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Uniqueness in Cauchy's problem for elliptic equations with double characteristics

By R. N. Pederson

1. Introduction

This paper is concerned with the question of local uniqueness of solutions of Cauchy's problem for elliptic partial differential equations with characteristics of multiplicity not greater than two. Let $P(x,\xi)$ be a homogeneous mth degree elliptic polynomial in $\xi=(\xi_1,\xi_2,\dots,\xi_n)$ with complex coefficients of class C^2 in a closed neighborhood D of the origin in R_n. For a given vector $0\neq N\in R_n$, we define $\mathcal{S}(N)$ to be the class of all C^m surfaces having normal N at the origin. We are interested in determining conditions on P which insure that solutions of Cauchy's problem for the equation

$$P(x, D)u+Q(x, D)u=0$$

are unique for every $(m-1)$st degree polynomial $Q(x, \xi)$ with bounded coefficients and for every initial surface $S\in\mathcal{S}(N)$. For equations with simple characteristics and real coefficients a relatively complete answer to this question was given by Calderón [1]. Various improvements were given by Hörmander [5], [6]. It follows from the work of Cohen [3] and Pliš [10] that one cannot in general allow characteristics of multiplicity exceeding two. Uniqueness results for P's with double characteristics have been obtained by the author [9], Mizohata [7], Hörmander [5] and Shirota [11]. The best result to date is that of Hörmander for the case where P is the product of two polynomials with simple characteristics. Pliš gave an example of a fourth order equation with real C^∞ coefficients having non-trivial solutions which vanish in a half-space.

The question arises as to whether or not there are any irreducible P's for which solutions of Cauchy's problem for (1.1) are unique. An examination of Pliš' counter-example shows that the characteristic roots have unbounded partial derivatives near the initial surface. Douglis [4] proved uniqueness for first order systems in two variables provided that the roots satisfy a smoothness condition and the double roots occur in pairs (that is two roots are either distinct or always equal).

In this paper we obtain a uniqueness theorem for equations with at most double characteristics by imposing a smoothness condition similar to that of Douglis. We do not, however, require that the double roots cannot separate. An example of an irreducible polynomial which satisfies our condition is given by

$$P(\xi_1, \xi_2, \xi_3) = (\xi_1^2+\xi_2^2+\xi_3^2)^2+\xi_2^2(\xi_2^2+\xi_3^2).$$

It is easily seen that the above equation has double characteristics with respect to $(1, 0, 0)$ (in the sense of condition A of section 2) if and only if $\xi_2 = 0$. Our smoothness condition precludes the possibility of an equation with double characteristics having non-singular characteristics or of a surface being strongly pseudo convex in the sense of Hörmander (see Lemma 5.1).

One novelty of our approach is that we reduce all integral inequalities which do not follow directly from pointwise inequalities to inequalities for first order ordinary differential operators. The method is too crude to handle the case of simple real characteristics.

In section 2 we state our smoothness condition and prove that it is invariant under a local change of coordinates which maps the initial surface onto a convex surface. In section 3 we state our uniqueness theorem and our version of Carleman's inequality. Sections 4–8 are devoted to proving the basic inequality stated in section 3.

As is becoming the custom, we denote by D_k the differential operator $-i\partial/\partial x_k$ and $D = (D_1, D_2, ..., D_n)$. If $P(x, \xi)$ is a polynomial in ξ whose coefficients depend on x, the polynomials P^k and P_k are defined by $\partial P/\partial \xi_k$ and $\partial P/\partial x_k$ respectively. If $\alpha = (\alpha_1, \alpha_2, ..., \alpha_q)$ is a multi-index of length q, the polynomials P^α and P_α are defined similarly. A brief exception to the latter convention is made in Lemmas 6.4 and 6.5 where for a polynomial $Q(\tau)$ we let $Q^{(k)}(\tau) = d^k Q/d\tau^k$. The letters C and K represent positive constants, not necessarily the same ones as in previous appearances. R_n and C_n denote the real and complex euclidean n-spaces. If $\xi = (\xi_1, \xi_2, ..., \xi_n) \in R_n$, $\xi' = (0, \xi_2, ..., \xi_n)$. For $u \in C_0^\infty$ we denote by $\hat{u}(x_1, \xi')$ and $\tilde{u}(\xi)$ the Fourier transforms of u with respect to x' and x.

2. The smoothness condition

If the initial surface is convex with normal $N \neq 0$ at the origin, we shall prove uniqueness in Cauchy's problem for elliptic operators which satisfy the following condition. This allows us to generalize the results of Hörmander [5] and Nirenberg [8] for equations with constant coefficients.

A. *The roots* $\tau_1(x, \xi), \tau_2(x, \xi), ..., \tau_m(x, \xi)$ *of* $P(x, \tau N + \xi)$ *have multiplicity at most two and are of class* C^1 *with respect to* (x, ξ) *for* $x \in D$ *and* $\xi \in R_n$ *not proportional to* N.

In case the initial surface is not convex we add the following condition.

B. *For each fixed* $x \in D$, $\tau_1(x, \xi), \tau_2(x, \xi), ..., \tau_m(x, \xi)$ *are analytic in* ξ *for each* $\xi \in R_n$ *not proportional to* N. *For each* ξ_0 *the radius of convergence of the power series expansion of* $\tau_j(x, \xi)$ *about* ξ_0 *has a positive lower bound* ϱ_0 *which is independent of* x.

We remark that we could have equivalently restricted ξ to be in any $n-1$ dimensional hyperplane not containing N (for example $N^\perp$). This amounts to translating each of the roots by the component of ξ in the direction N which cannot effect either the multiplicity or the smoothness. We also remark that A and B are to be understood in the local sense only. We do not require that the roots be globally single valued.

An example of a polynomial which satisfies A but which does not satisfy B is given by

$$P(x, \xi) = (\xi_1^4 + \xi_2^4 + \xi_3^4)^2 + x_1^2(\xi_2 - \xi_3)^3 (\xi_2 + \xi_3)^5.$$

The purpose of B is to insure that A is satisfied after a preliminary local change of coordinates which maps a non-convex surface onto a convex surface. Before proving this we examine the effect of a change of coordinates on the characteristic polynomial. For this purpose we write

$$P(x, D)=\sum_{i_1, i_2, \ldots, i_m=1}^{n} A^{i_1, i_2, \ldots, i_m}(x)\, Dx_{i_1}, Dx_{i_2}, \ldots, Dx_{i_m}. \tag{2.1}$$

By elementary calculus, if $x \to y(x)$ is of class C^m, then

$$Dx_{i_1}, Dx_{i_2}, \ldots, Dx_{i_m}=\sum_{j_1, j_2, \ldots, j_m=1}^{n} \frac{\partial y_{j_1}}{\partial x_{i_1}}, \frac{\partial y_{j_2}}{\partial x_{i_2}}, \ldots, \frac{\partial y_{j_m}}{\partial x_{i_m}}, Dy_{j_1}, Dy_{j_2}, \ldots, Dy_{j_m} \tag{2.2}$$

modulo a differential operator of order $m-1$. By substituting (2.2) into (2.1), we see that in the y-coordinate system the characteristic polynomial is given by

$$P(x(y), J\xi), \tag{2.3}$$

where J is the Jacobian matrix of the transformation $x \to y(x)$.

Theorem 2.1. *Let S_1 and S_2 be surfaces of Class C^m each having the same normal $N \neq 0$ at the origin. Let $P(x, \xi)$ be a polynomial satisfying A and B. There exists a local C^m change of coordinates of the form $x \to y = x + 0(|x|^2)$ which maps S_1 onto S_2 and such that $P(x(y), J\xi)$ satisfies A.*

Proof. The power series expansion of $\tau_j(x, \xi)$ about $\xi_0 \in N^\perp - \{0\}$ provides an analytic continuation of τ_j into $|\zeta-\xi_0| < \varrho_0$, $\zeta \in C_n$. By permanence of functional relations, $\tau_j(x, \zeta)$ is a root of $P(x, \tau N+\zeta)$ when $|\zeta-\xi_0| < \varrho_0$. Since, by A, $\tau_j(x, \xi) \in C'$, the coefficients in its power series expansion about ξ_0 are of class C' for $x \in D$, and hence $\tau_j(x, \zeta)$ is of class C' for $(x, \zeta) \in D \times \{|\zeta-\xi_0| < \varrho_0\}$.

Assume that S_i is parametrized by $\varphi_i(x)=0$ where $\varphi_i \in C^m$, $i=1,2$. Since $N = \operatorname{grad} \varphi_i(0) \neq 0$, one of its components, say N_1, is not zero. The transformation $T_i: y_1 = \varphi_i(x)$, $y_k = x_k$, $k \geqslant 2$, is locally $1-1$, and maps S_i onto $y_1=0$. The transformation $T_2^{-1} T_1$ then is locally $1-1$, maps S_1 onto S_2, and is of the form $x \to y = x + 0(|x|^2)$; hence $J = I + 0(x)$ where I is the identity and $0(x) \in C^{m-1}$. The fact that $x \to y$ is a perturbation of the identity allows us to interchange $0(x)$ and $0(y)$. It then follows from (2.3) that the characteristic polynomial in the y coordinate system is given by

$$Q(y, \xi) = P(x(y), (I + 0(y))\xi).$$

For y sufficiently small, τ in a neighborhood of $\tau_j(0, \xi_0)$ and ξ in a neighborhood of ξ_0, the quantity

$$\zeta = \xi + 0(y)(\tau N + \xi)$$

satisfies $|\zeta - \xi_0| < \varrho_0$. It then follows from the implicit function theorem that for each $j = 1, 2, \ldots, m$, there exists a unique solution $\tau = \tau_j'(y, \xi)$ of

$$\tau = \tau_j(x(y), \xi + 0(y)(\tau N + \xi))$$

which is of class C' for (y, ξ) in a neighborhood of $(0, \xi_0)$. Evidently, $\tau_j'(y, \xi)$ is, for each j, a root of

$$Q(y, \tau N+\xi) \equiv P(x(y), \tau N+\xi+0(y)(\tau N+\xi)).$$

It follows from continuity and $\tau_j'(0, \xi)=\tau_j(0, \xi)$ that the roots τ_j' have multiplicity no greater than two. Hence $Q(y, \xi)$ satisfies A. This completes the proof.

3. The uniqueness theorem and basic inequality

The main result of this paper is

Theorem 3.1. *Let $P(x, \xi)$ be an elliptic polynomial (that is $P(x, \xi) \neq 0$ for real $\xi \neq 0$) with complex coefficients of class C^2 in a closed neighborhood D of the origin. Suppose that $S: \varphi(x)=0$ is a surface of class C^m with normal $N \neq 0$ at the origin. If P satisfies A and B, or if S is convex and P satisifies A, then solutions of* (1.1) *which vanish for $\varphi<0$ are identically zero in a neighborhood of the origin.*

The smoothness condition is, of course, superfluous if the characteristics are simple. If the characteristics are double, the roots of $P(x, \tau N+\xi)$ are algebraic functions and we would expect trouble at the branch points. The crux of our smoothness condition is that ξ is a real vector. That is, we can prove uniqueness for initial surfaces N such that the branch points of $P(x, \tau N+\zeta)$ occur only for complex values of ζ.

Suppose that for $\zeta_0=\xi_0+i\eta_0(\eta_0 \neq 0)$, $P(x, \tau N+\zeta_0)$ has a branch point at $\tau=x_0+iy_0$. Then if $\tilde{N}=y_0 N+\eta_0$ and $\tilde{\xi}=u_0 N+\xi_0$, the polynomial $P(x, \tau\tilde{N}+\tilde{\xi})$ has a root which fails to be analytic at $\tau=i$. Our uniqueness theorem then does not apply to initial surfaces with normal proportional to $\tilde{N}$. Hörmander's result may be the best possible for uniqueness with respect to every initial surface.

The essential tool in uniqueness proofs to date has been a weighted L_2 inequality analogous to an L_1 inequality used by Carleman [2]. Our version of Carleman's inequality is given by

Theorem 3.2. *Let $P(x, \xi)$ be an elliptic polynomial with complex coefficients of class C^2 in a neighborhood D of the origin in R_n and which satisfies A with respect to $N=(1, 0, \ldots, 0)$. There exists a $K>0$ and $\delta_0<1$ such that, if $\delta<\delta_0$ and $\lambda>\delta^{-3}$, then*

$$\int e^{\lambda(x_1-\delta)^2}|P(x, D)u|^2\,dx \geqslant \frac{K}{\delta^2}\sum_{|\alpha|\leqslant m-1}\int e^{\lambda(x_1-\delta)^2}|D_\alpha u|^2\,dx$$

for all $$u \in C_0^m[0 \leqslant x_1 \leqslant \delta/2].$$

Proof of Theorem 3.1. We may, without loss of generality, assume that $N=(1, 0, \ldots, 0)$. By Theorem 2.1 we may assume that the initial surface is convex. Suppose then that u is a solution of $P(x, D)u+Q(x, D)u=0$ which vanishes, say, for $x_1<x_2^2+\ldots+x_n^2$. Let ζ be a C^∞ function of x_1 such that $\zeta=1$ for $x<\delta/4$, and $\zeta=0$ for $x_1>\delta/2$. Since Q has bounded coefficients there exists a constant C such that

$$|P(x, D)u|^2 \leqslant C\sum_{|\alpha|\leqslant m-1}|D_\alpha u|^2.$$

The function $\tilde{u}=\zeta u$ has compact support in $0\leqslant x_1\leqslant\delta/2$. By Theorem 3.2 we have

$$\int_{0<x_1<\delta/2} e^{\lambda(x_1-\delta)^2}|P(x,D)\tilde{u}|^2\,dx \geqslant \frac{K}{\delta^2}\sum_{|\alpha|\leqslant m-1}\int_{0<x_1<\delta/2} e^{\lambda(x_1-\delta)^2}|D_\alpha\tilde{u}|^2\,dx.$$

After splitting the range of integration at $x_1=\delta/4$ and using the fact that $\tilde{u}=u$ when $x_1<\delta/4$ we obtain

$$\int_{\delta/4<x_1<\delta/2} e^{\lambda(x_1-\delta)^2}|P(x,D)\tilde{u}|^2\,dx \geqslant \left(\frac{K}{\delta^2}-C\right)\int_{0<x_1<\delta/4} e^{\lambda(x_1-\delta)^2}\sum_{|\alpha|\leqslant m-1}|D_\alpha u|^2\,dx.$$

We next choose δ so that $K/\delta^2-C>1$. We then shrink the region on the right side of the above inequality to $0<x_1<\delta/8$ and use the monotonicity of $(x_1-\delta)^2$ to obtain

$$e^{\lambda(\frac{3}{4}\delta)^2}\int_{\delta/4<x_1<\delta/2}|P(x,D)\tilde{u}|^2\,dx \geqslant e^{\lambda(\frac{7}{8}\delta)^2}\int_{0<x_1<\delta/8}\sum_{|\alpha|\leqslant m-1}|D_\alpha u|^2\,dx$$

which is clearly impossible for large λ unless u is zero for $x_1\in[0,\delta/8]$. This completes the proof of Theorem 3.1.

4. Reduction of the proof of Theorem 3.2

In this section we reduce the proof of Theorem 3.2 to proving L_2 inequalities between differential operators with constant coefficients acting on functions whose support is small with respect to $\lambda^{-1/2}$. We use a slight modification of the partition of unity used by Hörmander.

Lemma 4.1. *There exists a sequence of points $\{x_g\}$ in R_n and a partition of unity $\{\Psi_g^2\}$ such that $x_g\in\operatorname{supp}\Psi_g$ and $\operatorname{diam}\operatorname{supp}\Psi_g<K\lambda^{-1/2}\delta^{-1/4}$. At most 2^n of the supports of the x_g's intersect at one point. There exists a constant K_m such that if $|\alpha|<m$, then $|D^\alpha\Psi_g|\leqslant K_m(\lambda^{1/2}\delta^{1/4})^{|\alpha|}$.*

Proof. Let $0\leqslant\chi\in C_0^\infty[0\leqslant x_k\leqslant 1/2]$ and assume that

$$\int_{R_n}\chi(x)\,dx=1. \tag{4.1}$$

The non-negative function $\Omega(x)$ defined by

$$\Omega^2(x)=\int_{0\leqslant y_k\leqslant 1}\chi(x-y)\,dy$$

is in $C_0^\infty[-1/2\leqslant x_k\leqslant 3/2]$. It follows from (4.1) that if $\{\mathfrak{L}\}$ denotes the set of lattice points in R_n, then

$$\sum_{\mathfrak{L}}\Omega^2(x-\mathfrak{L})=1.$$

It is clear from the fact that the support of Ω is in the cube $[-1/2\leqslant x_k\leqslant 3/2]$ that at most 2^n of the supports of the functions $\Omega(x-\mathfrak{L})$ intersect at a point. There exists a common bound K_m for $\Omega(x-\mathfrak{L})$ and its first m derivatives. The sequence $\{\Psi_g^2\}$ with $\Psi_g=\Omega(\lambda^{1/2}\delta^{1/4}x-\mathfrak{L})$ and $x_g=\lambda^{-1/2}\delta^{-1/4}\mathfrak{L}$ satisfies the requirements of Lemma 4.1.

Lemma 4.2. *Let $u \in C_0^\infty[R_n]$ and define $u_g = \Psi_g u$. If $P(x, D)$ is an m-th order homogeneous differential operator with C^2 coefficients, then there exists a K such that*

$$|P(x,D)\,u|^2 \geqslant 1/2 \sum_g \{|P(x_g,D)\,u_g|^2 - K\lambda\delta^{1/2}\sum_{|\alpha|=1}|P^\alpha(x_g,D)\,u_g|^2$$
$$-K\lambda^{-1}\delta^{-1/2}\sum_{|\alpha|=1}|P_\alpha(x_g,D)\,u_g|^2 - K\lambda^{-2}\delta^{-1}\sum_{|\alpha|=m}|D_\alpha u_g|^2$$
$$-K\sum_{|\alpha|=m-1}|D_\alpha u_g|^2\} - K\sum_{|\alpha|\leqslant m-2}(\lambda\delta^{1/2})^{m-|\alpha|}\,|D_\alpha u|^2. \tag{4.2}$$

Proof. It follows from Leibniz formula that

$$\Psi_g P(x,D)\,u = P(x,D)\,u_g - \sum_{|\alpha|\geqslant 1}\frac{1}{|\alpha|!}D_\alpha\Psi_g\cdot P^\alpha(x,D)\,u.$$

Since $\sum \Psi_g^2 = 1$, $|D_\alpha\Psi_g| \leqslant K_m(\lambda^{1/2}\delta^{1/4})^{|\alpha|}$ and at most 2^n of the supports of the Ψ_g's intersect at a point, it follows from the inequality $2xy \leqslant \varepsilon x^2 + \varepsilon^{-1}y^2$ that

$$|P(x,D)\,u|^2 \geqslant 1/2\sum_g |P(x,D)\,u_g|^2 - K\sum_{|\alpha|\geqslant 1}(\lambda\delta^{1/2})^{|\alpha|}\,|P^\alpha(x,D)\,u|^2.$$

for some large K.

After re-introducing the partition of unity into each of the above terms where $|\alpha| = 1$, we obtain by similar reasoning

$$|P(x,D)\,u|^2 \geqslant 1/2\sum_g\{|P(x,D)\,u_g|^2 - K\sum_{|\alpha|=1}|P^\alpha(x,D)\,u_g|^2\}$$
$$-K\sum_{|\alpha|\geqslant 2}(\lambda\delta^{1/2})^{|\alpha|}\,|P^\alpha(x,D)\,u|^2. \tag{4.3}$$

By expanding each of the coefficients of $P(x, D)$ to two terms about $x = x_g$, we obtain

$$P(x,D)\,u_g = P(x_g,D)\,u_g + \sum_k (x_k - x_{g,k})\,P_k(x_g,D)\,u_g$$
$$+\tfrac{1}{2}\sum_{j,k}(x_j - x_{g,j})\,(x_k - x_{g,k})\,P_{jk}(\bar{x},D)\,u_g.$$

Within the support of u_g we have $|x - x_g| \leqslant K\lambda^{-1/2}\delta^{-1/4}$. Hence

$$|P(x,D)\,u_g|^2 \geqslant |P(x_g,D)\,u_g|^2 - K\lambda^{-1}\delta^{-1/2}\sum_{|\alpha|=1}|P_\alpha(x_g,D)\,u_g|^2 - K\lambda^{-2}\delta^{-1}\sum_{|\alpha|=m}|D_\alpha u_g|^2. \tag{4.4}$$

Similarly, by expanding the coefficients of $P^\alpha(x, D)$ to one term about x_g, we obtain

$$\sum_{|\alpha|=1}|P^\alpha(x,D)\,u_g|^2 \leqslant \sum_{|\alpha|=1}|P^\alpha(x_g,D)\,u_g|^2 + K\lambda^{-1}\delta^{-1/2}\sum_{|\alpha|=m-1}|D_\alpha u_g|^2. \tag{4.5}$$

The proof of Lemma 4.2 is completed by substituting (4.4) and (4.5) into (4.3) and using the fact that $P^\alpha(x, D)$ is a homogeneous operator of degree $m - |\alpha|$ with bounded coefficients.

5. Pointwise inequalities between polynomials

In this section we derive pointwise inequalities between various polynomials related to a given elliptic polynomial $P(x, \xi)$ which satisfies condition A and has at most double characteristics. The analyticity condition B is needed only to insure that conditon A is preserved after a preliminary change of variables. Once that change of variables has been made, we make no further use of analyticity.

We assume, as we may, that $N=(1, 0, ..., 0)$ and that the coefficient of ξ_1^m in $P(x, \xi)$ is equal to one. Condition A then implies that we may write

$$P(x, \xi)=\prod_{k=1}^{m}(\xi_1-\tau_k(x, \xi')), \tag{5.1}$$

where $\tau_k(x, \xi')$ is of class C^1 with respect to $x \in D$ and $\xi'=(0, \xi_2, ..., \xi_n) \in R_{n-1}$. We shall have use for the Lagrange interpolation polynomials defined by

$$_kP(x, \xi)=P(x, \xi)/(\xi_1-\tau_k(x, \xi'))$$

and $$_{jk}P(x, \xi)=P(x, \xi)/(\xi_1-\tau_j(x, \xi'))(\xi_1-\tau_k(x, \xi')), \quad j \neq k.$$

We begin by proving an identity relating the polynomials $P^{(k)}=\partial P/\partial \xi_k$ and the polynomials $_jP(x, \xi)$. It will be convenient to rewrite $(x, \xi)=(x, \xi_1, \xi')$.

Lemma 5.1. *There exist bounded functions $A_{jk}(x, \xi')$ such that $P^{(k)}(x, \zeta_1, \xi')=\sum_{j=1}^{m} A_{jk}(x, \xi')\,{}_jP(x, \zeta_1, \xi')$ for $(x, \zeta_1, \xi') \in D \times C_1 \times R_{n-1}$.*

Proof. If $k \geqslant 2$, we differentiate (5.1) in order to obtain

$$P^{(k)}(x, \xi_1, \xi')=\sum_{j=1}^{m} \frac{\partial \tau_j(x, \xi')}{\partial \xi_k}\,{}_jP(x, \xi_1, \xi').$$

The functions $\tau_j(x, \xi')$, being roots of a homogeneous polynomial in (ζ_1, ξ'), are homogeneous of degree one in ξ'. Hence their derivatives, $A_{jk}=\partial \tau_j/\partial \xi_k$, are homogeneous of degree zero in ξ' and are therefore bounded by a bound for their restrictions to the compact set $D \times \{|\xi'|=1\}$. If $k=1$, we have $A_{j1}=1$. This completes the proof of Lemma 5.1.

Corollary 5.2. *There exists a constant K such that if $\eta \in R_1$, $\xi' \in R_{n-1}$, $y \in D$ and $u \in C^{m-1}$, then*

$$|P^{(k)}(y, D_1+i\eta, \xi') u(x_1)|^2 \leqslant K \sum_j |{}_jP(y, D_1+i\eta, \xi') u(x_1)|^2. \tag{5.2}$$

Proof. Since the A_{jk}'s of Lemma 5.1 are independent of ξ_1, we have the identity

$$P^k(y, D_1+i\eta, \xi') u(x_1)=\sum_j A_{jk}(x, \xi')\,{}_jP(y, D_1,+i\eta, \xi') u(x_1).$$

The proof of Corollary 5.2 is then an immediate consequence of the boundedness of the A_{jk}'s.

Lemma 5.3. *There exists a constant K such that if $(x, \zeta) = (x, \zeta_1, \xi') \in D \times C_1 \times R_{n-1}$, then*

$$|\zeta|^{2(m-1)} \leqslant K\{\sum_j |{}_jP(x, \zeta)|^2 + (\operatorname{Im} \zeta_1)^2 \sum_{j \neq k} |{}_{jk}P(x, \zeta)|^2\}, \tag{5.3}$$

$$|\zeta|^{2m} \leqslant K\{|P(x, \zeta)|^2 + (\operatorname{Im} \zeta_1)^4 \sum_{j \neq k} |{}_{jk}P(x, \zeta)|^2\}, \tag{5.4}$$

and

$$|P_k(x, \zeta)|^2 \leqslant K\{|P(x, \zeta)|^2 + (\operatorname{Im} \zeta_1)^2 \sum_j |{}_jP(x, \zeta)|^2\}. \tag{5.5}$$

Proof. By homogeneity it is sufficient to prove (5.3) and (5.4) when $|\zeta| = 1$. If $\operatorname{Im} \zeta_1 = 0$ (and hence ζ is real), the interpolation polynomials ${}_jP$ are not zero since $P(x, \xi)$ has no real zeros. If $\operatorname{Im} \zeta_1 \neq 0$ the polynomials ${}_{jk}P(x, \zeta)$ have no common zero since P has at most double characteristics. It follows that the right-hand side of (5.3) is never zero and hence by continuity has a positive lower bound on $D \times \{|\zeta| = 1\}$. Similarly, using the fact that $P(x, \xi) \neq 0$ for real ξ, it is seen that the right-hand side of (5.4) has a positive lower bound on $D \times \{|\zeta| = 1\}$.

In proving (5.5) we assume, by homogeneity, that $|\xi'| = 1$. Since, by ellipticity, $\operatorname{Im} \tau_k(x, \xi') \neq 0$ for $\xi' \neq 0$, and $\tau_k(x, \xi)$ is continuous, there exists an $\eta > 0$ such that

$$|\operatorname{Im} \tau_k(x, \xi')| > \eta$$

for $(x, \xi') \in D \times \{|\xi'| = 1\}$. We obtain by differentiating (5.1) that

$$P_k(x, \zeta_1, \xi') = -\sum_j \frac{\partial \tau_j(x, \xi')}{\partial x_k} \quad {}_jP(x, \xi_1, \xi').$$

Since $\partial \tau_j / \partial x_k$ is bounded on $D \times \{|\xi'| = 1\}$, there exists a K such that

$$|P_k(x, \zeta_1, \xi')|^2 \leqslant \frac{K}{\eta^2} (\operatorname{Im} \zeta_1)^2 \sum_j |{}_jP(x, \zeta_1, \xi')|^2, \tag{5.6}$$

if $|\operatorname{Im} \zeta_1| > \eta$ and $(x, \xi') \in D \times \{|\xi'| = 1\}$. If $|\operatorname{Im} \zeta_1| < \eta$, $P(x, \zeta_1, \xi') \neq 0$. As a polynomial in ζ_1 P is of higher degree than P_k. It follows that there exists a constant K such that

$$|P_k(x, \zeta_1, \xi')|^2 \leqslant K |P(x, \zeta_1, \xi')|^2 \tag{5.7}$$

if $|\operatorname{Im} \zeta_1| < \eta$ and $(x, \xi') \in D \times \{|\xi'| = 1\}$. The proof of (5.5) (with a different K) is concluded by combining (5.6) and (5.7). This completes the proof of Lemma 5.3.

6. L_2 inequalities between ordinary differential operators

In section 4 we reduced the proof of Theorem 3.2 to proving inequalities between expressions of the form

$$\int e^{\lambda(x_1 - \delta)^2} |Q(D) u|^2 \, dx,$$

where Q is a partial differential operator with constant coefficients. By taking the Fourier transform with respect to $x' = (x_2, \ldots, x_n)$ and applying Parseval's identity the problem can be further reduced to considering weighted L_2 norms of ordinary differential operators. In this section, we therefore prove some preliminary L_2 inequalities between ordinary differential operators. The first is the simplest case of a general inequality of Trèves [12] p. 137. For completeness we repeat his proof.

Lemma 6.1. *If* $\mu \in C_1$ *and* $\delta \in R_1$, *then for every* $u \in C_0^\infty(R_1)$

$$\int |(D_t + i\lambda(t-\delta) + \mu)\, u|^2\, dt \geqslant 2\lambda \int |u|^2\, dt.$$

Proof. Let the operator A be defined by $A = D_t + i\lambda(t-\delta) + \mu$. The commutator of A and its formal adjoint $\bar{A} = D_t - i\lambda(t-\delta) + \bar{\mu}$ is given by

$$\bar{A}A - A\bar{A} = 2\lambda.$$

It follows that if $u \in C_0^\infty$, then

$$\|Au\|^2 = \|\bar{A}u\|^2 + 2\lambda\|u\|^2 \geqslant 2\lambda\|u\|^2.$$

This completes the proof of Lemma 6.1.

We note that Lemma 6.1 places no restriction on μ or on the support of u. If $\mu = 0$ and the support of u is suitably restricted, we can obtain an inequality which is stronger for large λ.

Lemma 6.2. *If* $u \in C_0^\infty[0, \delta/2]$, *then*

$$\int |(D_t + i\lambda(t-\delta)\, u|^2\, dt \geqslant \frac{(\lambda\delta)^2}{4} \int |u|^2\, dt \quad \textit{for all } \lambda \geqslant 0.$$

Proof. The right-hand side of the above inequality is equal to (recall that $D_t = -i\,(d/dt)$

$$\int \{|u'(t)|^2 - 2\lambda(t-\delta)\ \mathrm{Re}\ u'\bar{u} + \lambda^2(t-\delta)^2|u|^2\}\, dt.$$

By writing $2\ \mathrm{Re}\ u'\bar{u} = d/dt\,|u|^2$ and integrating by parts, we see that the second term is non-negative for $\lambda \geqslant 0$. The proof is completed by noting that $(t-\delta)^2 \geqslant \delta^2/4$ within the support of u. We obtain from Lemma 6.1 immediate relations between a given polynomial with constant coefficients and the corresponding Lagrange interpolation polynomials.

Lemma 6.3. *Let* $P(\tau)$ *be a polynomial of degree* m *with roots* $\tau_1, \tau_2, \ldots, \tau_m$. *If* ${}_jP(\tau) = P(\tau)/(\tau - \tau_j)$ *and* ${}_{jk}P(\tau) = P(\tau)/(\tau-\tau_j)(\tau-\tau_k)$, $j \neq k$, *then*

$$\int |P(D_t + i\lambda(t-\delta))\, u|^2\, dt \geqslant \frac{2\lambda}{m} \sum_j \int |{}_jP(D_t + i\lambda(t-\delta))\, u|^2\, dt, \tag{6.1}$$

$$\int |{}_jP(D_t + i\lambda(t-\delta)\, u|^2\, dt \geqslant \frac{2\lambda}{(m-1)} \sum_k \int |{}_{jk}P(D_t + i\lambda(t-\delta))\, u|^2\, dt, \tag{6.2}$$

and $$|P(D_t+i\lambda(t-\delta))\,u|^2\,dt \geqslant \frac{4\lambda^2}{m(m-1)}\sum_{j\neq k}\int|_{jk}P(D_t+i\lambda(t-\delta))\,u|^2\,dt \tag{6.3}$$

for all $u\in C_0^\infty(R_1)$.

Proof. (6.1) is proved by writing $P(\tau)=(\tau-\tau_j)\,_jP(\tau)$, applying Lemma 6.1 with $\mu=-\tau_j$, and summing with respect to j. The proofs of (6.2) and (6.3) are similar.

The next two lemmas will be used to determine the error resulting when an operator $R(D_t+i\lambda(t-\delta)$ is replaced by the constant coefficient operator $R(D_t+i\lambda(t_0-\delta))$.

Lemma 6.4. *Let $Q(\tau)$ be a polynomial of degree m. There exist constants $C_{k,j}$ such that for any quadratic polynomial $\omega(t)$ we have*

$$Q(D_t+i\omega'(t))=\sum_{k=0}^{m}\sum_{j=0}^{[k/2]}C_{kj}[\omega'(t)]^{k-2j}[\omega''(t)]^j\,Q^{(k)}(D_t).$$

(Here $Q^{(k)}(\tau)=d^kQ/dt^k$.)

Proof. We first note that $Q(D_t+i\omega'(t))\,u=e^{\omega(t)}Q(D_t)\,e^{-\omega(t)}u$. We then apply Leibniz formula in order to obtain

$$Q(D_t)\,e^{-\omega(t)}u=\sum_{k=0}^{m}\frac{1}{k!}D_t^k\,e^{-\omega(t)}\cdot Q^{(k)}(D_t)\,u. \tag{6.4}$$

It is easily verified that there exist constants C_{kj} such that

$$\frac{1}{k!}D_t^k\,e^{-\omega(t)}=e^{-\omega(t)}\sum_{j=0}^{[k/2]}C_{kj}[\omega'(t)]^{k-2j}[\omega''(t)]^j. \tag{6.5}$$

(It can be shown that $C_{kj}=(-1)^k/2^j j!(k-2j)!$.) The proof of Lemma 6.4 is completed by substituting (6.5) into (6.4).

Lemma 6.5. *Let $Q(\tau)$ be a polynomial and let $I_{\lambda\eta}$ be an interval of length $C(\lambda\eta)^{-1/2}$, $0<\eta<1$. If $t_0\in I_{\lambda\eta}$ and $u\in C_0^\infty[I_{\lambda\eta}]$, then there exists a constant $K>0$ such that*

$$\tfrac{3}{2}|Q(D_t+i\lambda(t-t_0))\,u|^2+K\sum_{k\geqslant 1}\left(\frac{\lambda}{\eta}\right)^k|Q^{(k)}(D_t+i\lambda(t-t_0))\,u|^2\geqslant|Q(D_t)\,u|^2.$$

$$\geqslant\tfrac{1}{2}|Q(D_t+i\lambda(t-t_0))\,u|^2-K\sum_{k\geqslant 1}\left(\frac{\lambda}{\eta}\right)^k|Q^{(k)}(D_t+i\lambda(t-t_0))\,u|^2.$$

Proof. Let $\omega(t)=\tfrac{1}{2}\lambda(t-t_0)^2$. If $t\in I_{\lambda\eta}$, we have $|\omega'(t)|\leqslant C(\lambda/\eta)^{1/2}$ and $|\omega''(t)|\leqslant\lambda$. It follows that

$$\left|\sum_{j=0}^{[k/2]}C_{kj}[\omega'(t)]^{k-2j}[\omega''(t)]^j\right|\leqslant K(\lambda/\eta)^{k/2}. \tag{6.6}$$

Hence, by Lemma 6.4, we have

$$|Q(D_t+i\lambda\omega'(t))\,u-Q(D_t)\,u|\leqslant K\sum_{k\geqslant 1}\left(\frac{\lambda}{\eta}\right)^{k/2}|Q^{(k)}(D_t)\,u| \tag{6.7}$$

for $t \in I_{\lambda\delta}$. By applying (6.7) to each $Q^{(k)}$, $k \geqslant 1$, we obtain (with a larger K)

$$|Q(D_t + i\lambda\omega'(t))\,u - Q(D_t)\,u| \leqslant K \sum_{k\geqslant 1} \left(\frac{\lambda}{\eta}\right)^{k/2} |Q^{(k)}(D_t + i\lambda\omega'(t))\,u|. \tag{6.8}$$

The proof of Lemma 6.5 follows by squaring both sides of 6.8 and using the inequality between geometric and arithmetic means in the form $2xy = \varepsilon x^2 + \varepsilon^{-1} y^2$.

7. L_2 inequalities between partial differential operators

We now use the results of the previous section to prove inequalities suggested by Lemma 4.2. We begin with an inequality which gives an estimate for lower order derivatives in terms of $(m-1)$st derivatives.

Lemma 7.1. *If $u \in C_0^\infty[0 \leqslant x_1 \leqslant \delta/2]$, then there exists a constant K such that for all $\lambda \geqslant 0$ and $k < m-1$.*

$$\sum_{|\alpha|=m-1} \int e^{\lambda(x_1-\delta)^2} |D_\alpha u|^2\,dx \geqslant K(\lambda\delta)^{2(m-1-k)} \sum_{|\alpha|=k} \int e^{\lambda(x_1-\delta)^2} |D_\alpha u|^2\,dx$$

for all $\lambda \geqslant 0$ and $k < m-1$.

Proof. By applying Lemma 6.2 to $v = e^{\lambda(x_1-\delta)^2/2} D_\alpha u$, we obtain

$$\sum_{|\alpha|=j} \int e^{\lambda(x_1-\delta)^2} |D_1 D_\alpha u|^2\,dx \geqslant \frac{(\lambda\delta)^2}{4} \sum_{|\alpha|=j} \int e^{\lambda(x_1-\delta)^2} |D_\alpha u|^2\,dx.$$

If we sum the left side of the above inequality over all $(j+1)$st derivatives, we obtain an inequality which, when interated, yields Lemma 7.1.

The next lemma shows that Theorem 3.2 is true for functions whose support is small with respect to λ.

Lemma 7.2. *Suppose that $\Omega_{\lambda\delta} \in R_n$ is a subdomain of $[0 < x_1 < \delta/2]$ whose support is of diameter $0(\lambda^{-1/2}\delta^{-1/4})$. There exist constants $K > 0$ and $\delta_0 < 1$ such that if $\delta < \delta_0$ and $\lambda > \delta^{-3}$, then*

$$\int e^{\lambda(x_1-\delta)^2} |P(y, D)\,u|^2\,dx \geqslant \frac{K}{\delta^2} \sum_{|\alpha|=m-1} \int e^{\lambda(x_1-\delta)^2} |D_\alpha u|^2\,dx$$

for all $u \in C_0^\infty(\Omega_{\lambda\delta})$ and $y \in \Omega_{\lambda\delta}$.

Proof. Let $v = e^{\lambda(x_1-\delta)^2/2} u$, $\hat{v}(x_1, \xi')$ the Fourier transform of v with respect to $(x_2, \ldots, x_n)$, and $\tilde{v}(\xi)$ the Fourier transform of v with respect to $(x_1, \ldots, x_n)$. It is a consequence of Parseval's identity that

$$\int e^{\lambda(x_1-\delta)^2} |P(y, D)\,u|^2\,dx = \int |P(y, D_1 + i\lambda(x_1-\delta), \xi')\,\hat{v}|^2\,dx_1\,d\xi'. \tag{7.1}$$

By using (6.1) and (6.3), we see that, if $\lambda > 1/\delta^2$, the right-hand side of (7.1) dominates

$$\frac{K}{\delta^2}\int\{\sum_j |{}_jP(y,D_1+i\lambda(x_1-\delta),\xi')\,\hat{v}|^2+(\lambda\delta)^2\sum_{j\neq k}|{}_{jk}P(y,D_1+i\lambda(x_1-\delta),\xi')\,\hat{v}|^2\}\,dx_1\,d\xi' \quad (7.2)$$

for some $K>0$. We next multiply both sides of (5.3) by $|\tilde{v}(\xi)|^2$ and integrate in order to obtain

$$\int\sum_j |{}_jP(y,\xi+i\lambda(y_1-\delta))|^2+\lambda^2(y_1-\delta)^2\sum_{j\neq k}|{}_{jk}P(y,\xi+i\lambda(y_1-\delta))|^2\}\,|\tilde{v}(\xi)|^2\,d\xi$$

$$\geqslant K\int|\xi+i\lambda(y_1-\delta)|^{2(m-1)}|\tilde{v}(\xi)|^2\,d\xi. \quad (7.3)$$

By using Parseval's identity (with respect to x_1) and $(y_1-\delta)^2\leqslant\delta^2$ in (7.3) we obtain

$$\int\{\sum_j |{}_jP(y,D_1+i\lambda(y_1-\delta),\xi')\,\hat{v}|^2+(\lambda\delta)^2\sum_{j\neq k}|{}_{jk}P(y,D_1+i\lambda(y_1-\delta),\xi')\,\hat{v}|^2\}\,dx_1\,d\xi'$$

$$\geqslant K\int|(D_1+i\lambda(y_1-\delta),\xi')\,\hat{v}|^{2(m-1)}\,dx_1\,d\xi'. \quad (7.4)$$

Again using Parseval's identity, we see that

$$\int|(D_1+i\lambda(x_1-\delta),\xi')\,\hat{v}|^{2(m-1)}\,dx_1\,d\xi'=\sum_{|\alpha|=m-1}\int e^{\lambda(x_1-\delta)^2}|D_\alpha u|^2\,dx. \quad (7.5)$$

We note that the left side of (7.5) is the right side of (7.4) with y_1 replaced by x_1. In order to complete the proof of Lemma 7.2 it therefore suffices to examine the error introduced by replacing y_1 by x_1 in (7.4). Since diam supp $\Omega_{\lambda\delta}=0(\lambda^{-1/2}\delta^{-1/4})$, it follows from Lemma 6.5 with $\eta=\delta^{1/2}$, that this error is bounded by

$$K\left\{\sum_{|\alpha|\leqslant m-2}\left(\frac{\lambda}{\delta^{1/2}}\right)^{m-1-|\alpha|}\int e^{\lambda(x_1-\delta)^2}|D_\alpha u|^2\,dx\right.$$

$$\left.+(\lambda\delta)^2\sum_{|\alpha|\leqslant m-3}\left(\frac{\lambda}{\delta^{1/2}}\right)^{m-2-|\alpha|}\int e^{\lambda(x_1-\delta)^2}|D_\alpha u|^2\,dx\right\}. \quad (7.6)$$

By using Lemma (7.1) together with $\lambda^{-1}\leqslant\delta^3$ we see that (7.6) is bounded by

$$K\delta^{1/2}\sum_{|\alpha|=m-1}\int e^{\lambda(x_1-\delta)^2}|D_\alpha u|^2\,dx. \quad (7.7)$$

The proof of Lemma 7.2 is completed by using (7.1)–(7.7) together with the remark following (7.5).

The next lemma shows that when the characteristics are smooth we can get a better estimate for P_k than we can for arbitrary mth derivatives (compare with Lemma 7.4).

Lemma 7.3. *Suppose that $\Omega_{\lambda\delta}\subset R_n$ is a subdomain of $[0<x_1<\delta/2]$ whose support has diameter of order $0(\lambda^{-1/2}\delta^{-1/4})$. There exist constants $K>0$ and $\delta_0<1$ such that if $\delta<\delta_0$ and $\lambda>\delta^{-3}$ then*

$$\int e^{\lambda(x_1-\delta)^2} |P_k(y,D)u|^2 dx \leq K\lambda\delta^2 \int e^{\lambda(x_1-\delta)^2} |P(y,D)u|^2 dx + \frac{K\lambda}{\delta^{1/2}} \sum_{|\alpha|=m-1} \int e^{\lambda(x_1-\delta)^2} |D_\alpha u|^2 dx$$

for all $u \in C_0^\infty(\Omega_{\lambda\delta})$ *and* $y \in \Omega_{\lambda\delta}$.

Proof. Let v, $\hat{v}$, and $\tilde{v}$ be defined as in the proof of Lemma 7.2. We multiply both sides of (5.3) by $|\tilde{v}(\xi)|^2$ and integrate in order to obtain

$$\int |P_k(y,\xi+i\lambda(y_1-\delta))\tilde{v}(\xi)|^2 d\xi \leq K\int |P(y,\xi+i\lambda(y_1-\delta))\tilde{v}(\xi)|^2 d\xi + K\lambda^2(y_1-\delta)^2 \sum_j \int |{}_jP(y,\xi+i\lambda(y_1-\delta))\tilde{v}(\xi)|^2 d\xi. \tag{7.8}$$

As in the previous proof we apply Parseval's formula with respect to x_1 and then use Lemma 6.5 to estimate the error introduced by replacing $(y_1-\delta)$ by $(x_1-\delta)$. The result is

$$\begin{aligned}\int |P_k(y,D_1+i\lambda(x_1-\delta),\xi')\hat{v}|^2 dx_1 d\xi' &\leq K\int |P(y,D_1+i\lambda(x_1-\delta),\xi')\hat{v}|^2 dx, d\xi' \\ &+ K\lambda^2\delta^2 \sum_j \int |{}_jP(y,D_1+i\lambda(x_1-\delta),\xi')\hat{v}|^2 dx_1 d\xi' \\ &+ K\sum_{|\alpha|\leq m-1}\left(\frac{\lambda}{\delta^{1/2}}\right)^{m-|\alpha|}\int e^{\lambda(x_1-\delta)^2}|D_\alpha u|^2 dx \\ &+ K(\lambda\delta)^2\sum_{|\alpha|\leq m-2}\left(\frac{\lambda}{\delta^{1/2}}\right)^{m-1-|\alpha|}\int e^{\lambda(x_1-\delta)^2}|D_\alpha u|^2 dx.\end{aligned} \tag{7.9}$$

The proof of Lemma 7.3 is completed by using Lemma 6.3 to estimate the term involving ${}_jP$ and by using Lemma 7.1 to estimate the error terms.

The next lemma gives an estimate for arbitrary mth derivatives in terms of $P(y, D)u$.

Lemma 7.4. *If* $\Omega_{\lambda\delta} \subset [0 < x_1 < \delta]$ *and has support of diameter* $0(\lambda^{-1/2}\delta^{-1/4})$, *there exist constants* $K > 0$ *and* $\delta_0 < 1$ *such that if* $\delta < \delta_0$ *and* $\lambda > \delta^{-3}$ *then*

$$\sum_{|\alpha|=m} \int e^{\lambda(x_1-\delta)^2}|D_\alpha u|^2 dx \leq K\lambda^2\delta^4 \int e^{\lambda(x_1-\delta)^2}|P(y,D)u|^2 dx.$$

for all $u \in C_0^\infty[\Omega_{\lambda\delta}]$ *and* $y \in \Omega_{\lambda\delta}$.

Starting with (5.4), the proof of Lemma 7.4 follows the same line as the two previous proofs. We leave the details to the reader.

Finally we state a lemma which shows that when the characteristics are smooth we can obtain better estimates for $P^k(x, D)u$ then we can for arbitrary $(m\text{-}1)$st derivatives.

Lemma 7.5. *There exists a constant $K>0$ such that*

$$\int e^{\lambda(x_1-\delta)^2}|P(y,D)u|^2\,dx \geqslant K\lambda \int e^{\lambda(x_1-\delta)^2}|P^k(y,D)u|^2\,dx$$

for all $u \in C_0^\infty[0<x_1<\delta/2]$ and $k=1,2,\ldots,n$.

Proof. Corollary 5.2 applied to $\hat{v}(x_1,\xi')$ gives

$$\int |P^k(y, D_1+i\lambda(x_1-\delta),\xi')\hat{v}|^2\,dx_1\,d\xi' \leqslant K\sum_j \int |_jP(y, D_1+i\lambda(x_1-\delta),\xi')\hat{v}|^2\,dx_1\,d\xi'.$$

The proof is completed by applying Lemma 6.3 to the right side of the inequality and using Parseval's identity.

8. Completion of the proof of Theorem 3.2

After multiplying both sides of (4.2) by $e^{\lambda(x_1-\delta)^2}$, integrating, and applying Lemma's 7.2–7.5, we obtain

$$\int e^{\lambda(x_1-\delta)^2}|P(x,D)u|^2\,dx \geqslant \sum_g \Big\{(1/4-O(\delta^{1/2}))\int e^{\lambda(x_1-\delta)^2}|P(x_g,D)u|^2\,dx + \frac{K}{\delta^2}(1-O(\delta))\sum_{|\alpha|=m-1}\int e^{\lambda(x_1-\delta)^2}|D_\alpha u_g|^2\,dx\Big\} - K\sum_{|\alpha|\leqslant m-2}(\lambda\delta^{1/2})^{m-|\alpha|}\int e^{\lambda(x_1-\delta)^2}|D_\alpha u|^2\,dx. \tag{8.1}$$

It follows from Leibniz formula as in the proof of Lemma 4.2 that

$$\sum_{|\alpha|=m-1}|D_\alpha u_g|^2 \geqslant 1/2\sum_{|\alpha|=m-1}\Psi_g^2|D_\alpha u|^2 - K\sum_{|\alpha|\leqslant m-2}(\lambda\delta^{1/2})^{m-1-|\alpha|}|D_\alpha u|^2. \tag{8.2}$$

We next substitute (8.2) into (8.1), use Lemma 7.1 to estimate the derivatives of order $\leqslant m\text{-}2$ and choose δ sufficiently small.

The result is

$$\int e^{\lambda(x_1-\delta)^2}|P(x,D)u|^2\,dx \geqslant \frac{K}{\delta^2}\sum_{|\alpha|=m-1}\int e^{\lambda(x_1-\delta)^2}|D_\alpha u|^2\,dx.$$

The proof of Theorem 3.2 is completed by using Lemma 7.1 together with the above inequality to show that

$$\int e^{\lambda(x_1-\delta)^2}|P(x,D)u|^2\,dx \geqslant \frac{K}{\delta^2}\sum_{|\alpha|\leqslant m-1}\int e^{\lambda(x_1-\delta)^2}|D_\alpha u|^2\,dx.$$

This completes the proof.

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Carnegie Institute of Technology, Pittsburgh, Pa. 15213, U.S.A.

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