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Multi-dimensional integral limit theorems

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1. Introduction

Let $X=(X_1, \ldots, X_k)$ be a random vector (r.v.) in the k-dimensional Euclidean space R_k, $k>1$, with zero mean and non-singular covariance matrix M. Further, let $X^{(1)}, \ldots, X^{(n)}$ be a sequence of independent r.v.'s in R_k with the same distributions as X. Then the normed sum $Y_n=n^{-\frac{1}{2}}\sum_{\nu=1}^{n}$ is approximately normally distributed with zero mean and covariance matrix M. Bergström [3] has shown that if $F_n(x)$, $x\in R_k$, is the d.f. of Y_n, and $\Phi(x)$ is the corresponding normal d.f. then, if the moments of the third order are finite:

$$|F_n(x)-\Phi(x)|\leqslant C\,n^{-\frac{1}{2}} \tag{1}$$

where C is a constant only depending on the moments of X. Esseen [8] has studied $F_n(A)=\int_A dF_n(x)$, where A is a closed sphere in R_k with its center in the origin: $A=\{x:|x|\leqslant a\}$ $(|x|=(x_1^2+\ldots+x_k^2)^{\frac{1}{2}})$ in the case $M=E_k$ (identity matrix of order $k\times k$). His result is that, if the moments of the fourth order are finite, then,

$$|F_n(A)-\Phi(A)|\leqslant C\,n^{-k/(k+1)}$$

Under the same condition, R. R. Rao [7] has announced without proof the result

$$|F_n(B)-\Phi(B)|\leqslant C\,n^{-1/2}(\log n)^{\beta}$$

where $\beta=(k-1)/2(k+1)$, valid uniformly for all convex Borel sets $B\subset R_k$, and also the expansion of $F_n(B)$ in powers of $n^{-1/2}$ given in Theorem 4, but with the remainder term $0(n^{-(s-2)/2}(\log n)^{(k-1)/2})$.

If the d.f. of X either has an absolutely continuous component or is of lattice type, it is possible to prove local limit theorems, that is, limit theorems for the density function of the absolutely continuous component of $F_n(x)$ or for probabilities corresponding to the lattice points of $F_n(x)$. By integrating (summing) the remainder terms in theorems of this type, A. Bikjalis [4, 5] has obtained integral limit theorems for arbitrary Borel sets and for arbitrary subsets of the lattice set of $F_n(x)$ respectively.

In the present paper, I shall prove two generalizations of (1) (Theorems 1 and 2) by a method which is entirely different from the one used by Bergström, who considers an expansion of $(F_n(x)-\Phi(x))\ast\Phi(x/\varepsilon)$ (convolution), together with an estimation of Weierstrass's singular integral. Theorems 3 and 4 give, as mentioned above, estimates of $F_n(B)$ for arbitrary Borel sets and for convex Borel sets respectively, when the moments of order r, $2<r\leqslant 3$, or of order s, $s\geqslant 3$, are finite.

2. Convergence of characteristic functions

The basic fact, upon which all my estimations are based, is the convergence of the characteristic function (ch.f.) of Y_n towards that of $\Phi(x)$.

If $F(x)$, $x \in R_k$ and $f(t)$, $t \in R_k$ are the d.f. and ch.f. of X, that is

$$f(t) = \int_{R_k} e^{i(t,x)} dF(x), \quad (t, x) = \sum_{j=1}^{k} t_j x_j$$

then $f_n(t) = f^n(t/\sqrt{n})$ is the ch.f. of Y_n. Denoting the rth absolute moment of X by $\beta_r = E|X|^r = E(X_1^2 + \ldots + X_k^2)^{r/2}$, we state the following lemma.

Lemma 1. (a) *If $\beta_r < \infty$ for some r, $2 < r \leqslant 3$, then*

$$|f_n(t) - e^{-\frac{1}{2}(t, Mt)}| \leqslant C \cdot n^{-(r-2)/2} |t|^r e^{-\alpha|t|^2}$$

for all t with $|t| \leqslant K\sqrt{n}$.

(b) *If $\beta_s < \infty$ for some integer $s \geqslant 3$, then*

$$\left| f_n(t) - \left(1 + \sum_{\nu=1}^{s-2} n^{-\nu/2} P_\nu(it)\right) e^{-\frac{1}{2}(t, Mt)} \right| \leqslant C \cdot d(n, t) n^{-(s-2)/2} |t|^s e^{-\alpha|t|^2}$$

for all t with $|t| \leqslant K\sqrt{n}$.

By K, α and C we denote here and in what follows unspecified positive constants only depending on k, s and the moments of X. $d(n,t)$ is bounded by one for all n and t, and $\lim_{n\to\infty} d(n,t) = \lim_{t\to 0} d(n,t) = 0$. $P_\nu(it)$ are polynomials in it, the coefficients of which are independent of n and functions of the moments of X (cf. von Bahr [1]).

We shall also need estimates for the derivatives of $f_n(t)$. We define for each k-tuple of non-negative integers $m = (m_1, \ldots, m_k)$ the differential operator

$$\mathcal{D}_m = \prod_{j=1}^{k} \left(\frac{\partial}{\partial t_j}\right)^{m_j} \quad \text{of order} \quad |m| = \sum_{j=1}^{k} m_j.$$

Lemma 2. *If for some k-tuple of non-negative integers $m = (m_1, \ldots, m_k)$, the moments $E|\prod_{j=1}^{k} X_j^{l_j}| < \infty$ for all $l = (l_1, \ldots, l_k)$ with $0 \leqslant l_j \leqslant m_j$, $1 \leqslant j \leqslant k$, then, for all t with $|t| \leqslant K\sqrt{n}$, the following inequalities hold.*

(a) *If $\beta_r < \infty$, $2 < r \leqslant 3$*

$$|\mathcal{D}_m(f_n(t) - e^{-(t, Mt)/2})| \leqslant C n^{-(r-2)/2} |t|^{(r-|m|)^+} e^{-\alpha|t|^2}$$

(b) *If $\beta_s < \infty$, s integer $\geqslant 3$*

$$\left| \mathcal{D}_m \left(f_n(t) - \left(1 + \sum_{\nu=1}^{s-2} n^{-\nu/2} P_\nu(it)\right) e^{-(t, Mt)/2} \right) \right| \leqslant C \cdot d(n, t) n^{-(s-2)/2} |t|^{(s-|m|)^+} e^{-\alpha|t|^2}$$

($x^+ = x$ *when* $x \geqslant 0$ *and* $= 0$ *when* $x \leqslant 0$).

In order to prove this lemma, we use and expansion of $\mathcal{D}_m f_n(t)$ in terms of $f(t)$ and its derivatives. Putting $\mathcal{D}_m = \prod_{\nu=1}^{|m|} \delta_\nu$, where each δ_ν is one of the operators $\partial/\partial t_j$, we have

$$\mathcal{D}_m f_n(t) = f_n(t) n^{-|m|/2} \sum_{\nu=1}^{|m|} n_\nu f^{-\nu}(t/\sqrt{n}) T_\nu$$

where $n_\nu = n(n-1) \dots (n-\nu+1)$ and

$$T_\nu = \sum_I \prod_{\eta=1}^{\nu} \left(\left(\prod_{j \in I_\eta} \delta_j \right) f(t/\sqrt{n}) \right)$$

the sum being taken over all possible partitions $I = (I_1, \dots, I_\nu)$ of the index set $(1, 2, \dots, |m|)$ into ν non-empty subsets I_η. Now we obtain

$$\mathcal{D}_m h_n(t) = (g_n(t) + h_n(t)) n^{-|m|/2} \sum_{\nu=1}^{|m|} n_\nu f^{-\nu}(t/\sqrt{n}) T_\nu - \mathcal{D}_m g_n(t)$$

where $h_n(t) = f_n(t) - g_n(t)$. ($g_n(t)$ is defined by (2), see below.) We now use Lemma 1 for $h_n(t)$ and the usual Taylor expansions for $f(t)$ and its derivatives. By integration and comparison with Lemma 1, we see that $\mathcal{D}_m g_n(t)$ and certain parts of the Taylor polynomials of $n^{-|m|/2} \sum_{\nu=1}^{|m|} n_\nu f^{-\nu}(t/\sqrt{n}) T_\nu$ multiplied by $g_n(t)$ are identical and thus vanish. The rest of the expansion is easily estimated, and the lemma follows.

In the general case, it is not possible to approximate $f_n(t)$ by $g_n(t)$ for $|t| > K\sqrt{n}$. However, if $f(t)$ satisfies Cramér's condition

$$\overline{\lim_{|t| \to \infty}} |f(t)| < 1 \tag{C}$$

then following lemma holds.

Lemma 3. *If $f(t)$ satisfies the condition (C), and if $\beta_s < \infty$, s integer ≥ 3, then Lemma* 1 b *and Lemma* 2 b *hold for all t with $|t| < n^{(s-1)/2}$, if $e^{-\alpha|t|^2}$ is replaced by* $\exp(-\alpha |t|^{2/(s-1)})$.

Proof. The condition (C) implies that there exists a constant $\alpha > 0$ such that $|f(t)| < e^{-2\alpha}$ for $|t| > K$, that is $|f_n(t)| < e^{-2\alpha n}$ for $|t| > K\sqrt{n}$. If $K\sqrt{n} < |t| \leq n^{(s-1)/2}$, then $n > |t|^{2/(s-1)}$ and thus $|f_n(t)| < e^{-\alpha n} e^{-\alpha |t|^{2/(s-1)}}$. Similar inequalities hold for $\mathcal{D}_m f_n(t)$, $g_n(t)$ and $\mathcal{D}_m g_n(t)$, and the lemma follows.

We shall use these three lemmas to estimate $F_n(A)$ by $G_n(A)$, where $G_n(x)$, $x \in R_k$ is a function of bounded variation with the Fourier Stieltjes Transform (F.S.T.) $g_n(t)$, that is

$$g_n(t) = \int_{R_k} e^{i(t,x)} dG_n(x).$$

Now, since the F.S.T. of $\partial G_n(x)/\partial x_j$ is $-it_j g_n(t)$, it follows that if

$$g_n(t) = \left(1 + \sum_{\nu=1}^{s-2} n^{-\nu/2} P_\nu(it)\right) e^{-(t, Mt)/2} \tag{2}$$

then
$$G_n(x) = \left(1 + \sum_{\nu=1}^{s-2} n^{-\nu/2} P_\nu(-D)\right) \Phi(x) \tag{3}$$

where $P_\nu(-D)$ is the differential operator obtained from $P_\nu(it)$ by replacing it_j by $-\partial/\partial x_j$.

3. Main formula

Let $H(x) = F(x) - G(x)$, where $F(x)$ is a d.f. and $G(x)$ is of bounded variation in R_k, and take two positive integrable functions $Q(x)$, $x \in R_k$ and $q(t)$, $t \in R_k$ such that $q(t)$ is the Fourier Transform (F.T.) of $Q(x)$:

$$q(t) = \int_{R_k} e^{i(t,x)} Q(x)\, dx.$$

We then define function $H_T(x)$ for $T > 0$ by

$$H_T(x) = \int_{R_k} Q(y) H(x + y/T)\, dy$$

and for every Borel set B

$$H_T(B) = \int_B dH_T(x) = \int_{R_k} Q(y) H(B + y/T)\, dy \tag{4}$$

where $B + y/T$ is the translate of B by y/T.

$H_T(x)$ is of bounded variation in R_k, its F.S.T. being

$$h_T(t) = q(-t/T)\, h(t)$$

and thus $H_T(x)$ is absolutely continuous with the "density function"

$$p_T(x) = (2\pi)^{-k} \int_{R_k} e^{-i(t,x)} q(-t/T) h(t)\, dt.$$

If the indicator function of B

$$V_B(x) = \begin{cases} 1, & x \in B \\ 0, & x \notin B \end{cases}$$

is integrable, and its F.T. is $v_B(t)$, we obtain from Parseval's relation

$$H_T(B) = \int_{R_k} V_B(x) p_T(x)\, dx = (2\pi)^{-k} \int_{R_k} v_B(-t) q(-t/T) h(t)\, dt. \tag{5}$$

This formula is fundamental, and will be used to estimate $H(B)$ out of $h(t)$, when $H(x) = H_n(x) = F_n(x) - G_n(x)$ and $h(t) = h_n(t)$. First, however, we must produce a relationship between $H_T(B)$, $H(B)$ and T, and we shall do this in different ways when B is a "rectangle" with the sides parallel to the coordinate planes, and when B is an arbitrary integrable Borel set.

4. Rectangles: preliminaries

We put
$$Q(x)=\prod_{j=1}^{k} Q_1(x_j),\ q(t)=\prod_{j=1}^{k} q_1(t_j) \tag{6}$$

where $Q_1(x)$ and $q_1(t)$ are functions of one variable satisfying the conditions

$$Q_1(x)\geqslant 0$$
$$0\leqslant q_1(t)\leqslant q_1(0)=1$$
$$q_1(t)=0 \quad \text{when} \quad |t|>1.$$

We may, for instance, take

$$Q_1(x)=(2\pi)^{-1}(2/x)^2\sin^2 x/2$$

and
$$q_1(t)=(1-|t|)^+$$

Let $\mathcal{R}$ be the class of bounded rectangles $R\subset R_k$ with the sides parallel to the coordinate planes and put
$$\delta=\sup_{R\in\mathcal{R}}|H(R)|$$

and
$$\delta_T=\sup_{R\in\mathcal{R}}|H_T(R)|.$$

Then the following lemma holds.

Lemma 4. *If* $|\operatorname{grad} G(x)|\leqslant L$, *then*
$$\delta\leqslant 3\delta_T+cL/T$$
where c *is a constant only depending on* k.

Proof. For every $\delta'<\delta$, there is a rectangle $R=\{x:a_j\leqslant x_j\leqslant b_j,\ 1\leqslant j\leqslant k\}$, such that $|H(R)|>\delta'$. If $H(R)>0$, we take the rectangle $R_1=\{x:a_j-a/T\leqslant x_j\leqslant b_j+a/T,\ 1\leqslant j\leqslant k\}$, where a is an absolute constant to be determined later.

From (4) we obtain

$$H_T(R_1)=\int_{R_k}Q(y)H(R_1+y/T)\,dy=\int_K+\int_{K'}=I_1+I_2$$

where K is the cube $\{y:|y_j|\leqslant a,\ 1\leqslant j\leqslant k\}$ and K' is its complement. If $y\in K$, then $R_1+y/T\supset R$, and thus $F(R_1+y/T)\geqslant F(R)$.

Further, by simple calculations

$$G(R_1+y/T)\leqslant G(R)+2^kL\frac{2a}{T}\sqrt{k}$$

and thus
$$H(R_1+y/T)\geqslant H(R)-cL/2T,\quad y\in K.$$

We now choose a so that $\int_K Q(y)\,dy=\frac{2}{3}$, $\int_{K'}Q(y)\,dy=\frac{1}{3}$ and obtain

$$I_1\geqslant\tfrac{2}{3}(H(R)-cL/2T)\geqslant\tfrac{2}{3}(\delta'-cL/2T).$$

$|H(R_1+y/T)|\leqslant\delta$ for all y, and thus $|I_2|<\delta/3$. We now get

$$\delta_T\geqslant|H_T(R_1)|\geqslant\tfrac{2}{3}(\delta'-cL/2T)-\delta/3$$

for every $\delta'<\delta$, that is

$$\delta\leqslant 3\delta_T+cL/T.$$

If $H(R)\leqslant 0$, we take $R_1=\{x:a_j+a/T\leqslant x_j\leqslant b_j-a/T,\ 1\leqslant j\leqslant k\}$ (which may be empty) and proceed in a similar way.

5. Rectangles: results

We are now ready to prove the following two generalizations of Bergström's result (1).

Theorem 1. *If* $\beta_r<\infty$, $2<r\leqslant 3$, *then*

$$|F_n(x)-\Phi(x)|\leqslant Cn^{-(r-2)/2}.$$

Proof. Since $\sup|H(x)|\leqslant\sup|H(R)|$, it suffices to show that $\delta\leqslant Cn^{-(r-2)/2}$ with $H(x)=F_n(x)-\Phi(x)$. We take $T=K\sqrt{n/k}$, and thus by Lemma 4, it remains to show that $\delta_T\leqslant Cn^{-(r-2)/2}$. If $R=\{x:a_j\leqslant x_j\leqslant b_j\}$

then
$$v_R(t)=\prod_{j=1}^{k}\frac{e^{it_jb_j}-e^{it_ja_j}}{it_j}$$

and thus from (5) and (6)

$$H_T(R)=(2\pi)^{-k}\int_{R_k}\left(\prod_{j=1}^{k}\frac{e^{-it_jb_j}e-{}^{-it_ja_j}}{-it_j}\,q_1(-t_j/T)\right)h(t)\,dt.$$

We now define projection operators P_j, $1\leqslant j\leqslant k$, such that P_jt, $t\in R_k$, is the projection of t in the plane $t_j=0$. We also define for every function $a(t)$, $t\in R_k$, $P_ja(t)=a(P_jt)$. The operators P_j evidently satisfy the following relations:

$$P_iP_ja(t)=P_jP_ia(t)$$

$$P_j(a(t)b(t))=(P_ja(t))(P_jb(t))$$

$$P_jc(t)=c(t)\text{ if } c(t)\text{ is independent of } t_j$$

We now put

$$h(t)=\prod_{j=1}^{k}[(1-e^{-\alpha t_j^2}P_j)+e^{-\alpha t_j^2}P_j]h(t)=\sum_{(\Gamma,\Lambda)}\prod_{\gamma\in\Gamma}(1-e^{-\alpha t_\gamma^2}P_\gamma)\prod_{\lambda\in\Lambda}e^{-\alpha t_\lambda^2}P_\lambda h(t)\qquad(7)$$

where the summation is taken over all different partitions (Γ,Λ) of the index set $(1,2,\dots,k)$. We put $\Pi_{\lambda\in\Lambda}P_\lambda t=t_\Gamma$ (the projection of t in the subspace R_Γ spanned by t_γ, $\gamma\in\Gamma$). It suffices in (7) to sum over all non-empty Γ, for if Γ is empty, then $t_\Gamma=0$ and $h(t_\Gamma)=0$ according to Lemma 1a. We thus get

$$H_T(R)=(2\pi)^{-k}\sum_{(\Gamma,\Lambda)}'\int_{R_\Lambda}\prod_{\lambda\in\Lambda}\frac{e^{-ib_\lambda t_\lambda}-e^{-ia_\lambda t_\lambda}}{-it_\lambda}\,q_1(-t_\lambda/T)^{-\alpha t_\lambda^2}\,dt_\Lambda$$
$$\times\int_{R_\Gamma}\left(\prod_{\gamma\in\Gamma}\frac{e^{-ib_\gamma t_\gamma}-e^{-ia_\gamma t_\gamma}}{-it_\gamma}\,q_1(-t_\gamma/T)\right)\prod_{\gamma\in\Gamma}(1-e^{-\alpha t_\gamma^2}P_\gamma)h(t_\Gamma)\,dt_\Gamma.$$

The integral over R_Γ is independent of t_λ, $\lambda\in\Lambda$, and the integral over R_Λ is uniformly bounded, according to the inversion formula for d.f.'s. Because $q_1(-t_\gamma/T)=0$ when $|t_\gamma|>T$, we get

$$|H_T(R)|\leqslant C\sum_{(\Gamma,\Lambda)}'\int_{\substack{|t_\gamma|\leqslant T\\ \gamma\in\Gamma}}\frac{\left|\prod_{\gamma\in\Gamma}(1-e^{-\alpha t_\gamma^2}P_\gamma)h(t_\Gamma)\right|}{\prod_{\gamma\in\Gamma}|t_\gamma|}\,dt_\Gamma. \tag{8}$$

Now, for $\mu\in\Gamma$

$$(1-e^{-\alpha t_\mu^2}P_\mu)h(t_\Gamma)=(1-e^{-\alpha t_\mu^2})h(t_\Gamma)+e^{-\alpha t_\mu^2}(1-P_\mu)h(t_\Gamma).$$

From Lemma 1 a and Lemma 2 a with $\mathcal{D}_m=\partial/\partial t_\mu$ we easily get

$$|(1-e^{-\alpha t_\mu^2}P_\mu)h(t_\Gamma)|\leqslant Cn^{-(r-2)/2}|t_\mu|e^{-\alpha|t_\Gamma|^2}\ \text{for}\ |t_\Gamma|\leqslant T\sqrt{k}$$

and thus with a new C

$$\left|\prod_{\gamma\in\Gamma}(1-e^{-\alpha t_\gamma^2}P_\gamma)h(t_\Gamma)\right|\leqslant Cn^{-(r-2)/2}|t_\mu|e^{-\alpha|t_\Gamma|^2}.$$

Taking the geometrical mean over all $\mu\in\Gamma$, finally get

$$\left|\prod(1-e^{-\alpha t_\gamma^2}P_\gamma)h(t_\Gamma)\right|\leqslant Cn^{-(r-2)/2}\prod|t_\mu|^\beta e^{-\alpha|t_\Gamma|^2}$$

where $\beta\geqslant 1/k$. By using this estimation in (8), we immediately obtain the desired estimate of $|H_T(R)|$. The proof is concluded.

Putting

$$H(x)=F_n(x)-\left(1+\sum_{\nu=1}^{s-2}n^{-\nu/2}P_\nu(-D)\right)\Phi(x)\ \text{and}\ T=n^{(s-1)/2}/\sqrt{k},$$

and using Lemma 3, we obtain in the same way the following theorem.

Theorem 2. *If $\beta_s<\infty$, s integer $\geqslant 3$, and if $f(t)$ satisfies the condition (C), then*

$$\left|F_n(x)-\left(1+\sum_{\nu=1}^{s-2}n^{-\nu/2}P_\nu(-D)\right)\Phi(x)\right|\leqslant Cd(n)n^{-(s-2)/2}$$

where $d(n)\leqslant 1$ and $\lim_{n\to\infty}d(n)=0$.

6. Borel sets: introduction

$F_n(B)$ is a positive measure, defined at least on the class $\mathcal{B}$ of Borel sets $B\subset R_k$. It may be natural to expect that the difference $F_n(B)-\Phi(B)$ tends to zero for all $B\in\mathcal{B}$ when $n\to\infty$. This, however, is not the fact, as is shown by the following example.

Let X be purely discontinuous and let B be the denumerable set of all points $x \in R_k$ with $F_n(\{x\}) > 0$, some $n \geq 1$. Then $\Phi(B) = 0$, but $F_n(B) = 1$ for all $n \geq 1$. Nevertheless, we shall now estimate the difference $F_n(B) - \Phi(B)$ for arbitrary Borel sets B, but convergence to zero will thus heavily depend on the set B.

7. Parallel sets

Let $B \in \mathcal{B}$ and $\varepsilon > 0$. We define the exterior parallel set B_ε as

$$B_\varepsilon = \bigcup_{u \in U} (B + \varepsilon u)$$

where $B + \varepsilon u$ is the translate of B by εu, and the union is taken over all $u \in U =$ the open unit sphere in R_k. Now B_ε can be written

$$B_\varepsilon = \bigcup_{b \in B} (\varepsilon U + b)$$

and thus B_ε is an open set.

The interior parallel set $B_{-\varepsilon}$ is defined as

$$B_{-\varepsilon} = ((B')_\varepsilon)' = \bigcap_{u \in U} (B + \varepsilon u)$$

where B' is the complement of B. Clearly

$$B_{-\varepsilon} \subset B \subset B_\varepsilon \text{ and further}$$

$$(B_\varepsilon)_{-\varepsilon} = \bigcap_{u \in U} \bigcup_{v \in U} (B + \varepsilon u + \varepsilon v) \supset \bigcap_{u \in U} (B + \varepsilon u - \varepsilon u) = B.$$

In the same way $(B_{-\varepsilon})_\varepsilon \subset B$ and thus

$$B_{-\varepsilon} \subset (B_{-\varepsilon})_\varepsilon \subset B \subset (B_\varepsilon)_{-\varepsilon} \subset B_\varepsilon.$$

From the definitions, it readily follows that for $\varepsilon > 0$, $h > 0$

$$(B_\varepsilon)_h = B_{\varepsilon + h}, \quad (B_{-\varepsilon})_{-h} = B_{-(\varepsilon + h)}.$$

We denote by δB the set of boundary points of B and $\bar{B}$ by the closure of B. Then $(\bar{B})_\varepsilon = B_\varepsilon$ and $\bar{B} = \bigcap_{\varepsilon > 0} B_\varepsilon$. It is easy to show that $(\delta B)_\varepsilon = B_\varepsilon - B_{-\varepsilon}$ and $\delta((\delta B)_\varepsilon) = \delta(B_\varepsilon) \cup \delta(B_{-\varepsilon})$. For every non-empty set $B \subset R_k$ and every point $x \notin B$, the shortest distance $d(x,B)$ from x to B is defined by $d(x,B) = \inf_{b \in B} |x - b|$, and there exists at least one projection point $p(x,B) \in \delta B$ such that $|x - p(x,B)| = d(x,B)$. The following lemma gives a characterisation of the boundary points of a parallel set.

Lemma 5. *Let $B \subset R_k$, $\varepsilon > 0$ and $p \in \delta(B_\varepsilon)$. Then the set of points $x \notin \overline{B_\varepsilon}$ with projection point $p(x, B_\varepsilon) = p$ is either empty or a line segment with p as an end point, and for every y in the interior of this line segment, the projection point $p(y, B_\varepsilon) = p$ is unique.*

Proof. Suppose there is a point $x \notin \overline{B_\varepsilon}$ with $p(x, B_\varepsilon) = p$. Then if $y = \lambda x + (1 - \lambda)p$, $0 < \lambda < 1$, $|y - p| = \lambda |x - p|$, and if $p_1 \in \delta(B_\varepsilon)$, $p_1 \neq p$ and $|y - p_1| \leq \lambda |x - p|$, then $|x - p_1| < |x - y| + |y - p_1| \leq |x - p|$, which gives a contradiction. Thus $p(y, B_\varepsilon) = p$

uniquely. It remains to show that, for every point x_1 outside the straight line through x and p, $p(x_1,B_\varepsilon)\neq p$. Now, there is a point $q\in\delta B$ such that $|p-q|=\varepsilon$. If $d=|x-p|$, then $|x-q|=\varepsilon+d$, for otherwise $|x-q|=\varepsilon+d-\eta$, some $\eta>0$, that is $x\in B_{\varepsilon+d-\eta/2}=(B_\varepsilon)_{d-\eta/2}$, and $d(x, B_\varepsilon)<d$, which is false. Consequently x, p and q lie on a straight line. If x_1 lies outside this line, $p(x_1, B_\varepsilon)=p$ and $|x_1-p|=d_1$, then $|x_1-q|<|x_1-p|+|p-q|=d_1+\varepsilon$, that is $d(x_1, B_\varepsilon)<d_1$, which is false. The lemma is proved.

Corollary. *It follows that for every projection point $p\in\delta(B_\varepsilon)$, the point $q\in\delta B$ is uniquely determined, and the line joining p and q is a normal to the surface $\delta(B_\varepsilon)$, for $\delta(B_\varepsilon)$ lies outside both the spheres $\{y:|y-x|<d\}$ and $\{y:|y-q|<\varepsilon\}$.*

8. Borel sets: preliminaries

We now choose the two functions $Q(x)$ and $q(t)$ in (4) as follows (see von Bahr [1]):

$$Q(x)=Q_2(|x|)$$

and

$$q(t)=q_2(|t|)$$

where $Q_2(r)$, $r\geqslant 0$, and $q_2(s)$, $s\geqslant 0$ are two functions satisfying

$$Q_2(r)\geqslant 0$$
$$0\leqslant q_2(s)\leqslant q_2(0)=1$$
$$q_2(s)=0 \quad \text{when} \quad s\geqslant 1$$
$$Q_2(r)=0(e^{-\sqrt{r}}) \quad \text{when} \quad r\to\infty.$$

According to the inversion formula for F.T.'s, $q(t)$ and all its derivatives are continuous, and thus vanish when $|t|\geqslant 1$.

Now, if $H(x)=F(x)-G(x)$, $x\in R_k$, where $F(x)$ is a d.f. and $G(x)$ is of bounded variation, and if $H_T(x)$ is given by (4), we define for every Borel set B:

$$a(B)=\sup_{z\in R_k} H(B+z)$$

$$-b(B)=\inf_{z\in R_k} H(B+z)$$

$$d_T(B)=\sup_{z\in R_k} |H_T(B+z)|$$

$$\alpha(B,\varepsilon)=\sup_{z\in R_k}\int_{(\delta B)_\varepsilon}|dG(x+z)|.$$

The following lemma gives relationships between these quantities.

Lemma 6. *For every $B\in\mathcal{B}$ the following inequalities hold:*

$$a(B)\leqslant \tfrac{1}{2}b(B_{a/T})+\tfrac{3}{2}d_T(B_{a/T})+\alpha(B, 2a/T) \tag{9}$$

and

$$b(B)\leqslant \tfrac{1}{2}a(B_{-a/T})+\tfrac{3}{2}d_T(B_{-a/T})+\alpha(B, 2a/T) \tag{10}$$

where a is a constant only depending on k.

Proof. We have from (4)

$$H_T(B_{a/T}) = \int_{R_k} Q(y) H(B_{a/T} + y/T)\, dy = \int_{|y|<a} + \int_{|y|\geqslant a} = I_1 + I_2$$

where the constant a is chosen so that

$$\int_{|y|<a} Q(y)\, dy = \tfrac{2}{3}, \quad \int_{|y|\geqslant a} Q(y)\, dy = \tfrac{1}{3}$$

Now if $|y| < a$, then $B_{a/T} + y/T \supset B$ and

$$F(B_{a/T} + y/T) \geqslant F(B).$$

Further
$$|G(B_{a/T} + y/T) - G(B)| \leqslant \alpha(B, 2a/T)$$

and thus
$$I_1 \geqslant \tfrac{2}{3}(H(B) - \alpha(B, 2a/T))$$

Since $H(B_{a/T} + y/T) \geqslant -b(B_{a/T})$, we get

$$I_2 \geqslant -\tfrac{1}{3} b(B_{a/T})$$

and finally

$$d_T(B_{a/T}) \geqslant H_T(B_{a/T}) \geqslant \tfrac{2}{3}(H(B) - \alpha(B, 2a/T)) - \tfrac{1}{3} b(B_{a/T}).$$

This relation holds even if $H(B)$ is replaced by $H(B+z)$, $z \in R_k$, and (9) follows. (10) is proved in a similar way, starting from $H_T(B_{-a/T})$.

We now define a sequence of Borel sets B^ν, $\nu = 0, \pm 1, \pm 2, \ldots$ in the following way: $B^0 = B$, and for $n \geqslant 0$

$$B^{2n+1} = (B^{2n})_{a/T}, \quad B^{2n+2} = (B^{2n+1})_{-a/T}$$

$$B^{-2n-1} = (B^{-2n})_{-a/T}, \quad B^{-2n-2} = (B^{-2n-1})_{a/T}.$$

Using (9) for B^{2n} and (10) for B^{2n+1}, we get

$$a(B^{2n}) \leqslant \tfrac{1}{4} a(B^{2n+2}) + \alpha(B^{2n}, 2a/T) + \tfrac{1}{2}\alpha(B^{2n+1}, 2a/T)$$
$$+ 3[\tfrac{1}{2} d_T(B^{2n+1}) + \tfrac{1}{4} d_T(B^{2n+2})] \quad n = 0, 1, 2, \ldots$$

and thus by induction

$$a(B) \leqslant 2^{-2N} a(B^{2N}) + \sum_{n=0}^{2N-1} 2^{-n} \alpha(B^n, 2a/T) + 3 \sum_{n=1}^{2N} 2^{-n} d_T(B^n).$$

Since a(B) is uniformly bounded, we obtain, by letting $N \to \infty$,

$$a(B) \leqslant \sum_{n=0}^{\infty} 2^{-n} \alpha(B^n, 2a/T) + 3 \sum_{n=1}^{\infty} 2^{-n} d_T(B^n).$$

In the same way we get

$$b(B) \leqslant \sum_{n=0}^{\infty} 2^{-n} \alpha(B^{-n}, 2a/T) + 3 \sum_{n=1}^{\infty} 2^{-n} d_T(B^{-n}).$$

Now, from Section 7 we obtain the relations

$$B^{-1} \subset B^{-2n-1} \subset B^{-2n} \subset B^{0} \subset B^{2n} \subset B^{2n+1} \subset B^{1}$$

and thus $B^{\nu}_{-2a/T} \supset B_{-3a/T}$ and $B^{\nu}_{2a/T} \subset B_{3a/T}$.

Consequently, for every ν

$$\alpha(B^{\nu}, 2a/T) \leqslant \alpha(B, 3a/T)$$

and finally

$$a(B) \leqslant 2\alpha(B, 3a/T) + 3 \sum_{n=1}^{\infty} 2^{-n} d_T(B^{n}) \tag{11}$$

and

$$b(B) \leqslant 2\alpha(B, 3a/T) + 3 \sum_{n=1}^{\infty} 2^{-n} d_T(B^{-n}). \tag{12}$$

9. Estimation of $\alpha(B, \varepsilon)$: Borel sets

The rest of this paper is devoted to the estimation of $\alpha(B, \varepsilon)$ and $d_T(B^{\nu})$, when $F(x) = F_n(x)$, d.f. of the normed sum Y_n, and $G(x)$ is given by (3), $s \geqslant 2$. In this section we shall examine $\alpha(B, \varepsilon)$ for a subclass $\mathcal{B}_1$ of Borel sets B. In Theorem 4 we need the condition

$$\lim_{\varepsilon \to 0} \alpha(B, \varepsilon) = 0 \tag{13}$$

to be able to show that $F_n(B) \to \Phi(B)$ when $n \to \infty$. The following lemma gives a necessary and sufficient condition for (13).

Lemma 7. *If $G(x)$ is absolutely continuous and B is a bounded Borel set, then $\lim_{\varepsilon \to 0} \alpha(B, \varepsilon) = 0$ if and only if $V(\delta B) = \int_{\delta B} dx = 0$.*

Proof: We write

$$\alpha(B, \varepsilon) = \sup_{z \in R_k} \int_{(\delta B)_\varepsilon + z} \psi(x)\, dx$$

where $|dG(x)| = \psi(x) dx$. For every $\eta > 0$, there exists a constant M such that, if $K = \{x : \psi(x) \leqslant M\}$, then

$$\int_{K'} \psi(x)\, dx \leqslant \eta.$$

Thus

$$\alpha(B, \varepsilon) \leqslant \sup_{z} \int_{K \cap (\delta B)_\varepsilon + z)} \psi(x)\, dx + \eta \leqslant M\, V((\delta B)_\varepsilon) + \eta$$

Since δB is closed, $\lim_{\varepsilon \to 0} (\delta B)_\varepsilon = \delta B$, and thus the "if" statement follows from the dominated convergence theorem. On the other hand, if $V(\delta B) > 0$ then there is a $z \in R_k$ such that $\int_{\delta B} \psi(x+z)\, dx = h > 0$, for otherwise we obtain a contradiction by integrating with respect to z and using Fubini's theorem, and thus $\alpha(B, \varepsilon) \geqslant h > 0$ for all ε.

Actually, we are chiefly interested in Borel sets B satisfying $\alpha(B, \varepsilon) = O(\varepsilon)$, when $\varepsilon \to 0$, and our purpose is to express $\alpha(B, \varepsilon)$ as ε times a surface integral over $\delta(B_h)$ and $\delta(B_{-h})$. I therefore make the following assumptions.

For every positive finite ϱ and every h, $0 < h \leqslant \varepsilon$, those parts of $\delta(B_h)$ and $\delta(B_{-h})$ which lie in the sphere ϱU are a finite disjoint union of subsets S_ν, each of which is representable in a system of rectangular coordinates $(y_1, \ldots y_{k-1}, w)$ by a relation $w = f(y)$, such that $f(y)$ has bounded continuous derivatives of the first two orders for every y in the interior of P_ν = the projection of S_ν in the hyperplane $w = 0$, and such that the set of boundary points δP_ν of P_ν is of $(k-1)$-dimensional Lebesque measure zero.

The class of sets B satisfying the above conditions and for which $V(\delta B) = 0$ is denoted $\mathcal{B}_1$.

Lemma 8. *If* $B \in \mathcal{B}_1$ *and* $|dG(x)| \leqslant \psi(x)dx$, *where* $\psi(x)$ *is continuous and bounded, then*

$$\alpha(B, \varepsilon) < \varepsilon \sup_{z \in R_k} \int_{0 < h \leqslant \varepsilon\ \delta(B_h) \cup \delta(B_{-h})} \psi(x)\,dS \tag{14}$$

and

$$\alpha(B, \varepsilon) \leqslant C \sup\,[S(B_h) + S(B_{-h})]$$

where dS *indicates surface integral and* $S(B) = \int_{\delta B} dS$.

Proof. Take $\varrho > 0$ and put

$$\mu(\varepsilon, \varrho) = \int_{\varrho U \cap (\delta B)_\varepsilon} \psi(x)\,dx$$

Let $h > 0$, $v > 0$, $h + v \leqslant \varepsilon$ and $A = (\delta B)_h$. Then

$$\mu(h+v, \varrho) - \mu(h, \varrho) = \int_{\varrho U \cap (A_v - A)} \psi(x)\,dx.$$

If $x \in \varrho U \cap (A_v - A)$ and $p(x)$ is the projection point of x on δA, then $|p(x)| \leqslant p + h$, and thus $p(x) \in (\varrho + h) U \cap \delta A$, where according to the assumptions,

$$(\varrho + h)\, U \cap \delta A = \sum_{\nu=1}^{N} S_\nu.$$

Let
$$V_\nu = A' \cap \{x : p(x) \in S_\nu,\ |x - p| \leqslant h\}.$$

Then
$$\varrho U \cap (A_\nu - A) \subset \bigcup_{\nu=1}^{N} V_\nu,$$

and thus
$$\mu(h+v, \varrho) - \mu(h, \varrho) \leqslant \sum_{\nu=1}^{N} \int_{V_\nu} \psi(x)\,dx.$$

For every point $p \in S_\nu$, let $q(p) \in P_\nu$ be the projection point of p in the hyperplane $w = 0$ of the νth coordinate system. Now the set of points $x \in V_\nu$ with $q(p(x)) \in \delta P_\nu$

is of k-dimensional Lebesque measure zero, because δP_ν is of $(k-1)$-dimensional Lebesque measure zero, and for every point $q \in P_\nu$ the set of points x with $q(p(x)) = q$ is a line segment of length at most h. Thus, if V_ν is replaced by the set W_ν of points $x \in V_\nu$ with $q(p(x)) \in P_\nu - \delta P_\nu$, then the value of the νth integral remains unchanged. If $q(p) \in P_\nu - \delta P_\nu$, then the normal $n(p)$ is defined, and thus for $x \in W_\nu$, $x = p(x) + tn(p(x))$, $0 \leqslant t \leqslant v$. By changing variables from x to (p, t) in the νth integral, we obtain

$$\int_{W_\nu} \psi(x)\,dx = \int_{\substack{q(p) \in P\nu - \delta P\nu \\ 0 \leqslant t \leqslant v}} \psi(p + tn(p))\,|J(p, t)|\,dp\,dt$$

where $J(p, t)$ is the Jacobian of the transformation. Both ψ and J are continuous, and since the latter is one when $t = 0$, we get

$$\mu(h+v, \varrho) - \mu(h, \varrho) \leqslant \sum_{\nu=1}^{N} \int_{S_\nu} \psi(p)\,dp + o(v) \leqslant v \int_{\delta A} \psi(x)\,dS + o(v).$$

The inequality gives an upper bound of the upper right derivative of $\mu(h, \varrho)$. Since $\mu(+0, \varrho) = 0$, we obtain by letting $\varrho \to \infty$ (cf. [7], p. 155)

$$\int_{(\delta B)_\varepsilon} \psi(x)\,dx \leqslant \varepsilon \sup_{0 < h \leqslant \varepsilon} \int_{\delta(B_h) \cup \delta(B_{-h})} \psi(x)\,dS.$$

Now (14) follows by changing $\psi(x)$ to $\psi(x+z)$ and taking the supremum over all $z \in R_k$. The second inequality follows from (14) because $\psi(x)$ is bounded.

It should be noted that all $B \in \mathcal{B}_1$ do not satisfy $\alpha(B, \varepsilon) = 0(\varepsilon)$. For example, if, in R_2, B is given in polar coordinates (r, φ) by

$$B = \{(x_1, x_2) : 1 - \varphi^{-1} \leqslant r \leqslant 1 - (\varphi + \pi)^{-1}, \varphi \geqslant 1\}$$

then $\alpha(B, \varepsilon) = 0(\sqrt{\varepsilon})$, $\varepsilon \to 0$.

10. Estimation of $\alpha(B, \varepsilon)$: convex sets

We also consider the class $\mathcal{C}$ of convex Borel sets in R_k. If $B \in \mathcal{C}$, then B_ε and $B_{-\varepsilon}$ are both convex, and for every $h > 0$ we can find a convex polyhedron P such that $P \subset B \subset P_h$ (cf. Valentine [10], p. 143) and thus $\alpha(B, \varepsilon)$ differs arbitrarily little from $\alpha(P, \varepsilon)$ if $\psi(x)$ is continuous. The surface integral of a convex surface exists (Busemann [6], p. 7) and equals the limit of the surface integrals of approximating convex polyhedral surfaces δP. Thus, the change in the right-hand side of (14) is arbitrarily small if B is replaced by P and thus (14) holds for $B \in \mathcal{C}$. The following lemma gives a uniform upper bound of the integral in the righthand side of (14) for $B \in \mathcal{C}$.

Lemma 9. *If $\psi(x) \leqslant \psi_1(r)$, when $|x| = r$, where $\psi_1(r)$ is differentiable, $\psi_1(r)\,r^{k-1} \to 0$, when $r \to \infty$, and*

$$\int_0^\infty |\psi_1'(r)|\,r^{k-1}\,dr = L,$$

then for every convex set $B \in R_k$

$$\int_{\delta B} \psi(x)\, dS \leqslant \omega_k \cdot L$$

where $\omega_k = S(U) = 2\pi^{k/2}/\Gamma(k/2)$.

Proof. We put $S(r) = \int_{\delta B \cap rU} dS$, where $rU = \{ru : u \in U\}$. Because $B \cap rU$ is convex, $S(r) \leqslant S(B \cap rU) \leqslant S(rU) = \omega_k r^{k-1}$ and thus

$$\int_{\delta B} \psi(x)\, dS \leqslant \int_0^\infty \psi_1(r)\, dS(r) \leqslant \int_0^\infty |\psi_1'(r)|\, S(r)\, dr \leqslant \omega_k \cdot L.$$

From Lemma 8 we now obtain, if B is convex,

$$\alpha(B, \varepsilon) \leqslant \varepsilon \sup_{\substack{0 < h \leqslant \varepsilon \\ z \in R_k}} \left\{ \int_{\delta(B_h + z)} \psi(x)\, dS + \int_{\delta(B_{-h} + z)} \psi(x)\, dS \right\} \leqslant 2\omega_k L \varepsilon \tag{15}$$

and since $S(B_{-h}) \leqslant S(B_h) \leqslant S(B_\varepsilon)$, we also get

$$\alpha(B, \varepsilon) \leqslant C \cdot \varepsilon \cdot S(B_\varepsilon). \tag{16}$$

11. Estimation of $d_T(B)$

In this section we shall give an immediate estimate of $d_T(B)$ for arbitrary Borel sets B with $V(B) < \infty$, when $H(x) = F_n(x) - G_n(x)$.

If $\beta_r < \infty$, $2 < r \leqslant 3$, we put $G_n(x) = \Phi(x)$ and use (5) and Lemma 1a with $T = K\sqrt{n}$. Observing that $q(-t/T) = 0$ when $|t| > T$ and that $|v_B(-t)| \leqslant \int_B dx = V(B)$, we easily obtain

$$|H_T(B)| \leqslant C \cdot V(B) \cdot n^{-(r-2)/2}$$

and this inequality evidently holds for $d_T(B)$ too:

$$d_T(B) \leqslant C \cdot V(B)\, n^{-(r-2)/2}. \tag{17}$$

If $\beta_s < \infty$, s integer $\geqslant 3$ and if $f(t)$ satisfies the condition (C), we take $G_n(x)$ according to (3) and obtain in the same way, using Lemma 1b and Lemma 3,

$$d_T(B) \leqslant C\, V(B)\, d(n)\, n^{-(s-2)/2} \tag{18}$$

where $d(n) \leqslant 1$ and $d(n) \to 0$ when $n \to \infty$.

12. Estimation of $d_T(B)$ when there exists a weight polynomial

The inequalities (17) and (18) have the quality that the quantity on the right-hand side is small when the volume of B is small. I shall use this fact in a subsequent paper for estimating the probabilities of large deviations. For large B, however, it would be more favourable to use inequalities of the types

$$d_T(B) \leq C n^{-(r-2)/2} \tag{19}$$

if $\beta_r < \infty$, and

$$d_T(B) \leq C d(n) n^{-(s-2)/2} \tag{20}$$

if $\beta_s < \infty$, and if $f(t)$ satisfies the condition (C).

In Section 5 we obtained such results for $B \in \mathcal{R}$ by making full use of the explicit form of the F.T. of the indicator function of a rectangle.

For arbitrary Borel sets B, we shall obtain (19) and (20) by imposing the additional condition that a number of moments of higher order exist. We proceed as follows.

We say that a polynomial $\varrho(z) = \varrho(z_1, ..., z_k)$ is a weight polynomial of the r.v. X, if

$$\int_{R_k} \frac{dx}{\varrho(|x_1|, \ldots, |x_k|)} < \infty$$

and if, for every term $C_m \prod_{j=1}^k z_j^{m_j}$ of $\varrho(z_1, \ldots, z_k)$, the moments $E\prod_{j=1}^k |X_j|^{l_j}$ exist for every k-tuple of integers $l = (l_1, \ldots, l_k)$ with $0 \leq l_j \leq m_j$, $1 \leq j \leq k$.

If, for example, $E|X_j|^{m_j} < \infty$ where m_j are positive integers satisfying

$$\sum_{j=1}^k m_j^{-1} < 1, \quad \text{then} \quad \varrho(z_1, \ldots, z_k) = 1 + \sum_{j=1}^k z_j^{m_j}$$

is a weight polynomial of X. This is the case if $\beta_s < \infty$, $s > k$. If the components X_j of X are independent, then

$$E\prod_{j-1}^k (1 + X_j^2) = \prod_{j-1}^k E(1 + X_j^2) < \infty$$

and thus $\varrho(z_1, \ldots, z_k) = \prod_{j=1}^k (1 + z_j^2)$ is a weight polynomial of X.

Assuming that X has a weight polynomial $\varrho(z)$, we put $B = \bigcup_{\nu=1}^{2^k} B_\nu$, where B_ν is the intersection of B and the νth "octant" O_ν of R_k, $1 \leq \nu \leq 2^k$. For every ν, we define the polynomial $\varrho_\nu(x)$, $x \in R_k$ by the relation $\varrho_\nu(x_1, ..., x_k) = \varrho(|x_1|, ..., |x_k|)$ when $x \in O_\nu$, and put $W_{B_\nu}(x) = V_{B_\nu}(x)/\varrho_\nu(x)$. $W_{B_\nu}(x)$ is integrable on R_k and its F.T. $w_{B_\nu}(t)$ is uniformly bounded (independent of B). The derivative

$$\prod_{j=1}^k \left(-i\frac{\partial}{\partial t_j}\right)^{l_j} f(t) = E\prod_{j=1}^k X_j^{l_j} e^{i(t,X)}$$

exists for every l with $0 \leq l_j \leq m_j$, and thus the derivatives

$$\prod_{j=1}^k \left(-i\frac{\partial}{\partial t_j}\right)^{l_j} f_n(t/\sqrt{n})$$

and

$$\prod_{j=1}^k \left(-i\frac{\partial}{\partial t_j}\right)^{l_j} \{q(-t/T) h_n(t)\}$$

where $h_n(t) = f_n(t/\sqrt{n}) - g_n(t)$, also exist. Further

$$\int \varrho_\nu(x) |dH_T(x)| < \infty$$

and
$$\int \varrho_\nu(x)\,|dH_T(x)| < \infty$$

and thus $\varrho_\nu(x)p_T(x)$ is integrable in R_k, its F.T. being $\varrho_\nu(-iD)\{q(-t/T)h(t)\}$.

Using Parseval's relation, we now obtain

$$H_T(B_\nu) = \int_{R_k} W_{B_\nu}(x)\varrho_\nu(x)p_T(x)\,dx = (2\pi)^{-k}\int_{R_k} w_{B_\nu}(-t)\varrho_\nu(-iD)\{q(-t/T)h(t)\}\,dt.$$

If $\beta_r < \infty$, $2 < r \leqslant 3$, we put $G(x) = \Phi(x)$, and obtain from Lemma 2 a

$$|H_T(B_\nu| \leqslant Cn^{-(r-2)/2}, \quad \nu = 1, 2, \dots, 2^k.$$

By summing over ν from 1 to 2^k, we get

$$d_T(B) \leqslant 2^k C n^{-(r-2)/2} \tag{21}$$

If $\beta_s < \infty$, s integer $\geqslant 3$ and if $f(t)$ satisfies the condition (C), we obtain in the same way with $G(x) = G_n(x)$ given by (3) and using Lemma 2b and Lemma 3:

$$d_T(B) \leqslant Cd(n)\,n^{-(s-2)/2}. \tag{22}$$

13. Results

We sum up our results in the following theorems.

Theorem 3 (a) *If $\beta_r < \infty$, $2 < r \leqslant 3$, then for every Borel set $B \subset R_k$*

$$|F_n(B) - \Phi(B)| \leqslant 2\alpha(B, c/\sqrt{n}) + C \cdot n^{-(r-2)/2}V(B_{c/\sqrt{n}}) \tag{23}$$

and if there exists a weight polynomial of X,

$$|F_n(B) - \Phi(B)| \leqslant 2\alpha(B, c/\sqrt{n}) + Cn^{-(r-2)/2}.$$

(b) *If $\beta_s < \infty$, s integer $\geqslant 3$ and if $f(t)$ satisfies the condition (C), then for every Borel set $B \subset R_k$ and with $G_n(x)$ given by* (3)

$$|F_n(B) - G_n(B)| \leqslant 2\alpha(B, 3an^{-(s-1)/2}) + Cd(n)\,n^{-(s-2)/2}V(B_{an^{-(s-1)/2}})$$

and if there exists a weight polynomial of X

$$|F_n(B) - G_n(B)| \leqslant 2\,\alpha(B, 3an^{-(s-1)/2}) + Cd(n)\,n^{-(s-2)/2}$$

$\alpha(B, \varepsilon)$ is given by

$$\alpha(B, \varepsilon) = \sup_{z \in R_k} \int_{(\delta B)_\varepsilon} |dG(x+z)|$$

where $G(x) = \Phi(x)$ in a *and $G(x) = G_n(x)$ in* b, *and satisfies the inequalities of Lemma 8, if $B \varepsilon \mathfrak{B}_1$.*

Proof. We start from (11) and (12), and put $T=K\sqrt{n}$ in a. Because $V(B^{\nu})\leqslant V(B^{1})=V(B_{c/\sqrt{n}})$ with $c=a/K$, and according to (17), both sums are less than $CV(B_{c/\sqrt{n}})\,n^{-(r-2)/2}$, and thus

$$|H(B)|\leqslant \max\,(a(B)),\, b(B))\leqslant 2\alpha(B,3c/\sqrt{n})+3CV(B_{c/\sqrt{n}})n^{-(r-2)/2}.$$

The remaining inequalities are proved in the same way, using (21), (18) and (22) respectively.

Specializing to convex Borel sets B and using (15) and (16), we obtain in the same way the following theorem.

Theorem 4. *Let B be a convex Borel set. Then if $\beta_r<\infty$, $2<r\leqslant 3$,*

$$|F_n(B)-\Phi(B)|\leqslant C\left(n^{-\frac{1}{2}}S(B_{c/\sqrt{n}})+n^{-(r-2)/2}V(B_{c/\sqrt{n}})\right) \tag{24}$$

and if $\beta_s<\infty$, s integer $\geqslant 3$ and $f(t)$ satisfies the condition (C)

$$|F_n(B)-G_n(B)|\leqslant C(n^{-(s-1)/2}S(B_{an^{-(s-1)/2}})+d(n)n^{-(s-2)/2}V(B_{an^{-(s-1/2)}}))$$

where $G_n(x)$ is given by (3). Both $S(B_{c/\sqrt{n}})$ and $S(B_{an^{-(s-1)/2}})$ may be replaced by 1, and furthermore, if there exists a weight polynomial of X, even $V(B_{c/\sqrt{n}})$ and $V(B_{an^{-(s-1)/2}})$ may be replaced by 1.

Corollary. *If $\beta_s<\infty$, s integer $>k>1$, then*

$$|F_n(B)-\Phi(B)|\leqslant\frac{C}{\sqrt{n}}$$

uniformly for all convex Borel sets $B\subset R_k$.

Application 1. *We can use (24) to estimate the probability that Y_n falls into a bounded Borel set E contained in an affine manifold $L\subset R^k$ of dimension $h<k$. Taking B equal to the convex hull of E, we get*

$$S(B_{c/\sqrt{n}})=O(n^{-(k-h+1)/2})$$

and

$$V(B_{c/\sqrt{n}})=O(n^{-(k-h)/2})$$

and thus, because $\Phi(B)=0$,

$$F_n(E)\leqslant F_n(B)=O(n^{-(k-h)/2}).$$

It is easy to show by an example that this order of magnitude can actually be attained.

Application 2. *If $B\in\mathfrak{B}_1$ is closed, then $\delta(\delta B)=\delta B$ and thus $\alpha(\delta B,\varepsilon)=\alpha(B,\varepsilon)$. If $\beta_r<\infty$, $2<r\leqslant 3$, then we obtain, because $G_n(\delta B)=0$,*

$$F_n(\delta B)\leqslant Cn^{-\frac{1}{2}}\sup_{0<h\leqslant c/\sqrt{n}}[S(B_h)+S(B_{-h})]$$

and if B is convex and X has a weight polynomial

$$F_n(\delta B) \leqslant Cn^{-(r-2)/2}.$$

Theorem 5. *If* $\beta_r < \infty$, $r > 2$, *and if* B *is a Borel set such that* $V(\delta B) = 0$, *then* $\lim_{n\to\infty} F_n(B) = \Phi(B)$.

Proof. Let M^{-1} be the inverse matrix of M, and put $(x, M^{-1}x) = \Sigma_{i,j} M_{ij}^{-1} x_i x_j$, $x \in R_k$. Then $E(Y_n, M^{-1}Y_n) = k$ and because $(x, M^{-1}x)$ is positive definite, the set $K = \{x : (x, M^{-1}x) \leqslant b^2\}$ is compact. From Chebyshev's inequality, we get

$$F_n(K') \leqslant \frac{k}{b^2}.$$

It follows that we can make b, independently of n, so large that both $F_n(B \cap K')$ and $\Phi(B \cap K')$ are arbitrarily small. For each finite b, the set $B \cap K$ is bounded, and thus by Lemma 7 and (23) applied to $B \cap K$, we obtain

$$\lim_{n\to\infty} (F_n(B \cap K) - \Phi(B \cap K)) = 0.$$

The theorem follows.

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