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On Bohr's spectrum of a function

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Let φ be a measurable and bounded function. We will examine the complement of the set of real values t for which

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\varphi(x)\,e^{itx}dx=0. \tag{A}$$

Eggleston [2] has shown that the exceptional set is not necessarily enumerable, but that it has Hausdorff measure zero with respect to $\left(\log\frac{1}{r}\right)^{-1-\varepsilon}$, $\varepsilon>0$. The following more general results are true.

Theorem 1. *The set where (A) is false has Hausdorff measure zero with respect to every increasing function $h(r)$, $h(0)=0$, such that*

$$\int_0^1\frac{h(r)}{r}\,dr<\infty.$$

Theorem 2. *There exists a closed set of positive logarithmic capacity where (A) fails to hold for a suitably chosen φ.*

Suppose $|\varphi(x)|<1$ and let us put

$$g_T(t)=\frac{1}{2T}\cdot\int_{-T}^{T}\varphi(x)\,e^{itx}dx$$

and note two preliminary relations

$$\int_{-\infty}^{\infty}|g_T(t)|^2\,dt<\frac{\pi}{T}. \tag{1}$$

$$\text{If } |g_T(t_0)|>b \text{ then } |g_T(t)|>\frac{b}{2} \text{ if } |t-t_0|<\frac{b}{T}. \tag{2}$$

$$2\,T\,g_T(t)=\int_{-\infty}^{\infty}\varphi_1(x)\,e^{itx}dx,$$

where $\varphi_1(x) = \varphi(x)$ for $|x| \leqslant T$ and $\varphi(x) = 0$ for $|x| > T$.

According to the Parseval relation we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |2\,T g_T(t)|^2\,dt = \int_{-\infty}^{\infty} |\varphi_1(x)|^2\,dx \leqslant 2\,T,$$

which gives (1). (2) follows from the fact that

$$|g_T(t) - g_T(t_0)| = \left|\frac{1}{2\,T} \int_{-T}^{T} \varphi(x)\,(e^{itx} - e^{it_0x})\,dx\right| \leqslant$$

$$\leqslant \frac{1}{2\,T} \int_{-T}^{T} |e^{itx} - e^{it_0x}|\,dx = \frac{2}{T} \int_0^T \left|\sin\left|\frac{t-t_0}{2}\right|x\right|dx < \frac{T}{2}|t-t_0| < \frac{b}{2}.$$

if $|t - t_0| < \dfrac{b}{T}$.

Proof of theorem 1. Let $h(r)$ be an arbitrary function satisfying the conditions in theorem 1. It is enough to show that the set where $\overline{\lim\limits_{T\to\infty}}\,|g_T(t)| > a$ has zero Hausdorff measure with respect to $h(r)$ for arbitrarily small values of a. Suppose a is chosen, $0 < a < 1$. If $m_T(a)$ is the Lebesgue measure of the set where $|g_T(t)| > \dfrac{a}{2}$, then (1) implies

$$m_T(a) < \frac{4\pi}{a^2\,T}. \tag{3}$$

Let $N_T(a)$ be the maximal number of points t for which $|g_T(t)| > \dfrac{a}{2}$ and which are furthermore situated at a distance of at least $\dfrac{a}{T}$ from each other. According to (1) and (2) we have

$$N_T(a) \cdot \left(\frac{a}{4}\right)^2 \cdot \frac{a}{T} < \frac{\pi}{T},$$

$$N_T(a) < \frac{16\pi}{a^3}. \tag{4}$$

Hence, $N_T(a) \leqslant N(a)$ which is independent of T.

It follows from (3) and (4), since $\dfrac{4\pi}{a^2\,T} > 2 \cdot \dfrac{a}{T}$, that for every T $|g_T(t)| > \dfrac{a}{2}$ in at most $N(a)$ different intervals of length $\dfrac{4\pi}{a^2\,T}$ each.

Now let us consider the set of values t where

$$\overline{\lim_{n\to\infty}}\,|g_{T_n}(t)|>\frac{a}{2}, \tag{5}$$

and where $\{T_n\}_1^\infty$ is chosen so that

$$\frac{T_{n+1}}{T_n}=k=k(a)>1.$$

(5) being true for a certain t means that, for this value of t, $|g_{T_n}(t)|>\frac{a}{2}$ for infinitely many n. This obviously means that (5) can be satisfied only in such sets E which can be covered infinitely many times by intervals $\{I_n\}$, where I_n has length $\frac{4\pi}{a^2 T_n}$ and each I_n may be used only $N(a)$ times at the covering. It is thus possible, for n_0 arbitrarily large, to cover E with intervals I_n, $n>n_0$. But

$$\sum_{n=n_0+1}^{\infty} N(a)\,h(c\,T_n^{-1})\leqslant\frac{N(a)}{\log k}\int_0^{c/T_{n_0}}\frac{h(r)}{r}\,dr,$$

where $c=\frac{4\pi}{a^2}$. The right hand side, however, is arbitrarily small, and so (5) can only be satisfied on sets E of h-measure zero. Now suppose t_0 is such that (5) is not satisfied. Choose $n=n(T)$ such that $T_{n+1}>T\geqslant T_n$. For T large enough we have

$$|g_T(t_0)|\leqslant|g_{T_n}(t_0)|+\Big|\frac{1}{2T}\int_{-T}^{-T_n}\varphi(x)\,e^{it_0x}dx\Big|+\Big|\frac{1}{2T}\int_{T_n}^{T}\varphi(x)\,e^{it_0x}dx\Big|\leqslant$$

$$\leqslant\frac{a}{2}+\frac{T-T_n}{T}\leqslant\frac{a}{2}+\frac{T_{n+1}-T_n}{T_n}=\frac{a}{2}+k-1=a,$$

if $k=1+\frac{a}{2}$. This yields $\overline{\lim}\,|g_T(t_0)|\leqslant a$. With this it is shown that the set where $\overline{\lim_{T\to\infty}}\,|g_T(t)|>a$, has vanishing h-measure.

Proof of theorem 2. Let us consider $g_{2^n}(t)$, where n is a positive integer. We can always find a bounded function φ such that $|g_{2^n}(t_0)|\geqslant\frac{1}{2}$, where n and t_0 are arbitrary. Namely, if φ is already chosen in $(-2^n, 2^n)$ such that $|g_{2^n}(t_0)|\geqslant\frac{1}{2}$ and we want to have the inequality $|g_{2^{n+1}}(t_1)|\geqslant\frac{1}{2}$ satisfied, we choose $\varphi(x)=e^{i\theta_1}\cdot e^{-it_1x}$ for $2^n<|x|\leqslant 2^{n+1}$, where $\theta_1=\arg\{g_{2^n}(t_1)\}$. By (2) it then follows that such sets which can be covered infinitely many times by intervals $\{I_n\}_1^\infty$, where I_n has length 2^{-n}, are exceptional sets. There is a closed set with this quality which has positive capacity (Carleson [1]), and so the theorem is shown.

Eggleston's result about the set where (A) is false is shown as a consequence of theorems, due to Erdös and Taylor [3], concerning the set of values of t for which the nonintegral parts of the sequence $\{n_k t\}_{k=1}^\infty$ are not equidistributed, where n_k is an increasing sequence of integers for which $n_{k+1} - n_k < C$, some constant C.

The theorems of Erdös and Taylor say that the set where the non-integer parts of $\{n_k t\}$ are not equidistributed may be nonenumerable, but that it has Hausdorff measure zero with respect to $\left(\log \frac{1}{r}\right)^{-1-\varepsilon}$, $\varepsilon > 0$, if $\{n_k\}$ satisfies the conditions above. The theorems (1) and (2) above say, according to Eggleston's result that the exceptional set, where the non-integer parts are not equidistributed, may have positive capacity, but that it has vanishing Hausdorff measure with respect to $h(r)$, where $h(r)$ satisfies the conditions in theorem 1.

It may be noted that if, instead of boundedness, we assume that $\varphi(x) = O(|x|^\beta)$, $\beta < \frac{1}{2}$, when $|x| \to \infty$ and that φ is bounded in every finite interval, we get an analogous result. Thus it can be shown by a similar method as above that in this case the exceptional set where (A) is false, has vanishing α-capacity if $\alpha > 2\beta$. The fact that (A) holds almost everywhere, if $\beta < \frac{1}{2}$, is a result of Wintner [4], who proved that if φ belongs to L^2 in every finite interval and

$$\int_{-T}^{T} |\varphi(x)|^2 \, dx = O(T^{2-\varepsilon})$$

for $T \to \infty$ and some $\varepsilon > 0$, then the exceptional set has Lebesgue measure zero.

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