half plane $Y > 0$.

Now the relation (4) gives the following equations connecting X , Y , x , y ;

$$(9) \quad X[(p_n - q_n x)^2 + q_n^2 y^2] = \eta_{n+1}[(p_{n-1} - q_{n-1} x)(p_n - q_n x) + q_{n-1} q_n y^2],$$

$$(10) \quad Y[(p_n - q_n x)^2 + q_n^2 y^2] = y.$$

From the second of these equations we see that if the T -points, which correspond respectively to

$$\tau' = \xi + \frac{i}{q_n}, \quad \tau'' = \xi + \frac{i}{q_{n+1}},$$

are $X' + iY'$ and $X'' + iY''$, then

$$(11) \quad Y' \left[(p_n - q_n \xi)^2 + \frac{1}{q_n^2} \right] = \frac{1}{q_n^2},$$

$$(12) \quad Y'' \left[(p_n - q_n \xi)^2 + \frac{q_n^2}{q_{n+1}^4} \right] = \frac{1}{q_{n+1}^2}$$

The T -curve that corresponds to the part of the straight line $x = \xi$ between $y = \frac{1}{q_n^2}$ and $y = \frac{1}{q_{n+1}^2}$ is the part of the semi-circle (8) which is intercepted between $X = X'$ and $X = X''$. We shall call this circular arc C_n , and shall prove that it lies in the half plane $Y > \frac{1}{2}$. Since C_n has its centre on the X -axis and is therefore concave towards this axis, it is clear that in order to prove that C_n lies wholly in the half plane $Y > \frac{1}{2}$ it is sufficient to show that¹

$$Y' > \frac{1}{2}, \quad Y'' > \frac{1}{2}.$$

Now by a known property of continued fractions

$$|p_n - q_n \xi| < \frac{1}{q_{n+1}}$$

so the coefficient of Y' in (11) is less than

$$\frac{1}{q_{n+1}^2} + \frac{1}{q_n^2} < \frac{2}{q_n^2},$$

from which it follows that $Y' > \frac{1}{2}$. Similarly the coefficient of Y'' in (12) is less than

$$\frac{1}{q_{n+1}^2} + \frac{q_n^2}{q_{n+1}^4} < \frac{1}{q_{n+1}^2} + \frac{1}{q_{n+1}^2} = \frac{2}{q_{n+1}^2},$$

and so $Y'' > \frac{1}{2}$.

Let us now consider the points

$$T = T_n(\sigma) = X_n(\sigma) + i Y_n(\sigma) \quad (n \geq 2)$$

obtained by effecting the transformations S_n on the points

$$\tau = \tau_n(\sigma) = \xi + i y_n(\sigma),$$

where

$$y_n(\sigma) = \frac{\sigma}{q_n} + \frac{1 - \sigma}{q_{n+1}}.$$

¹ Cf. HARDY and LITTLEWOOD, loc. cit., p. 229.

After multiplying each of the equations (9) and (10) by q_n^2 , and after a slight rearrangement we get

$$(13) \quad \left\{ \begin{aligned} & X_n(\sigma) \left[(p_n q_n - \xi q_n^2)^2 + \left\{ \sigma + (1 - \sigma) \frac{q_n^2}{q_{n+1}^2} \right\}^2 \right] \\ & = \eta_{n+1} \left[\frac{q_n}{q_{n-1}} (p_{n-1} q_{n-1} - \xi q_{n-1}^2) (p_n q_n - \xi q_n^2) + \frac{q_{n-1}}{q_n} \left\{ \sigma + (1 - \sigma) \frac{q_n^2}{q_{n+1}^2} \right\}^2 \right] \end{aligned} \right.$$

$$(14) \quad Y_n(\sigma) \left[(p_n q_n - \xi q_n^2)^2 + \left\{ \sigma + (1 - \sigma) \frac{q_n^2}{q_{n+1}^2} \right\}^2 \right] = \sigma + (1 - \sigma) \frac{q_n^2}{q_{n+1}^2}.$$

We now keep σ fixed in the interval $0 \leq \sigma < 1$ and allow n to tend to infinity by taking all values congruent to a fixed $\varrho \pmod{g}$. Since n has the same parity, η_{n+1} has the constant value $\eta_{\varrho+1}$. The coefficients of $X_n(\sigma)$, $Y_n(\sigma)$ and the absolute terms on the right hand sides of the equations (13) and (14) tend to finite limits by Lemma 1. Hence $X_n(\sigma)$, $Y_n(\sigma)$ tend respectively to limits $\mathfrak{X}_\varrho(\sigma)$, $\mathfrak{Y}_\varrho(\sigma)$ given by

$$\mathfrak{X}_\varrho(\sigma) [\mathcal{A}_\varrho^2 + \{\sigma + (1 - \sigma) L_{\varrho+1}^2\}^2]$$

$$= \eta_{\varrho+1} \left[\frac{\mathcal{A}_{\varrho-1} \mathcal{A}_\varrho}{L_\varrho} + L_\varrho \{\sigma + (1 - \sigma) L_{\varrho+1}^2\}^2 \right],$$

$$\mathfrak{Y}_\varrho(\sigma) [\mathcal{A}_\varrho^2 + \{\sigma + (1 - \sigma) L_{\varrho+1}^2\}^2] = \sigma + (1 - \sigma) L_{\varrho+1}^2;$$

or, on writing for shortness,

$$\sigma + (1 - \sigma) L_{\varrho+1}^2 = \mathcal{J}_{\varrho+1}(\sigma) = \mathcal{J}_{\varrho+1},$$

$$(15) \quad \left\{ \begin{aligned} & \mathfrak{X}_\varrho(\sigma) (\mathcal{A}_\varrho^2 + \mathcal{J}_{\varrho+1}^2) = \eta_{\varrho+1} \left[\frac{\mathcal{A}_{\varrho-1} \mathcal{A}_\varrho}{L_\varrho} + L_\varrho \mathcal{J}_{\varrho+1}^2 \right], \\ & \mathfrak{Y}_\varrho(\sigma) (\mathcal{A}_\varrho^2 + \mathcal{J}_{\varrho+1}^2) = \mathcal{J}_{\varrho+1}. \end{aligned} \right.$$

Further, since $\tau_n(\sigma)$ lies on the line $x = \xi$ between the points τ' and τ'' , it follows that $T_n(\sigma)$ lies on the arc C_n ; and since for all values of n , C_n lies in the half-plane $Y > \frac{1}{2}$, it is clear that for all values of n and σ , $Y_n(\sigma) > \frac{1}{2}$; and, therefore $\mathfrak{Y}_\varrho(\sigma) \geq \frac{1}{2}$ for all values of ϱ and σ . We have thus proved the following lemma.

Lemma 2. *If σ has a fixed value in the interval $0 \leq \sigma < 1$, and n tends to infinity by taking all values congruent to a fixed $\varrho \pmod{g}$, then the points $T_n(\sigma)$ tend to the definite limiting position*

$$\mathfrak{T}_\varrho(\sigma) = \mathfrak{X}_\varrho(\sigma) + i\mathfrak{Y}_\varrho(\sigma);$$

and this limiting point lies in the half plane $Y \geq \frac{1}{2}$.

It is of interest to observe that, if we let n tend to infinity in the manner described above, the arcs C_n tend to a limiting arc \mathfrak{C}_ϱ , whose equation is

$$\mathcal{A}_\varrho(X^2 + Y^2) + D_\varrho X + \mathcal{A}_{\varrho-1} = 0,$$

and which is intercepted between the lines $X = \Xi'_\varrho$ and $X = \Xi''_\varrho$, where D_ϱ , Ξ'_ϱ , Ξ''_ϱ are given by

$$D_\varrho = 1 + \frac{2\eta_\varrho \mathcal{A}_{\varrho-1}}{L_\varrho},$$

$$\Xi'_\varrho = \frac{\eta_{\varrho+1} \left[\frac{\mathcal{A}_{\varrho-1} \mathcal{A}_\varrho}{L_\varrho} + L_\varrho \right]}{\mathcal{A}_\varrho^2 + 1},$$

$$\Xi''_\varrho = \frac{\eta_{\varrho+1} \left[\frac{\mathcal{A}_{\varrho-1} \mathcal{A}_\varrho}{L_\varrho} + L_\varrho L_{\varrho+1}^4 \right]}{\mathcal{A}_\varrho^2 + L_{\varrho+1}^4}.$$

It is easily seen that, if ϱ is kept fixed, and σ varies between 0 and 1, the points $\mathfrak{T}_\varrho(\sigma)$ describe the arc \mathfrak{C}_ϱ ; so that the points $\mathfrak{T}_\varrho(\sigma)$ (for all values of ϱ and σ under consideration) lie on a finite number of circular arcs \mathfrak{C}_ϱ , all of which lie in the half-plane $Y \geq \frac{1}{2}$.

III.

Periodicity of Certain Sequences.

7. This section is devoted to the proof of some congruence properties of p_n , q_n , the numerators and denominators of the convergents of the continued fraction for ξ , and also to the proof of some properties of the numbers

$$\left[\frac{q_{n-1}}{\eta_n q_n} \right], \quad (n = 2, 3, \dots)$$

where the symbol $\left[\frac{a}{b} \right]$ denotes a generalisation of the LEGENDRE-JACOBI symbol $\left(\frac{a}{b} \right)$. The symbol $\left[\frac{a}{b} \right]$ appears in the formulae of the linear transformation of the Theta functions, and its definition and fundamental properties are given by TANNERY and MOLK.¹

8. We shall begin with a few definitions, which will facilitate the description of what follows. Let us write the continued fraction for ξ in the two forms

$$\begin{aligned} & [b_1, b_2, \dots, b_m, \overline{a_1, a_2, \dots, a_r}], \\ & [c_1, c_2, \dots, c_n, \dots]. \end{aligned}$$

We have to consider some infinite sequences for which it will be convenient to adopt two notations. Let

$$\gamma_1, \gamma_2, \dots, \gamma_n, \dots$$

be the sequence, written in the usual way, the n th term being denoted by γ_n . Now we associate this sequence with the sequence formed by the partial quotients of the continued fraction for ξ , so that γ_1 corresponds to c_1 , γ_2 to c_2 , and in general γ_n to c_n . For $n > m$, this establishes a correspondence between the γ 's and the a 's. We shall regard a period of the partial quotients of the continued fraction to begin² with a_1 and end with a_r ; so that, when we speak of the partial quotient a_s ($1 \leq s \leq r$) in the j th period, the rank of the corresponding c in the c -sequence will be unambiguously determined. In fact the corresponding c will be $c_{m+(j-1)r+s}$. Now returning to the correspondence between the γ 's and the a 's, we shall denote by $\Gamma_s^{(j)}$ the member of the γ -sequence which corresponds to the partial quotient a_s ($1 \leq s \leq r$) in the j th period; that is to say,

$$\Gamma_s^{(j)} = \gamma_{m+(j-1)r+s}.$$

¹ F. E. pp. 109—111.

² Owing to the cyclic order in which the a 's appear as partial quotients in the continued fraction, one may regard (in the notation given in Lemma 1) a period as beginning with any d_t ($1 \leq t \leq r$) and ending with d_{t+r-1} . The convention adopted here is necessary to make our definitions, that follow, unambiguous.

To indicate that the terms of the γ -sequence are denoted by the alternative Γ -notation we shall write

$$\{\gamma_n\} = \{\Gamma_s^{(j)}\}.$$

We next define a *periodic sequence*. A sequence

$$\gamma_1, \gamma_2, \dots, \gamma_{n_1}, \dots$$

is said to be periodic, if there are two positive integers n_0 and E , so that for every $n > n_0$

$$\gamma_{n+E} = \gamma_n.$$

E is called a *period* of the sequence. Obviously, if E is a period, any integral multiple of E is also a period.¹

9. **Lemma 3.** *Suppose for $n = 1, 2, 3, \dots$, \mathfrak{p}_n is the integer such that*

$$1 \leq \mathfrak{p}_n \leq 8$$

$$\mathfrak{p}_n \equiv p_n \pmod{8}.$$

Then the sequence

$$(16) \quad \mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n, \dots$$

is a *periodic sequence*.²

Let

$$\{p_n\} = \{P_s^{(j)}\}$$

$$\{\mathfrak{p}_n\} = \{\mathfrak{P}_s^{(j)}\}.$$

¹ We can easily prove the following properties of periodic sequences. (i) If E_1, E_2 are two periods of a periodic sequence (γ_n) , and if d is the highest common divisor of E_1, E_2 , then d is also a period. (ii) If l is the least period of the sequence, then every other period is a multiple of l . To prove (i) we observe that there exist two integers n_1, n_2 such that $n_1 E_1 - n_2 E_2 = d$. If n is sufficiently large we have

$$\begin{aligned} \gamma_n &= \gamma_{n+n_1 E_1} && \text{(since } E_1 \text{ is a period)} \\ \gamma_{n+n_1 E_1} &= \gamma_{n+n_1 E_1 - n_2 E_2} && \text{(since } E_2 \text{ is a period)} \\ &= \gamma_{n+d}; \end{aligned}$$

and so $\gamma_n = \gamma_{n+d}$, showing that d is a period. To prove (ii) let $E > l$ be a period, and let d be the highest common divisor of E and l . Then $d \leq l$; also by (i) d is a period. If $d < l$, l would not be the least period. Therefore $d = l$ and E is a multiple of l .

² The sequence formed by the residues of p_1, p_2, \dots to any fixed modulus M (the residues lying between 1 and M) is also periodic. The proof is the same as that for the case $M = 8$ given above.

We shall understand that the congruences which appear in the course of this lemma are all to modulus 8. To prove the lemma, let us suppose for the moment that there are two integers h, k , ($k > h$) so that

$$(17) \quad \begin{cases} P_1^{(h)} \equiv P_1^{(k)} \\ P_2^{(h)} \equiv P_2^{(k)}. \end{cases}$$

Then since¹

$$\begin{cases} P_3^{(h)} = a_3 P_2^{(h)} + P_1^{(h)} \\ P_3^{(k)} = a_3 P_2^{(k)} + P_1^{(k)}, \end{cases}$$

it follows from (17) that

$$P_3^{(h)} \equiv P_3^{(k)}.$$

Now from

$$\begin{cases} P_2^{(h)} \equiv P_2^{(k)} \\ P_3^{(h)} \equiv P_3^{(k)} \end{cases}$$

we deduce similarly that

$$P_4^{(h)} \equiv P_4^{(k)};$$

and so on, till we prove that

$$\begin{cases} P_{r-1}^{(h)} \equiv P_{r-1}^{(k)} \\ P_r^{(h)} \equiv P_r^{(k)}. \end{cases}$$

From these we deduce successively as before

$$\begin{cases} P_1^{(h+1)} \equiv P_1^{(k+1)} \\ P_2^{(h+1)} \equiv P_2^{(k+1)}. \end{cases}$$

It is clear the argument can be repeated indefinitely; so that what we have proved is that, if in the sequence

$$p_1, p_2, \dots$$

there are two sets of consecutive numbers $(\mathfrak{P}_1^{(h)}, \mathfrak{P}_2^{(h)})$, $(\mathfrak{P}_1^{(k)}, \mathfrak{P}_2^{(k)})$ so that

$$(18) \quad \begin{cases} \mathfrak{P}_1^{(h)} = \mathfrak{P}_1^{(k)} \\ \mathfrak{P}_2^{(h)} = \mathfrak{P}_2^{(k)}, \end{cases}$$

¹ In order to show clearly the contents of the proof it is supposed here that $r \geq 4$. The formal alterations necessary when $r < 4$ can be easily seen. When $r = 1$ it will be convenient to regard the period as consisting of two equal partial quotients; this is clearly permissible.

then the sequence (16) is a periodic one, and $(k-h)r$ is a period of the sequence. To complete the proof of the lemma it is only necessary to show that there are two such sets satisfying (18). Now this is obvious since, if we consider the sets

$$(\mathfrak{P}_1^{(l)}, \mathfrak{P}_2^{(l)}), \quad (l=1, 2, \dots)$$

there can only be a finite number of distinct sets, as every \mathfrak{P} is one or other of the numbers 1, 2, ... 8.

The argument used above enables us to prove the following lemma also.

Lemma 4. *Suppose for $n=1, 2, 3, \dots$, q_n is the integer such that*

$$1 \leq q_n \leq 8 \\ q_n \equiv q_n \pmod{8}.$$

Then the sequence

$$(19) \quad q_1, q_2, \dots$$

is a periodic sequence.¹

10. We shall proceed to some lemmas on the numbers $\left[\frac{q_{n-1}}{r_n q_n} \right]$. If a, b are two integers prime to each other (they may be positive or negative) the symbol $\left[\frac{a}{b} \right]$ denotes a certain fourth root of unity. The method of finding the root, when a, b are given, is described by TANNERY and MOLK. In the applications that follow a will always be positive. In quoting the following properties of the symbol for future reference the fact that a is positive is taken into account.²

$$(20) \quad \left[\frac{a}{b} \right] \left[\frac{a}{-b} \right] = 1$$

$$(21) \quad \left[\frac{a+b}{b} \right] = \left[\frac{a}{b} \right] \exp \left\{ \frac{\pi i}{12} b(b^2-1)(2c+d) \right\}$$

$$(22) \quad \left[\frac{a}{b} \right] \left[\frac{b}{a} \right] = \exp \left\{ -\frac{\pi i}{4} (a-1)(b-1) \right\}$$

In (21) c, d are any two integers³ such that $ad-bc=1$.

¹ The periods of the p - and the q -sequences are, of course, not necessarily the same.

In the course of the proof of Lemma 3 the period whose existence is proved is a multiple of r . But smaller periods may very well exist, which are not divisible by r . Thus, for example, if every a is a multiple of 8, it is easily seen that 2 is a period.

² F.E., p. 109.

³ The dependence of $\left[\frac{a+b}{b} \right]$ on c, d is only apparent. See the remarks in F.E., p. 108.

Lemma 5. We have

$$\left[\frac{q_{n+1}}{q_n} \right] = \left[\frac{q_{n-1}}{q_n} \right] e^{\frac{\pi i}{12} v_n},$$

where

$$v_n = \eta_n c_{n+1} q_n (q_n^2 - 1) (p_{n-1} + p_{n+1}).$$

We apply the formula (21) with the following sets of values of a, b, c, d :

$$\begin{aligned} a &= q_{n-1} + \nu q_n, & b &= q_n, \\ c &= \eta_n (p_{n-1} + \nu p_n), & d &= \eta_n p_n. \end{aligned}$$

Here ν is given successively the values $0, 1, 2, \dots, (c_{n+1} - 1)$. Obviously, for every value of ν

$$ad - bc = \eta_n (p_n q_{n-1} - p_{n-1} q_n) = 1,$$

and so formula (21) is applicable.

We have

$$\begin{aligned} \left[\frac{q_{n-1} + q_n}{q_n} \right] &= \left[\frac{q_{n-1}}{q_n} \right] \exp \left\{ \frac{\pi i}{12} \eta_n q_n (q_n^2 - 1) (2p_{n-1} + p_n) \right\}, \\ \left[\frac{q_{n-1} + 2q_n}{q_n} \right] &= \left[\frac{q_{n-1} + q_n}{q_n} \right] \exp \left\{ \frac{\pi i}{12} \eta_n q_n (q_n^2 - 1) (2p_{n-1} + 3p_n) \right\}, \\ \left[\frac{q_{n-1} + 3q_n}{q_n} \right] &= \left[\frac{q_{n-1} + 2q_n}{q_n} \right] \exp \left\{ \frac{\pi i}{12} \eta_n q_n (q_n^2 - 1) (2p_{n-1} + 5p_n) \right\}, \\ &\dots \dots \dots \\ \left[\frac{q_{n-1} + c_{n+1} q_n}{q_n} \right] &= \left[\frac{q_{n-1} + (c_{n+1} - 1) q_n}{q_n} \right] \exp \left\{ \frac{\pi i}{12} \eta_n q_n (q_n^2 - 1) (2p_{n-1} + \overline{2c_{n+1} - 1} p_n) \right\}. \end{aligned}$$

On multiplying these equations and cancelling the common factors which appear on both sides, and remembering that

$$q_{n-1} + c_{n+1} q_n = q_{n+1},$$

$$1 + 3 + 5 + \dots + (2c_{n+1} - 1) = c_{n+1}^2,$$

we get

$$\left[\frac{q_{n+1}}{q_n} \right] = \left[\frac{q_{n-1}}{q_n} \right] \exp \left\{ \frac{\pi i}{12} \eta_n q_n (q_n^2 - 1) (2c_{n+1} p_{n-1} + c_{n+1}^2 p_n) \right\};$$

and using the identity

$$\begin{aligned} 2c_{n+1}p_{n-1} + c_{n+1}^2 p_n &= c_{n+1}(p_{n-1} + p_{n-1} + c_{n+1}p_n) \\ &= c_{n+1}(p_{n-1} + p_{n+1}), \end{aligned}$$

we get the result of the lemma.

Lemma 6. *We have*

$$\left[\frac{q_n}{\eta_{n+1} q_{n+1}} \right] = \left[\frac{q_{n-1}}{\eta_n q_n} \right] e^{\frac{\pi i}{12} w_n}$$

where

$$w_n = c_{n+1} q_n (q_n^2 - 1) (p_{n-1} + p_{n+1}) + 3 \eta_n (q_{n+1} - 1) (q_n - 1).$$

First, let n be even, so that

$$\left[\frac{q_n}{\eta_{n+1} q_{n+1}} \right] = \left[\frac{q_n}{-q_{n+1}} \right], \quad \left[\frac{q_{n-1}}{\eta_n q_n} \right] = \left[\frac{q_{n-1}}{q_n} \right]$$

Now using formulae (20) and (22)

$$\begin{aligned} \left[\frac{q_n}{-q_{n+1}} \right] \left[\frac{q_n}{q_{n+1}} \right] &= 1 \\ \exp \left\{ -\frac{\pi i}{4} (q_{n+1} - 1) (q_n - 1) \right\} &= \left[\frac{q_n}{q_{n+1}} \right] \left[\frac{q_{n+1}}{q_n} \right] \end{aligned}$$

From these two we get

$$\left[\frac{q_n}{-q_{n+1}} \right] = \left[\frac{q_{n+1}}{q_n} \right] \exp \left\{ \frac{\pi i}{4} (q_{n+1} - 1) (q_n - 1) \right\};$$

and so by Lemma 5

$$(23) \quad \left[\frac{q_n}{-q_{n+1}} \right] = \left[\frac{q_{n-1}}{q_n} \right] \exp \left\{ \frac{\pi i}{12} v_n + \frac{\pi i}{4} (q_{n+1} - 1) (q_n - 1) \right\}$$

Remembering that n is even, $\eta_n = 1$, it is seen that (23) is equivalent to the result of the lemma.

Next, let n be odd, so that

$$\left[\frac{q_n}{\eta_{n+1} q_{n+1}} \right] = \left[\frac{q_n}{q_{n+1}} \right], \quad \left[\frac{q_{n-1}}{\eta_n q_n} \right] = \left[\frac{q_{n-1}}{-q_n} \right].$$

By Lemma 5

$$\left[\frac{q_{n+1}}{q_n} \right] = \left[\frac{q_{n-1}}{q_n} \right] e^{\frac{\pi i}{12} v_n}.$$

By (20)

$$\left[\frac{q_{n-1}}{q_n} \right] \left[\frac{q_{n-1}}{-q_n} \right] = 1.$$

Also by (22)

$$\exp \left\{ -\frac{\pi i}{4} (q_{n+1} - 1)(q_n - 1) \right\} = \left[\frac{q_{n+1}}{q_n} \right] \left[\frac{q_n}{q_{n+1}} \right]$$

On multiplying these three equations we get

$$\left[\frac{q_{n-1}}{-q_n} \right] \exp \left\{ -\frac{\pi i}{4} (q_{n+1} - 1)(q_n - 1) \right\} = e^{\frac{\pi i}{12} v_n} \left[\frac{q_n}{q_{n+1}} \right],$$

and so

$$(24) \quad \left[\frac{q_n}{q_{n+1}} \right] = \left[\frac{q_{n-1}}{-q_n} \right] \exp \left\{ -\frac{\pi i}{12} v_n - \frac{\pi i}{4} (q_{n+1} - 1)(q_n - 1) \right\}.$$

Since n is odd, $\eta_n = -1$, and (24) is equivalent to the result of the lemma.

Lemma 7. *Let*

$$\zeta_n = \left[\frac{q_{n-1}}{\eta_n q_n} \right]^8;$$

then the sequence

$$\zeta_2, \zeta_3, \dots, \zeta_n, \dots$$

is a periodic sequence.

Let A, B be respectively the smallest periods of the p -sequence and the q -sequence. Let F be the least common multiple of 2, r, A, B . By Lemmas 3 and 4 there is an integer N' so that for $n \geq N'$

$$\left. \begin{aligned} p_{n+A} &\equiv p_n \\ q_{n+B} &\equiv q_n \end{aligned} \right\} \pmod{8}.$$

To prove the present lemma, suppose for the moment that there are two integers h, k satisfying the following conditions:

$$(25) \quad \left\{ \begin{aligned} k &> h > m \\ h &> N' \\ \zeta_h &= \zeta_k \\ k - h &= \alpha F, \end{aligned} \right.$$

where α is a positive integer.

By Lemma 6 we have

$$(26) \quad \begin{cases} \zeta_{h+1} = \zeta_h e^{\frac{\pi i}{4} w_h} \\ \zeta_{k+1} = \zeta_k e^{\frac{\pi i}{4} w_k} \end{cases}$$

where

$$\begin{aligned} w_h &= c_{h+1} q_h (q_h^2 - 1) (p_{h-1} + p_{h+1}) + 3 \eta_h (q_{h+1} - 1) (q_h - 1), \\ w_k &= c_{k+1} q_k (q_k^2 - 1) (p_{k-1} + p_{k+1}) + 3 \eta_k (q_{k+1} - 1) (q_k - 1). \end{aligned}$$

Now since $k - h$ is a multiple of r , and $k > h > m$,

$$c_{h+1} = c_{k+1};$$

since $k - h$ is even,

$$\eta_h = \eta_k;$$

since $k - h$ is a multiple of A , and $h > N'$, $h - 1 \geq N'$

$$\left. \begin{aligned} p_{h-1} &\equiv p_{k-1} \\ p_{h+1} &\equiv p_{k+1} \end{aligned} \right\} \pmod{8};$$

and lastly since $k - h$ is a multiple of B , and $h > N'$

$$\left. \begin{aligned} q_h &\equiv q_k \\ q_{h+1} &\equiv q_{k+1} \end{aligned} \right\} \pmod{8}.$$

Using these results we see that

$$w_h \equiv w_k \pmod{8},$$

and so

$$e^{\frac{\pi i}{4} w_h} = e^{\frac{\pi i}{4} w_k}.$$

Also by our assumption

$$\zeta_h = \zeta_k;$$

and therefore from (26)

$$\zeta_{h+1} = \zeta_{k+1}.$$

Starting from this result we can argue similarly and prove that

$$\zeta_{h+2} = \zeta_{k+2},$$

and so on; so that, the ζ -sequence will be a periodic sequence with period $k - h = aF$.

We will have therefore proved the lemma, if we establish the existence of two integers h, k , satisfying the conditions (25). To do this let¹

$$\{\zeta_n\} = \{Z_s^{(j)}\},$$

$$F = Gr,$$

so that G is an integer. Further, let β be an integer so large that, if

$$Z_1^{(\beta)} = \zeta,$$

then $t > N'$.

Let us consider the five numbers

$$(27) \quad Z_1^{(\beta)}, \quad Z_1^{(\beta+G)}, \quad Z_1^{(\beta+2G)}, \quad Z_1^{(\beta+3G)}, \quad Z_1^{(\beta+4G)}.$$

Since every ζ_n can have only one of the four values $+1, -1, +i, -i$, there must be at least two among the numbers (27) which are identical. There are therefore two integers $\gamma, \delta, (\delta > \gamma)$ among the numbers 0, 1, 2, 3, 4, so that

$$Z_1^{(\beta+\gamma G)} = Z_1^{(\beta+\delta G)}.$$

If these Z 's are respectively ζ_h, ζ_k when considered as members of the ζ -sequence, then clearly

$$k > h > m$$

$$h \geq t > N'$$

$$k - h = (\delta - \gamma) Gr = \alpha F,$$

α being the positive integer $\delta - \gamma$. The numbers h, k therefore satisfy the conditions (25), and the lemma is completely proved.

Since γ, δ are among the numbers 0, 1, 2, 3, 4, it follows that the ζ -sequence has a period αF where α is one of the numbers 1, 2, 3, 4.

IV.

Behaviour of $\mathcal{P}_3(\mathfrak{o} | \tau)$.

11. We shall apply the results established in the last two sections to obtain the behaviour of $\mathcal{P}_3(\mathfrak{o} | \tau)$ as $\tau \rightarrow \xi$ along the line $x = \xi$.

¹ The absence of the first term ζ_1 is of no importance. As usual ζ_n is supposed to correspond to c_n .

If

$$(28) \quad \tau = \frac{c + dT}{a + bT},$$

where a, b, c, d are integers such that $ad - bc = 1$, then it is known that¹

$$(29) \quad \mathcal{J}_3(\mathfrak{o} | \tau) = \varepsilon'' \sqrt{a + bT} \cdot \mathcal{J}_{\mu+1}(\mathfrak{o} | T),$$

where ε'' is an eighth root of unity, μ is one of the numbers 1, 2, 3, according to the type of the transformation (28), and the square root $\sqrt{a + bT}$ has that determination which has its real part positive.² The value of ε'' is given by the formulae³

$$(30) \quad \frac{\varepsilon''}{\varepsilon} = \exp \left\{ \frac{\pi i}{4} (ab + cd + 2bc + 2a + 2c + 2m'') \right\}$$

$$(31) \quad \varepsilon = \left[\frac{a}{b} \right]^3 \exp \left\{ \frac{\pi i}{4} (ab + ac + bd - acb^2 - 3b) \right\}.$$

The number m'' in (30) is either -1 or $b + d$ according to the type of the transformation (28).

There are six types of transformations according to the parity of the numbers a, b, c, d . The dependence of μ and m'' on a, b, c, d is shown in the following table.⁴ If a is odd, the number 1 is entered in the a -column; if a is even, the number 0 is entered. Similar notation applies to b, c, d .

Type	a	b	c	d	μ	m''
1°	1	0	0	1	2	-1
2°	1	0	1	1	3	$b + d$
3°	1	1	0	1	1	-1
4°	1	1	1	0	3	$b + d$
5°	0	1	1	0	2	-1
6°	0	1	1	1	1	-1

¹ F.E. p. 262, formula (3) with $v = V = 0$. I have slightly altered the notation and interchanged τ and T .

² F.E. p. 91.

³ F.E. p. 262 formulae (5) and (7).

⁴ F.E. p. 241 Table (6), and, p. 262 Table (8).

12. Let us now consider the transformations S_n mentioned in Section II,

$$(32) \quad \tau = \frac{p_{n-1} + \eta_n p_n T}{q_{n-1} + \eta_n q_n T} \quad (n = 2, 3, \dots).$$

They are all transformations of the modular group, since

$$\eta_n p_n q_{n-1} - \eta_n p_{n-1} q_n = \eta_n^2 = 1.$$

The numbers μ , ε'' , m'' corresponding to S_n will naturally depend on n , and to indicate this dependence we shall write

$$\mu + 1 = M_n,$$

$$\varepsilon'' = \omega_n,$$

$$m'' = m_n''.$$

As in Lemma 7, let F be the least common multiple of 2, r , A , B , and let $H = \alpha F$ be a period of the ζ -sequence. Then by Lemmas 7, 3, 4 there is an integer N'' such that for $n \geq N''$

$$\zeta_{n+H} = \zeta_n,$$

$$\left. \begin{aligned} p_{n+H} &\equiv p_{n+A} \equiv p_n \\ q_{n+H} &\equiv q_{n+B} \equiv q_n \end{aligned} \right\} \pmod{8}.$$

We have the following Lemma.

Lemma 8. *Let R be a fixed integer in $1 \leq R \leq H$. Let $n > N''$ and $n \equiv R \pmod{H}$. Then for all values of n which satisfy these two conditions ω_n has a constant value¹, and the transformations S_n are all of the same type.*

Since $n > N''$ and H is a multiple of A , all the numbers p_n are congruent to each other to modulus 8, and so their parity is the same; similarly since $n - 1 \geq N''$, the parity of the numbers p_{n-1} is the same. Since H is a multiple of B , the same argument shows that the parity of the numbers q_n and of the numbers q_{n-1} remain unaltered. So the transformations S_n are all of the same type; M_n therefore retains a constant value. If the common type of the transformations S_n is either 1° , 3° , 5° or 6° , m_n'' retains the constant value -1 . If the common type is either 2° or 4° , $m_n'' = \eta_n(p_n + q_n)$; since H is even, n has

¹ This implies that the ω -sequence is periodic with period H .

the same parity and η_n has the constant value η_R ; therefore from the congruence properties of p_n, q_n to modulus 8, we see that m_n'' takes values which differ by multiples of 8.

Now by (30) and (31)

$$\omega_n = \left[\frac{q_{n-1}}{\eta_n q_n} \right]^3 e^{\frac{\pi i}{4} W_n} = \zeta_n e^{\frac{\pi i}{4} W_n},$$

where

$$W_n = (ab + cd + 2bc + 2a + 2c + 2m_n'') \\ + (ab + ac + bd - acb^2 - 3b),$$

$$a = q_{n-1}, \quad b = \eta_n q_n, \quad c = p_{n-1}, \quad d = \eta_n p_n.$$

Since $n > N''$, $n \equiv R \pmod{H}$, ζ_n retains a constant value by Lemma 7. Also η_n has the constant value η_R . From the congruence properties (mod. 8) of $p_{n-1}, p_n, q_{n-1}, q_n, m_n''$, it follows that W_n takes values which differ by multiples of 8. Hence, finally, ω_n retains a constant value.

The constant values of M_n and ω_n will, in general, depend on R and we shall denote them respectively by $m = m(R)$ and $\Omega = \Omega(R)$. Further let $\lambda = \lambda(R)$ be equal to $+1$ or $-i$ according as R is even or odd.

13. We are now in a position to prove the following, which is the main theorem of this paper.

Theorem. *Suppose σ is a fixed number in the interval $0 \leq \sigma < 1$, and R is a fixed integer in $1 \leq R \leq H$. Let g be the least common multiple of 2 and r ; and let ϱ be the integer such that $1 \leq \varrho \leq g$, $\varrho \equiv R \pmod{g}$. Further let*

$$v = v_n(\sigma) = \xi + iy = \xi + iy_n(\sigma),$$

where

$$y_n(\sigma) = \frac{\sigma}{q_n} + \frac{1-\sigma}{q_{n+1}}.$$

Let n tend to infinity by taking all values congruent to $R \pmod{H}$. Then¹

$$\sqrt[4]{y} \mathfrak{P}_3(\circ | v) \rightarrow \frac{\lambda \Omega \sqrt[4]{J_{\varrho+1}}}{\sqrt{A_{\varrho} - i J_{\varrho+1}}} \mathfrak{P}_m(\circ | \mathfrak{X}_{\varrho}),$$

¹ The numbers $A_{\varrho}, J_{\varrho+1} = J_{\varrho+1}(\sigma), \mathfrak{X}_{\varrho} = \mathfrak{X}_{\varrho}(\sigma)$ are those defined in Section II.

where $\sqrt[4]{y}$, $\sqrt[4]{J_{\varrho+1}}$ denote the real positive fourth roots¹, and

$$\sqrt{A_{\varrho} - iJ_{\varrho+1}}$$

denotes that determination of the square root which has its real part positive.

We may clearly suppose that $n > N''$. Let $\tau = \tau_n(\sigma)$ and $T = T_n(\sigma)$ be connected by the relation (32) or its equivalent

$$(33) \quad T = \eta_{n+1} \frac{p_{n-1} - q_{n-1} \tau}{p_n - q_n \tau}.$$

We shall apply formula (29), the transformation (32) taking the place of (28). Since $n > N''$, $n \equiv R \pmod{H}$ we have, on using the result of Lemma 8,

$$(34) \quad \mathcal{D}_3(\circ | \tau) = \Omega \sqrt{q_{n-1} + \eta_n q_n T} \mathcal{D}_m(\circ | T),$$

the square root having its real part positive.

It is easily verified from (33) that

$$q_{n-1} + \eta_n q_n T = \frac{\eta_n}{(p_n - \xi q_n) - i q_n y}.$$

Remembering that η_n has the constant value η_R , we have

$$(35) \quad \sqrt{\eta}(q_{n-1} + \eta_n q_n T) = \frac{\eta_R q_n \sqrt{y}}{(p_n q_n - \xi q_n^2) - i q_n^2 y}.$$

Now

$$n \equiv R \pmod{H},$$

$$R \equiv \varrho \pmod{g},$$

$$H \equiv \circ \pmod{g},$$

and so

$$n \equiv \varrho \pmod{g}.$$

Therefore, by Lemma 1

$$q_n^2 y = \sigma + (1 - \sigma) \frac{q_n^2}{q_{n+1}} \rightarrow \sigma + (1 - \sigma) L_{\varrho+1}^2 = J_{\varrho+1},$$

$$(p_n q_n - \xi q_n^2) - i q_n^2 y \rightarrow A_{\varrho} - i J_{\varrho+1}.$$

¹ In what follows \sqrt{y} , $\sqrt[4]{y}$, $\sqrt{J_{\varrho+1}}$, $\sqrt[4]{J_{\varrho+1}}$ denote the real positive roots.

Using these in (35) we see that

$$\sqrt{y}(q_{n-1} + \eta_n q_n T) \rightarrow \frac{\eta_R \sqrt{J_{\varrho+1}}}{\mathcal{A}_{\varrho} - i J_{\varrho+1}},$$

and so

$$(36) \quad \sqrt[4]{y} \cdot \sqrt{q_{n-1} + \eta_n q_n T} \rightarrow \frac{\sqrt{\eta_R} \sqrt[4]{J_{\varrho+1}}}{\sqrt{\mathcal{A}_{\varrho} - i J_{\varrho+1}}},$$

where the left hand side has its real part positive. On the right hand side $\sqrt[4]{J_{\varrho+1}}$ is real and positive and

$$\sqrt{\mathcal{A}_{\varrho} - i J_{\varrho+1}} = \Re + i \Im$$

has its real part \Re positive. We have therefore to choose that determination of $\sqrt{\eta_R}$ which makes the real part of the right hand side in (36) positive. Observing that $J_{\varrho+1} > 0$ and $\Re > 0$, it is easily seen that $\Im < 0$, and so we should take

$$(37) \quad \sqrt{\eta_R} = \lambda(R).$$

Next since $n \equiv \varrho \pmod{g}$ we see from Lemma 2 that

$$T_n(\sigma) \rightarrow \mathfrak{X}_{\varrho}(\sigma) = \mathfrak{X}_{\varrho}(\sigma) + i \mathfrak{Y}_{\varrho}(\sigma),$$

and since by the same Lemma

$$\mathfrak{Y}_{\varrho}(\sigma) \geq \frac{1}{2},$$

it follows from the continuity of the Theta functions that

$$(38) \quad \mathfrak{J}_m(\circ | T) \rightarrow \mathfrak{J}_m(\circ | \mathfrak{X}_{\varrho}).$$

Combining (34), (36), (37), (38) we get

$$\sqrt[4]{y} \mathfrak{J}_3(\circ | \tau) \rightarrow \frac{\lambda \Omega \sqrt[4]{J_{\varrho+1}}}{\sqrt{\mathcal{A}_{\varrho} - i J_{\varrho+1}}} \mathfrak{J}_m(\circ | \mathfrak{X}_{\varrho}),$$

which is the result of the theorem.

14. Before we conclude, a few remarks may be made on the magnitude of the number H . Referring to the proof of Lemma 3 we see that a period of the

p-sequence exists which is not greater than Nr , where N is the number of distinct sets

$$(\mathfrak{P}_1^{(l)}, \mathfrak{P}_2^{(l)}) \quad (l=1, 2, 3 \dots),$$

the integers \mathfrak{P} being among the numbers $1, 2, 3, \dots, 8$. Now remembering that p_{n-1}, p_n are prime to each other (since $p_n q_{n-1} - p_{n-1} q_n = \pm 1$) we see that $\mathfrak{P}_1^{(l)}, \mathfrak{P}_2^{(l)}$ cannot both be even; and an easy calculation shows that $N=48$. Therefore there exists an $N_1 \leq 48$ such that $N_1 r$ is a period of the p-sequence. Similarly there is an $N_2 \leq 48$ such that $N_2 r$ is a period of the q sequence. The least common multiple of $2, r, N_1 r, N_2 r$ is easily seen to be not greater than $g \cdot 47 \cdot 48 = 2256 g$. Since the highest value of α is 4, it follows that there exists a suitable value of H not greater than $9024 g$.

An examination of the proof of the main theorem, however, shows that it is not the numbers A, B by themselves that play an essential role, but it is the period of the ω -sequence that is important. Of course, to ensure that the transformations S_n (when n is sufficiently large and $n \equiv R \pmod{H}$) are of the same type, we should take into consideration the periods of the sequences

$$(39) \quad p'_1, p'_2, \dots,$$

$$(40) \quad q'_1, q'_2, \dots,$$

where each p'_n and q'_n is either 1 or 2 and

$$\left. \begin{array}{l} p'_n \equiv p_n \\ q'_n \equiv q_n \end{array} \right\} \pmod{2}.$$

If A', B' are respectively the least periods of the sequences (39), (40), and if χ is the least period of the ω -sequence then the least common multiple of $2, r, A', B', \chi$ will provide a suitable value of H ; and this value will be, in many instances, much less than the bound $9024 g$ assigned above. It can be shown by arguments similar to those used for the p-sequence that $6r$ is a period of the sequences (39) and (40). Hence A', B' are divisors of $6r$ (see the first foot-note on page 155); and there is a value of H not greater than the least common multiple of $6r$ and χ .

