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# Asymptotic behavior of integrals connected with spectral functions for hypoelliptic operators

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ABSTRACT

In the first part of this paper are considered real polynomials $P(\xi)$, $\xi \in R^n$, complete and non-degenerate in the sense that there is a set of (even) multi-indices $\alpha^j$, $j=1, ..., N$, such that, for $|\xi| > K$, $\xi$ real,

$$cP(\xi) \leqslant \sum \xi^{\alpha^j} \leqslant CP(\xi).$$

(See V. P. Mihailov, *Soviet Math. Dokl. 164* (1965), MR 32: 6047.)

It is then proved by an explicit computation, for every given even multi-index $\gamma$, that there are a real number $\theta > 0$ and an integer $r$, $0 \leqslant r < n$, depending only on $\gamma$ and $\{\alpha^j\}$, and such that

$$\int \xi^\gamma \exp\{-tP(\xi)\}\, d\xi = K_\gamma(P)\, t^{-\theta} \left|\log t\right|^r (1+o(1))$$

as $t \to +0$. A Tauberian argument then leads to an asymptotic estimate of the integral

$$e_0^{(\beta,\beta)}(\lambda, 0) = \int_{P(\xi)\leqslant\lambda} \xi^{2\beta}\, d\xi,$$

where $e_0^{(\beta,\beta)}$ is a derivative of a certain spectral function. Less explicit results for a larger class of polynomials were given by N. Nilsson, *Ark. f. Mat. 5* (1965). In the second part of the paper, the explicit computations are extended to the larger class considered by Nilsson but under the restriction $n=2$.

## 0. Introduction

1. A polynomial $P(\xi)$, $\xi = (\xi_1, ..., \xi_n) \in R^n$, is called hypoelliptic if it is strictly stronger than all its derivatives $P^{(\alpha)}(\xi) = (\partial/\partial\xi_1)^{\alpha_1} ... (\partial/\partial\xi_n)^{\alpha_n} P(\xi)$, in the sense that $P^{(\alpha)}(\xi) = o(1)P(\xi)$ as $|\xi| \to \infty$, $\xi$ real. Consider now a hypoelliptic polynomial $P(\xi)$ with real coefficients. The sign of $P(\xi)$ will always be chosen so that

$$P(\xi) \to +\infty \quad \text{as} \quad |\xi| \to \infty, \quad \xi \text{ real.} \tag{0.1}$$

(We have to exclude the case, for $n=2$, when (0.1) cannot be made valid by a change of sign.) Let $P(D)$, $D = i^{-1}(\partial/\partial x_1, ..., \partial/\partial x_n)$ be the corresponding formally self-adjoint differential operator. Then there exists a unique self-adjoint realization $A_0$ of $P(D)$ in $L^2(R^n)$. The spectral resolution of $A_0$ is given by projection operators $E_0(\lambda)$, which can be expressed in terms of a kernel

$$e_0(\lambda, x-y) = \int_{P(\xi)\leqslant\lambda} \exp\{i\langle x-y, \xi\rangle\}\, d\xi,$$

the spectral function for $A_0$. We shall be concerned with the asymptotic behavior of the derivatives of $e_0$, in particular

$$e_0^{(\beta,\beta)}(\lambda, 0) = \int_{P(\xi)\leqslant\lambda} \xi^\gamma \, d\xi, \quad \gamma = 2\beta. \tag{0.2}$$

It was proved recently by N. Nilsson [9] that if $P(\xi)$ is a real polynomial satisfying the condition (0.1), then for every given even multi-index $\gamma$ there are real numbers $\theta$, $c>0$, and an integer $r>0$, such that

$$c^{-1}\lambda^\theta (\log \lambda)^r \leqslant \int_{P(\xi)\leqslant\lambda} \xi^\gamma \, d\xi \leqslant c\lambda^\theta (\log\lambda)^r, \quad \text{for} \quad \lambda > \lambda_0. \tag{0.3}$$

It was also shown in [9], that if $n=2$, then there is a sharp asymptotic estimate

$$\int_{P(\xi)\leqslant\lambda} \xi^\gamma \, d\xi = c\lambda^\theta (\log \lambda)^r \{1 + o(1)\}, \quad \text{as} \quad \lambda \to +\infty, \tag{0.4}$$

with $r=0$ or 1.

Since the proof of (0.3) and (0.4) in [9] is non-constructive, it remains to find the exact values of the parameters $\theta$ and $r$ for given $\gamma$ and $P$. Of course it is well known that $r\equiv 0$, $\theta = (n + |\gamma|)/m$ when $P(\xi)$ is elliptic (see L. Gårding [5], G. Bergendal [1]), and that $r\equiv 0$, $\theta = \sum_1^n q_i(1+\gamma_i)/m$, if $P$ is quasi-elliptic of weight $q=(q_1, \ldots, q_n)$ (see for instance F. Browder [2]). Or let

$$P(\xi) = \xi_1^{2m_1} + \xi_1^{2p_1}\xi_2^{2p_2} + \xi_2^{2m_2}, \tag{0.5}$$

with $m_1 > p_1$, $m_2 > p_2$, and $p_1/m_1 + p_2/m_2 > 1$. Then, as was announced in the note [6] by V. N. Gorčakov, for $\gamma=0$,

$$\left.\begin{aligned} &r = 0 \quad \text{if} \quad p_1 \neq p_2, \qquad r=1 \quad \text{if} \quad p_1 = p_2; \\ &\theta = \max\{(m_1+p_2-p_1)/2m_1p_2,\ (m_2+p_1-p_2)/2m_2p_1\}. \end{aligned}\right\} \tag{0.6}$$

A simple way to prove (0.6) is to compare $e_0(\lambda, 0)$, which is the volume of the set $\{\xi\in R^2;\ P(\xi)\leqslant\lambda\}$, with the volume of the set $\{\xi \in R^2;\ \max(\xi_1^{2m_1}, \xi_2^{2m_2}, \xi_1^{2p_1}\xi_2^{2p_2}) \leqslant \lambda\}$. The same idea (which I owe to a personal communication by L. Hörmander) can be used to show, for example, that if $P(\xi) = |\xi|^{2m} + (\xi_1 \ldots \xi_n)^{2p}$, $1/2p < n/2m$, and if $\gamma = 0$, then $r = n-1$, $\theta = 1/2p$.

2. Given a real polynomial $P(\xi) = \sum c_\alpha \xi^\alpha$, satisfying the condition (0.1), set $(P) = \{\alpha;\ c_\alpha \neq 0\}$, and let $(P)^*$ be the convex hull of $(P) \cup \{0\}$. Then $F(P)$, the Newton polyhedron for $P$, is the union $\cup F^k(P)$ of those $(n-1)$-dimensional flat pieces of the boundary of $(P)^*$ that are not contained in any coordinate hyperplane $x_i = 0$, $1\leqslant i \leqslant n$. Let $\{\alpha^j\}_1^N$ be the vertices of $F(P)$, and let $\nu^k$ be a normal for the face $F^k(P)$, normalized so that

$$tP(t^{-\nu_1^k}\xi_1, \ldots) = tP(t^{-\nu^k}\xi) = P_F^k(\xi) + o(1) \quad \text{as} \quad t\to 0, \tag{0.7}$$

where $P_F^k(\xi) = \sum c_\alpha \xi^\alpha$, $\alpha \in F^k(P)$. Then $P$ is called complete and non-degenerate (Mihailov [8]) if

$$\sum_{j=1}^N \xi^{\alpha^j} \leqslant CP(\xi), \quad \text{for} \quad \xi\in R^n, \quad |\xi| \text{ big enough.} \tag{0.8}$$

(If in addition $\nu^k > 0$ for all $k$, then $P(\xi)$ is a hypoelliptic polynomial, of the class called multi-quasielliptic in our previous papers [3], [4].) Using (0.1), (0.7), and (0.8), we can now show that if $P(\xi)$ is real, complete and non-degenerate, then for every even multi-index $\gamma$,

$$\int \xi^\gamma \exp\{-tP(\xi)\}\,d\xi = K_\gamma(P)\,t^{-\theta}\,|\log t|^r\,(1+o(1)) \quad \text{as} \quad t\to +0. \tag{0.9}$$

Here $\theta = \max \langle \nu^k, \gamma + e\rangle$, $e=(1, \dots, 1)$, and $r = n-1-s$ where $s$ is the dimension of a face of $F(P)$ defined in a unique way by $\gamma$. Since

$$\int \xi^\gamma \exp\{-tP(\xi)\}\,d\xi = \int e^{-t\lambda}\,de_0^{(\beta,\beta)}(\lambda, 0), \quad \gamma = 2\beta, \tag{0.10}$$

if $e_0^{(\beta,\beta)}(\lambda, 0)$ is given by (0.2), a simple Tauberian argument is all that is needed to arrive from (0.10) to an estimate like (0.4). This means that we have found a generalization of Gorčakov's result (0.6) to all real, complete and non-degenerate polynomials. It is interesting to notice that we always get $\theta \leqslant n-1$.

3. If $P(\xi)$ is an arbitrary real polynomial satisfying (0.1), then Nilsson's result can be used together with an Abelian theorem to derive an asymptotic extimate for $\int \xi^\gamma \exp\{-tP(\xi)\}d\xi$ as $t\to +0$. When $n=2$ it is again possible to find an algorithm for the actual computation of $\theta$ and $r$, because then we can use estimates for $P(\xi)$ based on expansions of the zeros of $P(\xi)$ in Puiseux series. (Cf. Friberg [4].)

## 1. The extremal case of a complete and non-degenerate polynomial

Consider a polynomial $P(\xi)$, $\xi \in R^n$, with real coefficients, and such that, say,

$$P(\xi) \to +\infty \quad \text{as} \quad |\xi| \to \infty, \quad \xi \text{ real}. \tag{1.1}$$

If $P(\xi) = \sum c_\alpha \xi^\alpha$, denote by $(P) = \{\alpha; c_\alpha \neq 0\}$ the index set of $P$, and let $(P)^*$ be the convex hull of $(P) \cup \{0\}$. As is well known, it follows from (1.1), that $P(\xi) \to +\infty$ at least as fast as a positive power of $|\xi|$, hence trivially that $(P)^*$ must contain a full neighborhood of the origin in $\overline{R^n_+}$. The newton polyhedron $F(P) = \cup F^k(P)$ is then defined as the union of those $(n-1)$-dimensional flat faces of the boundary of $(P)^*$ that are not parts of a coordinate hyperplane. It is possible to choose the normal $\nu^k$ of each $F^k(P)$ so that $\theta^k(\alpha) = \langle \nu^k, \alpha\rangle = 1$ for $\alpha \in F^k(P)$, and so that

$$(P)^* = \{\alpha \geqslant 0;\, \theta(\alpha) = \max_k \theta^k(\alpha) \leqslant 1\}. \tag{1.2}$$

Then $F(P) = \{\alpha \geqslant 0;\, \theta(\alpha) = 1\}$.

Now let $\{\alpha^j\}$, $1 \leqslant j \leqslant N$, be the vertices of $F(P)$. Then for all $\alpha \in \overline{R^n_+}$, we can find numbers $\lambda_1, \dots, \lambda_n$ such that

$$\alpha = \theta(\alpha) \sum_1^N \lambda_j \alpha^j, \quad \sum \lambda_j = 1, \quad \lambda_j \geqslant 0. \tag{1.3}$$

(In fact, $\alpha \in \theta(\alpha)\,F^k(P)$ for at least one value of $k$.) Since (1.1) implies that the components of each $\alpha^j$ are non-negative even integers, it follows from (1.3) that

$$|\xi^\alpha| \leqslant \left(\sum_{j=1}^{N} \xi^{\alpha^j}\right)^{\theta(\alpha)}, \quad \text{for } \xi \text{ real}, \quad \alpha \text{ arbitrary}. \tag{1.4}$$

In particular, since $\theta(\alpha) \leqslant 1$ for $\alpha \in (P)$, we have

$$P(\xi) \leqslant C(1 + \varrho_F(\xi)) \quad \text{for } \xi \text{ real}, \quad \varrho_F(\xi) = \sum_1^N \xi^{\alpha_j}. \tag{1.5}$$

If $P(\xi) \to +\infty$, and if not only $P(\xi) = O(1)\varrho_F(\xi)$, but also $\varrho_F(\xi) = O(1)P(\xi)$, when $|\xi| \to \infty, \xi$ real, then $P$ is called a *complete and non-degenerate* real polynomial (Mihailov [8]). Let $F^{s,j}(P)$ denote an arbitrary $s$-dimensional face of $F(P)$, $0 \leqslant s \leqslant n-1$, $j = 1, 2, \ldots$, and set

$$P_F^{s,j}(\xi) = \sum c_\alpha \xi^\alpha, \quad \alpha \in F^{s,j}(P),$$

where the $c_\alpha$ are the coefficients of $P(\xi)$. Then a necessary and sufficient condition for a real $P$ to be complete and non-degenerate is that, for all $s$, $j$,

$$P_F^{s,j}(\xi) \neq 0, \quad \text{for real } \xi = (\xi_1, \ldots, \xi_n) \quad \text{with all} \quad \xi_i \neq 0. \tag{1.6}$$

(Mihailov [8], see also Friberg [3]). Due to estimates like (1.4), where $\theta(\alpha) < 1$ when $\alpha \in (P)$, $\alpha \notin F(P)$, if $P$ is complete and non-degenerate then $P_F(\xi) = \sum c_\alpha \xi^\alpha$, $\alpha \in F(P)$, is in a natural sense the principal part of $P(\xi)$.

**Lemma 1.1.** *Let $\gamma = (\gamma_1, \ldots, \gamma_n)$, with $\gamma_i$ even non-negative integers, and suppose that the real polynomial $P(\xi)$ tends to $+\infty$ as $|\xi| \to \infty$, $\xi$ real, so that the integral*

$$I_\gamma(t) = \int \xi^\gamma \exp\{-tP(\xi)\}\, d\xi, \quad t > 0, \ \xi \in R^n$$

*is convergent. Let $e = (1, \ldots, 1)$, and set $\theta = \theta(\gamma + e) = \max \theta^k(\gamma + e)$. Then there are constants $c$, $C$, and $\theta' \geqslant \theta$, depending on $P$ and on $\gamma$, such that*

$$ct^{-\theta} \leqslant I_\gamma(t) \leqslant Ct^{-\theta'} \quad \text{for} \quad 0 < t \leqslant 1. \tag{1.7}$$

*If $P$ is also complete and non-degenerate, $\theta'$ can be chosen arbitrarily close to $\theta$.*

*Proof.* If $\alpha \in (P)$, then $\theta^k(\alpha) = \langle \nu^k, \alpha \rangle \leqslant 1$. Hence

$$P(\xi) \leqslant A\left(1 + \sum_{i=1}^{n} |\xi_i|^{1/\nu_i^k}\right), \quad \xi \in R^n.$$

But then trivially, for $0 < t \leqslant 1$, and for all $k$,

$$I_\gamma(t) \geqslant A_1 \prod_1^n \int \xi_i^{\gamma_i} \exp\left\{-t|\xi_i|^{1/\nu_i^k}\right\} d\xi_i = c_\gamma t^{-\langle \nu k, \gamma + e \rangle},$$

which proves the first of the estimates in (1.7). Next, choose $n$ linearly independent points $\beta^j \in (P)^*$ such that

$$P(\xi) \geqslant B \sum_1^n |\xi^{\beta^j}| - B_1 \quad \text{for } \xi \text{ real}, \quad \text{some} \quad B > 0, \tag{1.8}$$

and such that $$\gamma + e = \theta' \sum_1^n \lambda_j \beta^j, \quad \sum_1^n \lambda_j = 1, \quad \text{all} \quad \lambda_j > 0.$$

This can be done for some $\theta' \geqslant \theta(\gamma+e)$ when $P(\xi) \to +\infty$ as $|\xi| \to \infty$, and it can be done with $\theta' \leqslant \theta(\gamma+e)+\varepsilon$, for arbitrary $\varepsilon>0$, if $P$ is complete and non-degenerate. (It can even be done with $\theta'=\theta(\gamma+e)$ if we know that $\gamma+e$ is an interior point of $\theta(\gamma+e)F^k(P)$, for some $k$.) Now let us introduce as new independent variables $\eta_j = \xi^{\beta^j}$, $1 \leqslant j \leqslant n$. Let $\Lambda = (\lambda_j^i)$ be the inverse of the matrix $(\beta_j^i)$, and set $\lambda^i = (\lambda_1^i, \ldots, \lambda_n^i)$. Then $\xi_i = \eta^{\lambda^i}$ for $\xi \in R_+^n$, and the functional determinant is $d(\xi)/d(\eta) = \det(\xi_i \lambda_j^i/\eta^j) = \det(\Lambda)(\xi_1 \ldots \xi_n)/(\eta_1 \ldots \eta_n)$, $\det(\Lambda) = 1/\det(\beta_j^i)$. In view of (1.8), it follows that, for $0<t\leqslant 1$,

$$I_\gamma(t) \leqslant 2^n \int_{R_+^n} \xi^{\gamma+e} \exp\left\{-t\left(B\sum_1^n \xi^{\beta^j} - B_1\right)\right\} d\xi/(\xi_1 \ldots \xi_n)$$

$$\leqslant C_1 \int_{R_+^n} (\eta^\lambda)^{\theta'} \exp\left\{-Bt\sum_1^n \eta_j\right\} d\eta/(\eta_1 \ldots \eta_n) = C_j t^{-\theta'}, \tag{1.9}$$

which proves the remaining half of (1.7).

**Theorem 1.1.** *Let $P(\xi) = \sum c_\alpha \xi^\alpha$, $\xi \in R^n$, be a real complete and non-degenerate polynomial with $P(\xi) \to +\infty$ as $|\xi| \to \infty$, $\xi$ real. Suppose that, for a given even multi-index $\gamma \geqslant 0$, the point $\gamma+e$ is an interior point of $\theta F^k(P)$ for some $k$, $\theta = \theta(\gamma+e)$. Set $P_F^k(\xi) = \sum c_\alpha \xi^\alpha$, $\alpha \in F^k(P)$. Then, as $t \to +0$,*

$$I_\gamma(t) = \int \xi^\gamma \exp\{-tP(\xi)\}\, d\xi = t^{-\theta}\left[\int \xi^\gamma \exp\{-P_F^k(\xi)\}\, d\xi + o(1)\right]. \tag{1.10}$$

*Proof.* Let $\gamma+e \in \theta F^k(P)$, $\theta = \theta(\gamma+e)$, and let $\nu$ be the normal of $F^k(P)$, so that $\langle \nu, \gamma+e\rangle = \theta$. Let $t^{-\nu}\xi = (t^{-\nu_1}\xi_1, \ldots, t^{-\nu_n}\xi_n)$, and set

$$g(\xi, t) = \xi^\gamma \exp\{-tP(t^{-\nu}\xi)\}; \quad g(\xi) = \xi^\gamma \exp\{-P_F^k(\xi)\}.$$

Here $tP(t^{-\nu}\xi) = P_F^k(\xi) + O(1)t^\delta$, $\delta>0$, as $t \to +0$, for fixed $\xi$. It follows that, at least formally,

$$t^\theta I_\gamma(t) = \int g(\xi, t)\, d\xi \to \int g(\xi)\, d\xi = \int \xi^\gamma \exp\{-P_F^k(\xi)\}\, d\xi$$

as $t \to +0$. Now choose the $\beta^j$ of (1.8) as points on $F^k(P)$. Then, for $0<t\leqslant 1$,

$$0 \leqslant g(\xi, t) \leqslant \xi^\gamma \exp\{-B\sum_1^n |\xi^{\beta^j}| + B_1\} \in L^1(R^n).$$

(Cf. the proof of Lemma 1.1.) Therefore (1.10) will follow from Lebesgue's theorem on dominated convergence.

We can also give a direct proof that $g(\xi) \in L^1(R^n)$. If $P$ is complete and non-degenerate, then trivially (1.6) holds for all $\mu$, $j$. But (1.6) can be used to prove that, for some constants $C, c>0$,

$$c\varrho_F^k(\xi) \leqslant P_F^k(\xi) \leqslant C\varrho_F^k(\xi), \quad \text{when} \quad \xi \in R^n,$$

where $\varrho_F^k(\xi)=\sum \xi^{\alpha^j}$, summed over all $j$ with $\alpha^j \in F^k(P)$. (Cf. the proof of Theorem 4.3, Friberg [3].) We may therefore assume that $g(\xi)=\xi^\gamma \exp\{-\varrho_F^k(\xi)\}$. Obviously $\{\alpha^j;\ \alpha^j \in F^k\}$ is a basis for $R^n$. Choose, for $1 \leqslant i \leqslant n$, another basis $\{\alpha^{i,1}, ..., \alpha^{i,n-1}, e^i\}$, where $\{\alpha^{i,j}\}_1^{n-1}$ is subset of $\{\alpha^j;\ \alpha^j \in F^k\}$, and where $e^i$ is the $i$th coordinate vector $(0, ..., 1, ..., 0)$. Then $\gamma+e=\sum_1^{n-1} q_j^i \alpha^{i,j}+q_n^i e^i$. But all the $\alpha^{i,j}$, $1 \leqslant j \leqslant n$, are in a hyperplane $\langle \mu, \alpha\rangle=0$. Hence $q_n^i=\langle \mu, \gamma+e\rangle/\langle \mu, e^i\rangle$, and we can make $q_n^i<0$ by choosing the points $\alpha^{i,j}$ so that $\gamma+e$ and $e^i$ are on different sides of the hyperplane. To estimate $\int g(\xi)\,d\xi=\int \xi^\gamma \exp\{-\varrho_F^k(\xi)\}\,d\xi$, we now divide the domain of integration into subsets,

$$D_i:\quad \{\xi \in R^n;\ 1+\sum_j |\xi^{\alpha^{i,j}}| \leqslant |\xi_i|^\varepsilon\},\quad 1 \leqslant i \leqslant n,\quad \varepsilon>0,\quad \text{and}$$

$$D_{n+1}:\quad \{\xi \in R^n;\ 1+\sum_j |\xi^{\alpha^{i,j}}| \geqslant |\xi_i|^\varepsilon,\quad \text{for all } i\}.$$

Since $\varrho_F^k(\xi) \geqslant (\sum_1^n |\xi_i|)/n-1$ on $D_{n+1}$, the convergence of the integral over $D_{n+1}$ is obvious. But when $i=1$, for instance,

$$\int_{D_1} g(\xi)\,d\xi \leqslant \int_{D_1} \xi^{\gamma+e}\,d\xi/(\xi_1 \cdots \xi_n) \leqslant \int_{D_1} |\xi_1|^\delta\,d\xi/(\xi_1 \cdots \xi_n),$$

with $\delta=\varepsilon(\sum_1^{n-1} q_j^i)+q_n^i<0$ for $\varepsilon$ small enough. Moreover, on $D_1$ we have every $|\xi_j|$, $j>1$, bounded by a power of $|\xi_1|$. Consequently the integral over $D_1$ converges as $\int_1^\infty \xi_1^{\delta-1}(\log \xi_1)^{n-1}\,d\xi_1$.

**Theorem 1.2.** *Let $P(\xi)$ be as in Theorem 1.1, but suppose that $\gamma \geqslant 0$ is an even multi-index such that $\gamma+e$ is contained in $\theta F^{s,j}(P)$, $\theta=\theta(\gamma+e)$, for some $s$, $j$, with $s$ chosen as small as possible, $0 \leqslant s \leqslant n-1$. Then*

$$I_\gamma(t)=t^{-\theta}|\log t|^{n-1-s}[K_\gamma(P)+o(1)],\quad \text{as}\quad t \to +0, \tag{1.11}$$

*where the constant $K_\gamma(P)$ depends only on $F(P)$, $P_F^{s,j}(\xi)$, and $\gamma$. Also, for some constants $A_1, A_2>0$,*

$$A_2^{\theta+1}\Gamma(\theta) \leqslant K_\gamma(P) \leqslant A_1^{\theta+1}\Gamma(\theta),\quad \theta=\theta(\gamma+e). \tag{1.12}$$

*Proof.* Let $\gamma+e \in \theta F^{s,j}(P)$, $\theta=\theta(\gamma+e)$, and let $\nu$ be a normal of $F^{s,j}$, such that $\langle \nu, \alpha\rangle=1$ for $\alpha \in F^{s,j}$, and consequently $\langle \nu, \gamma+e\rangle=\theta$. If $s<n-1$, then $\nu$ is not uniquely determined, but varies over an affine manifold of dimension $r=n-1-s$. Let

$$v(t)=t^\theta I_\gamma(t)=\int \xi^\gamma \exp\{-tP(t^{-\nu}\xi)\}\,d\xi.$$

Obviously, in order to prove (1.11) it is enough to show that

$$\left(-t\frac{d}{dt}\right)^r v(t) \to K_\gamma(P) \neq 0 \quad \text{as} \quad t \to +0,\quad r=n-1-s. \tag{1.13}$$

The case $r=0$ was discussed in Theorem 1.1. Suppose now $r=1$. Then, since $tP(t^{-\nu}\xi)=\sum t^{1-\langle \nu,\alpha\rangle} c_\alpha \xi^\alpha$, we have

$$-tv'(t)=\int \xi^{\gamma}\{\sum(1-\langle \nu,\alpha\rangle)\,t^{1-\langle \nu,\alpha\rangle}c_{\alpha}\xi^{\alpha}\}\exp\{-tP(t^{-\nu}\xi)\}\,d\xi. \tag{1.14}$$

Let $F'$ be one of the $(n-1)$-dimensional faces of $F(P)$, passing through $F^{s,j}$. Then the normal $\nu'$ of $F'$ is such that $\langle \nu',\alpha\rangle=1$ for $\alpha\in F'$, and $\langle \nu',\gamma+e\rangle=\langle \nu,\gamma+e\rangle=\theta$. Therefore a change of coordinates $t^{-\nu}\xi\to t^{-\nu'}\xi$ transforms the integral in (1.14) into

$$\int \xi^{\gamma}\{\sum{}'(1-\langle \nu,\alpha\rangle)\,c_{\alpha}\xi^{\alpha}+o(1)\}\exp\{-tP(t^{-\nu'}\xi)\}\,d\xi, \tag{1.15}$$

where $o(1)$ stands for terms containing powers of $t$, while $\sum'$ contains the terms with $\langle \nu',\alpha\rangle=1$, $\langle \nu,\alpha\rangle<1$, i.e. with $\alpha\in F'$, $\alpha\notin F^{s,j}$. But for such $\alpha$ it is easy to check that $\gamma+e+\alpha$ is an interior point of $\theta(\gamma+e+\alpha)F'$. Thus, in view of Theorem 1.1, the integral

$$\int \xi^{\gamma}\{\sum{}'(1-\langle \nu,\alpha\rangle)\,c_{\alpha}\xi^{\alpha}\}\exp\{-tP(t^{-\nu'}\xi)\}\,d\xi \tag{1.16}$$

depends continuously on $t$ in the interval $[0, 1]$.

In order to show that the value of the integral for $t=0$ is independent of the $c_{\alpha}$ with $\alpha\notin F^{s,j}$, let us choose $n$ linearly independent points $\beta^1,\dots,\beta^n\in F'$, with $\beta^2,\dots,\beta^n\in F^{s,j}$, and such that $\gamma+e=\theta\sum_2^n\lambda_i\beta^i$ with $\sum\lambda_i=1$, $\lambda_2,\dots,\lambda_n>0$. We will get $\alpha=\sum_1^n\mu_i\beta^i$ with $\sum\mu_i=1$, $\mu_i\geq 0$, and $\mu_1>0$, when $\alpha\in F'$, $\alpha\notin F^{s,j}$. It follows that, for such $\alpha$,

$$1-\langle \nu,\alpha\rangle=\sum_1^n\mu_i(1-\langle \nu,\beta^i\rangle)=\mu_1(1-\langle \nu,\beta^1\rangle), \tag{1.17}$$

Also, we may always assume that $\langle \nu,\beta^1\rangle<1$, so that $1-\langle \nu,\beta^1\rangle\neq 0$. Now, as in the proof of Lemma 1.1, let us introduce new independent variables $\eta_i=\xi^{\beta^i}$, $1\leq i\leq n$. Since (for $\xi\in R_+^n$) $d(\xi)/d(\eta)=(\xi_1\dots\xi_n)/\{\det(\beta^1,\dots\beta^n)\eta_1\dots\eta_n\}$, we find that the limit of the integral in (1.16) as $t\to+0$ can be written as a sum of $2^n$ terms of the type

$$A'\int_{R_+^n}\eta^{\theta\lambda-e}\{\sum{}'\mu_1c'_{\mu}\eta^{\mu}\}\exp\{-\sum c'_{\mu}\eta^{\mu}\}\,d\eta,\quad \lambda=(0,\lambda_2,\dots,\lambda_n), \tag{1.18}$$

where $A'=(1-\langle \nu,\beta^1\rangle)/\det(\beta^1,\dots,\beta^n)$, and where the set of coefficients $\{c'_{\mu}\}$ is identical with the set $\{c_{\alpha};\,\alpha\in F'\}$ of coefficients for $P'_F(\xi)$ except possibly for a change of sign in some of them. Now let $\eta=(\eta_1,\dots,\eta_n)=(\eta_1,\eta')$, $\eta'\in R_+^{n-1}$, and set $\lambda'=(\lambda_2,\dots,\lambda_n)$, $e'=(1,\dots,1)\in R^{n-1}$. Then the integral in (1.18) is equal to

$$\int_{R_+^n}(\eta')^{\theta\lambda'-e'}\{-(\partial/\partial\eta_1)\}\exp\{-\sum c'_{\mu}\eta^{\mu}\}\,d\eta_1\,d\eta'$$
$$=\int_{R_+^{n-1}}(\eta')^{\theta\lambda'-e'}\exp\{-\sum c'_{\mu}\eta^{\mu}|_{\eta_1=0}\}\,d\eta'. \tag{1.19}$$

The method we have used above to take care of the terms in the sum in (1.15) corresponding to points $\alpha\in F'$, can of course also be used on the terms derived from points on the other $(n-1)$-dimensional face, call it $F''$, of $F(P)$ passing through $F^{s,j}$. Thus it remains only to consider the terms in (1.14) of the type

$$\int \xi^\gamma (1-\langle \nu, \alpha\rangle)\, t^{1-\langle \nu, \alpha\rangle} c_\alpha \xi^\alpha \exp\{-tP(t^{-\nu}\xi)\}\, d\xi, \tag{1.20}$$

with $\langle \nu', \alpha\rangle < 1$, $\langle \nu'', \alpha\rangle < 1$, hence also $\langle \nu, \alpha\rangle < 1$. After a substitution $t^{-\nu}\xi \to \xi$, (1.20) takes the form

$$(1-\langle \nu, \alpha\rangle)\, c_\alpha t^{\theta+1} \int \xi^{\gamma+\alpha} \exp\{-tP(\xi)\}\, d\xi = Ct^{\theta+1} I_{\gamma+\alpha}(t).$$

We can now use Lemma 1.1 to obtain the estimate

$$t^{\theta+1} I_{\gamma+\alpha}(t) \leqslant C_1 t^{-a}, \quad a = \theta(\gamma+e+\alpha)+\varepsilon-\theta(\gamma+e)-1,$$

for arbitrary $\varepsilon > 0$. But it is easy to check that $\theta(\gamma+e+\alpha) < \theta(\gamma+e)+1$, when $\langle \nu', \alpha\rangle < 1$, $\langle \nu'', \alpha\rangle < 1$. It follows that $a$ can be made negative, hence that the terms of type (1.20) do not influence the asymptotic behavior of $I_\gamma(t)$.

Consider now the case when $r > 1$ in (1.13). Let $\nu^1$ be a normal to $F^{s,j}$, with $\langle \nu^1, \alpha\rangle = 1$ for $\alpha \in (P)$ if and only if $\alpha \in F^{s,j}$. Set $\nu = \nu^1$ in (1.14), and split the integral into a sum of terms like

$$(1-\langle \nu^1, \alpha\rangle)\, c_\alpha t^{1-\langle \nu^1, \alpha\rangle} \int \xi^{\gamma+\alpha} \exp\{-tP(t^{-\nu^1}\xi)\}\, d\xi. \tag{1.21}$$

Obviously $\alpha \in F^{s,j}$ if we demand that $1-\langle \nu^1, \alpha\rangle \neq 0$. Suppose that $\alpha \in F^{s',j'}$, where $F$ is an $s'$-dimensional face of $F(P)$, passing through $F^{s,j}$, with $s' > s$, $s'$ chosen as small as possible. It is easy to check that $\gamma+e+\alpha$ is an interior point of $\theta(\gamma+e+\alpha)F^{s',j'}$. Let $\nu'$ be a normal to $F^{s',j'}$, with $\langle \nu', \alpha\rangle = 1$. Then (1.21) is equal to

$$(1-\langle \nu^1, \alpha\rangle)\, c_\alpha \int \xi^{\gamma+\alpha} \exp\{-tP(t^{-\nu'}\xi)\}\, d\xi.$$

We can now proceed by induction to show that the term (1.21) is of relevance to the asymptotic behavior of $I_\gamma(t)$ if and only if $\alpha \in F^{s',j'}$ for some $F^{s',j'}$ through $F^{s,j}$ with $s' = s+1$. Therefore, let us choose a nested sequence of faces of increasing dimension $F^{s,j} \subset F^{s+1,j'} \subset \ldots \subset F^{s+r,j_r} = F^{n-1,j_r}$ with corresponding normals $\nu^1, \ldots, \nu^r, \nu^{r+1}$. Finally, let us choose $n$ linearly independent points $\beta^1, \ldots, \beta^n$ with $\beta^{r+1}, \ldots, \beta^n \in F^{s,j}$, $\beta^r \in F^{s+1,j'} \ldots, \beta^1 \in F^{n-1,j_r}$. Then the same kind of argument that led to (1.19) will show us that the total contribution to $K_\gamma(P)$ due to any set of $r$ points $\alpha' \subset F^{s+1,j'}, \ldots, \alpha^r \in F^{n-1,j_r}$ on the chosen seuqence of faces is equal to a sum of $2^n$ terms of the type

$$A \int_{R_+^{n-r}} (\eta^\lambda)^\theta \exp\{-\sum c'_\mu \eta^\mu |_{\eta_1=\ldots=\eta_r=0}\}\, d\eta_1 \ldots d\eta_r/(\eta_1 \ldots \eta_r). \tag{1.22}$$

Here
$$A = \prod_{i=1}^{r} (1-\langle \nu^k, \beta^i\rangle)/\det(\beta^1, \ldots, \beta^n), \tag{1.23}$$

and $\lambda = (0, \ldots, 0, \lambda_{r+1}, \ldots, \lambda_n)$ is determined by the expansion $\gamma+e = \theta(\gamma+e)\sum_{r+1}^n \lambda_i \beta_i$, $\sum \lambda_i = 1$, $\lambda_i > 0$. Obviously, in (1.22) only the constant $A$ is dependent on the choice of the sequence $F^{s,j} \subset F^{s+1,j'} \subset \ldots$. This means that we have in fact proved (1.11), with $K_\gamma$ given by a sum of $2^n$ terms like (1.22), although with new constants $A$, equal to a sum of constants of the type (1.23).

It remains only to derive the estimate (1.12). But if $\eta_i = \xi^{\beta^i}$, $1 \leqslant i \leqslant n$, then

$$\sum c'_\mu \eta^\mu \big|_{\eta_1=\ldots=\eta_r=0} = \sum_{F^s,\,j} c_\alpha \xi^\alpha = P_F^{s,\,j}(\xi).$$

Further, it can be proved that

$$P_F^{s,\,j}(\xi) \geqslant c\varrho_F^{s,\,j}(\xi) = c\sum_{r+1}^{n} \xi^{\beta^i} \quad \text{for} \quad \xi \in R^n, \quad \text{some} \quad c > 0$$

(see Friberg [3], the proof of Theorem 4.3). It follows that

$$K_\gamma(P) \leqslant A \prod_{r+1}^{n} \int \eta_i^{\lambda_i\theta-1} \exp\{-c\eta_i\}\, d\eta_i$$

$$= A \prod_{r+1}^{n} \{c^{-\lambda_i\theta}\Gamma(\lambda_i\theta)\} \leqslant A_1^{\theta+1}\Gamma(\theta).$$

The second half of (1.12) follows in the same way from a trivial upper estimate of $P_F^{s,\,j}(\xi)$.

*Remark.* Let $P(\xi)$ be an arbitrary real polynomial with $P(\xi) \to \infty$ as $|\xi| \to \infty$, $\xi$ real. Let $\{\alpha^j\}_1^N$ be the vertices of $F(P)$, and set $\varrho_F(\xi) = \sum_1^N \xi^{\alpha^j}$. (The $\alpha^j$ are even, non-negative multi-indices.) Then $\varrho_F(\xi)$ is a complete and non-degenerate real polynomial, and $P(\xi) \leqslant C(1+\varrho_F(\xi))$ for $\xi$ real, so that

$$I_\gamma(t;P) = \int \xi^\gamma \exp\{-tP(\xi)\}\, d\xi \geqslant C_1 I_\gamma(c_2 t; \varrho_F)$$

for $0 < t \leqslant 1$. This means that in this general case Theorem 1.2 gives at least a lower bound for the singularity of $I_\gamma(t;P)$ as $t \to +0$.

## 2. The two-dimensional case

Let $P(\xi)$, $\xi \in R^2$, be a real polynomial in two variables, and write $P(\xi)$ in the form

$$P(\xi) = p_1(\xi_1) \prod_{i=1}^{m_2} (\xi_2 - \phi_i(\xi_1)), \quad \deg p_1(\xi_1) = m \geqslant 0. \tag{2.1}$$

Then there is a constant $A_1$ such that all the zeros $\phi_i(\xi_1)$ can be represented by Puiseux expansions of the type

$$\phi(\xi_1) = \sum_0^\infty c_j \xi_1^{\delta_j}, \; \delta_0 > \delta_1 > \ldots, \quad \text{for} \quad \xi_1 \geqslant A_1, \tag{2.2}$$

where either the sum is finite or $\delta_j \to -\infty$ as $j \to \infty$. Suppose, as in the preceding paragraph, that

$$P(\xi) \to +\infty \quad \text{as} \quad |\xi| \to \infty, \quad \xi \text{ real}. \tag{2.3}$$

It follows that the coefficients $c_i$ in the expansion (2.2) of a zero for $P(\xi)$ cannot all be real. Let $\phi$ be a fixed zero, and suppose that $c_J$ is the first non-real coefficient in (2.2), $J = J(\phi)$. Then, if

$$v_0 = \xi_2; \quad v_{\phi,k} = \xi_2 - \sum_0^{k-1} c_j \xi_1^{\delta_j}, \quad 1 \leqslant k \leqslant J, \tag{2.4}$$

each such $v_{\phi,k}$ will be called a *real truncated factor* of length $k$ for $P(\xi)$. Let $\phi' = \sum_0^\infty c_j' \xi_1^{\delta'_j}$ be a second zero of $P(\xi)$, with $v_{\phi',k} = v_{\phi,k}$, but with $v_{\phi',k+1} \neq v_{\phi,k+1}$ if $k+1 \leqslant J$. Then $\phi, \phi'$ will be called *conjugate at level k*. When $\phi'$ varies over all zeros conjugate to $\phi$ at level $k$, we will set $c_k' = c_{ki}$, $\delta_k' = \delta_{ki}$, $i = 1, 2, \ldots$. We shall also use the notations $\delta_{k,i} = \max(\delta_k, \delta_{ki})$, and $c_{k,i} = c_{ki}$, $c_{ki} - c_k$, or $-c_k$, depending on whether $\delta_{ki} > \delta_k$, $= \delta_k$, or $< \delta_k$.

**Lemma 2.1.** *Suppose $P(\xi)$ is a real polynomial* (2.1), *satisfying the condition* (2.3). *Let $v_{\phi,s} = \xi_2 - \sum_0^{s-1} c_j \xi_1^{\delta_j}$, $s \geqslant 1$, be a given real truncated factor for $P(\xi)$, and set*

$$M_{\phi,s}(\xi_1, v) = \xi_1^m \prod_{k<s} \prod_{c_{k,i} \neq 0} (|v| + \xi_1^{\delta_{k,i}}) \prod_i (|v| + \xi_1^{\delta_{si}}). \tag{2.5}$$

*Then there are constants $A$, $B$, $B' > 0$ such that*

$$B \leqslant P(\xi)/M_{\phi,s}(\xi_1, v_{\phi,s}) \leqslant B', \tag{2.6}$$

*when $\xi$ varies over a certain region $V_{\phi,s}$, defined by conditions of the type*

$$\left.\begin{array}{l} \text{(i)}\ \xi_1 \geqslant A > 0, \quad \text{(ii)}\ |v_{\phi,s}| < \varepsilon \xi_1^{\delta_{s-1}}, \\ \text{(iii)}\ |v_{\phi,s} - c_{si}\xi_1^{\delta_{si}}| \geqslant \varepsilon \xi_1^{\delta_{si}} \quad \textit{for all } i \textit{ with } c_{si} \textit{ real.} \end{array}\right\} \tag{2.7}$$

*Similarly, if*

$$M_0(\xi) = \xi_1^m \prod_i (|\xi_2| + \xi_1^{\delta_{0i}}),$$

*then*

$$B \leqslant P(\xi)/M_0(\xi) \leqslant B',$$

*when $\xi$ varies over a region $V_0$, defined by the conditions*

$$\text{(i)}\ \xi_1 \geqslant A > 0, \quad \text{(ii)}\ |\xi_2 - c_{0i}\xi_1^{\delta_{0i}}| \geqslant \varepsilon \xi_1^{\delta_{0i}} \quad \textit{for all } i \textit{ with } c_{0i} \textit{ real.}$$

*Proof.* Let $\phi'$ be an arbitrary zero, and let $v = v_{\phi,s}$.

Then
$$\xi_2 - \phi'(\xi_1) = v + \sum_0^{s-1} c_j \xi_1^{\delta_j} - \sum_0^\infty c_j' \xi_1^{\delta_j'}.$$

Hence if $\phi, \phi'$ are conjugate at level $k < s$, then

$$\xi_2 - \phi'(\xi_1) = (v - c_{k,i}\xi_1^{\delta_{k,i}}) + o(1)\, \xi_1^{\delta_{k,i}}$$

for some $i$, as $\xi_1 \to +\infty$. If $\phi, \phi'$ are conjugate at level $\geqslant s$, then instead

$$\xi_2 - \phi'(\xi_1) = (v - c_{si}\xi_1^{\delta_{si}}) + o(1)\, \xi_1^{\delta_{si}}$$

for some $i$, as $\xi_1 \to +\infty$. But obviously, for some $B_1 > 0$,

$$|v - c_{k,i}\xi_1^{\delta_{k,i}}| \geqslant B_1(|v| + \xi_1^{\delta_{k,i}}),$$

$$|v - c_{si}\xi_1^{\delta_{si}}| \geqslant B_1(|v| + \xi_1^{\delta_{si}}),$$

when $v = v_{\phi,s}$ and $\xi_1$ satisfy conditions (i)–(iii) of the lemma (with $\varepsilon$ small enough), i.e. when $\xi \in V_{\phi,s}$. Since $P(\xi) = p(\xi_1)\prod(\xi_2 - \phi'(\xi_1)) > 0$ for $\xi_1$ big enough, it is now easy to complete the proof of the lemma.

**Lemma 2.2.** *Let $P(\xi)$, $\xi \in R^2$, be a real polynomial satisfying* (2.3), *and define $M_{\phi,s}(\xi_1, v)$ as in Lemma 2.1. Let $\gamma = (\gamma_1, \gamma_2)$ be a given even multi-index, and set $\gamma_\phi = (\gamma_1 + \delta_0\gamma_2, 0)$, when $\phi(\xi_1) = c_0\xi_1^{\delta_0} + \dots$ . Then, as $t \to +0$, the singularity of*

$$I_{\gamma,A}(t; P) = \int_{\xi_1 > A} \xi^\gamma \exp\{-tP(\xi)\}\, d\xi$$

*with A big enough, is of the same order of magnitude as the highest singularity of anyone of the integrals*

$$I_{\gamma,A}(t; M_{\phi,s}) = \int_{\xi_1 > A} \xi_1^{\gamma_\phi} \exp\{-tM_{\phi,s}(\xi_1, v)\}\, d\xi_1\, dv$$

*for arbitrary $\phi$, $s \geqslant 1$, or of*

$$I_{\gamma,A}(t; M_0) = \int_{\xi_1 > A} \xi^\gamma \exp\{-tM_0(\xi)\}\, d\xi.$$

(A corresponding statement may be proved for

$$I'_{\gamma,A}(t; P) = \int_{\xi_1 < -A} \xi^\gamma \exp\{-tP(\xi)\}\, d\xi.)$$

*Proof.* Let $V_{\phi,s}$ be the set (2.7), for arbitrary $\phi$ and $s$. In view of the definition (2.4),

$$v_{\phi,s} - c_{si}\xi_1^{\delta_{si}} = v_{\phi', s+1},$$

for some $\phi'$ with $\phi$, $\phi'$ conjugate at level $s$. It follows that the union of the mutually disjoint sets $V_{\phi,s}$, for arbitrary $\phi$, $s$, and of $V_0$, is the entire set $\{\xi; \xi_1 > A\}$. Hence,

$$I_{\gamma,A}(t; P) = \sum_{\phi,s} \int_{V_{\phi,s}} \xi^\gamma \exp\{-tP(\xi)\}\, d\xi + \int_{V_0} \xi^\gamma \exp\{-tP(\xi)\}\, d\xi. \tag{2.8}$$

But for given $\phi$, there are $c_\gamma, c'_\gamma > 0$ such that

$$c_\gamma \xi_1^{\gamma_\phi} \leqslant \xi^\gamma = \xi_1^{\gamma_1}\left(v_{\phi,s} + \sum_0^{s-1} c_i \xi_1^{\delta_i}\right)^{\gamma_2} \leqslant c'_\gamma \xi_1^{\gamma_\phi}, \quad \xi \in V_{\phi,s}.$$

Together with the lower estimate in (2.6), (2.8) therefore shows that

$$I_{\gamma,A}(t; P) \leqslant \sum_{\phi,s} c'_\gamma I_{\gamma,A}(Bt; M_{\phi,s}) + I_{\gamma,A}(Bt; M_0).$$

On the other hand, the upper estimate in (2.6) is obviously valid not only in $V_{\phi,s}$ but for all $\xi$ with $\xi_1 > A$. This means that

$$I_{\gamma,A}(t; P) \geqslant \max\left(\max_{\phi,s} c_\gamma I_{\gamma,A}(B't; M_{\phi,s}),\ I_{\gamma,A}(B't; M_0)\right),$$

and the proof of the lemma is complete.

Although $M_0$ and all the $M_{\phi,s}$ are not necessarily polynomials, at least they tend to infinity as $|\xi| \to \infty, \xi_1 > A$, and it is easy to check that the results of section 1 are still valid if we give the natural meaning to $F(M_0)$, etc. Consequently each $I_{\gamma,A}(t; M_0)$ or $I_{\gamma,A}(t; M_{\phi,s})$ has a singularity of order $t^{-\theta}|\log t|^r$ as $t \to +0$, with $r=0$ or 1, and with $\theta$ defined by $\gamma$ and $F(M_0)$ or by $\gamma_\phi$ and $F(M_{\phi,s})$, respectively. But then, due to Lemma 2.2, $I_{\gamma,A}(t; P)$ must have a singularity of the same type, with

$$\theta = \max(\theta(\gamma + e; M_0), \quad \max_{\phi,s} \theta(\gamma_\phi + e; M_{\phi,s})),$$

and with $r=0$ or 1.

Now suppose, for given $\phi$, $s$, that $\theta = \theta(\gamma_\phi + e; M_{\phi,s})$, and let $\chi_{\phi,s} = 1$ on $V_{\phi,s}$, $=0$ outside $V_{\phi,s}$. Then we can find $v = v_{\phi,s}$ such that

$$t^\theta \int_{V_{\phi,s}} \xi^\gamma \exp\{-tP(\xi)\}\, d\xi$$
$$= \int \chi_{\phi,s}(t^{-\nu_1}\xi_1, t^{-\nu_2}v)\, \xi_1^{\gamma_\phi}(1+o(1)) \exp\{-P_{\phi,s}(\xi_1, v)(1+o(1))\}\, d\xi,$$

where $P_{\phi,s}$ is made up of the constant terms in the expansion of $tP(t^{-\nu_1}\xi_1, t^{-\nu_2}v + \sum_0^{s-1} c_j (t^{-\nu_1}\xi_1)^{\delta_j})$ in powers of $t$. Assuming for simplicity that $r=0$, we can now use Lebesgue's theorem on dominated convergence to show that

$$t^\theta \int_{V_{\phi,s}} \xi^\gamma \exp\{-tP(\xi)\}\, d\xi \to \int \chi^0_{\phi,s}(\xi_1, v)\, \xi_1^{\gamma_\phi} \exp\{-P_{\phi,s}(\xi_1, v)\}\, d\xi_1\, dv \qquad (2.9)$$

as $t \to +0$. Here $\chi^0_{\phi,s}(\xi, v_{\phi,s})$ is the characteristic function for the set $V^0_{\phi,s}$ defined as the limit, as $t \to +0$, of the set given by the conditions

$$\text{(i)}\ \xi_1 > At^{\nu_1}, \quad \text{(ii)}\ t^{\nu_1\delta_{s-1}-\nu_2}|v_{\phi,s}| < \varepsilon\xi_1^{\delta_{s-1}},$$
$$\text{(iii)}\ |t^{\nu_1\delta_{si}-\nu_2} v_{\phi,s} - c_{si}\xi_1^{\delta_{si}}| \geq \varepsilon\xi_1^{\delta_{si}} \text{ for } c_{si} \text{ real}.$$

Let for instance $\nu_2/\nu_1 = \delta_{sj}$, for some $j$. We may assume without restriction that $\delta_{sj} = \delta_s$, the exponent determined by the expansion (2.2) of $\phi$. Then it is easy to check that $V^0_{\phi,s}$ is given by the conditions

$$\text{(i)}\ \xi_1 > 0, \quad \text{(ii)}\ |v - c_{si}\xi_1^{\delta_{si}}| \geq \xi_1^{\delta_{si}} \text{ if } c_{si} \text{ is real}, \quad \delta_{si} = \delta_s. \qquad (2.10)$$

(We have to assume here that $\varepsilon < \min \delta_{si}$.) Further,

$$\theta = (\gamma_\phi + 1 + \delta_s)/m_{\phi,s}, \qquad (2.11)$$

where, as is easy to check,

$$m_{\phi,s} = m + \sum_{k<s} \sum_{c_{k,i} \neq 0} \delta_{k,i} + \sum_i \delta_{s,i}. \qquad (2.12)$$

In other words, (2.11) means that $\theta = \theta(\gamma_\phi + e; M_{\phi,s})$ in this case. If instead $\nu_2/\nu_1 = \delta_{s-1}$, then $V^0_{\phi,s}$ is given by

$$\text{(i)}\ \xi_1 > 0, \quad \text{(ii)}\ |v_{\phi,s}| < \varepsilon\xi_1^{\delta_{s-1}}, \qquad (2.13)$$

and $\theta=(\gamma_\phi+1+\delta_{s-1})/m_{\phi,s-1}$, again equal to $\theta(\gamma_\phi+e; M_{\phi,s})$. Finally, if $\nu_2/\nu_1>\delta_{s-1}$, then $V^0_{\phi,s}$ reduces to the half-line $\xi_1>0$, $v=0$. Hence this case does not contribute a relevant term to the asymptotic behavior of $I_{\gamma,A}(t;P)$.

Let now $\phi$, $s$ be given such that $\theta$ satisfies (2.11), and denote by $\phi_i$ the zeros of $P(\xi)$ for which

$$v_{\phi_i,s+1}(\xi_1)=v_{\phi,s}(\xi_1)-c_{si}\xi_1^{\delta_{si}},\quad c_{si}\text{ real},\quad \delta_{si}=\delta_s. \tag{2.14}$$

Then $\theta(\gamma_{\phi_i}+e; M_{\phi_i,s+1})=\theta(\gamma_\phi+e; M_{\phi,s})$, and while $V_{\phi,s}$ is given by (2.7), $V_{\phi_i,s+1}$ is given by the conditions

$$\text{(i) } \xi_1\geqslant A>0,\quad \text{(ii) } |v_{\phi,s}-c_{si}\xi_1^{\delta_{si}}|<\varepsilon\xi_1^{\delta_{si}},\quad \text{(iii) } \ldots$$

so that $V^0_{\phi_i,s+1}$ has to be the set

$$\text{(i) } \xi_1>0,\quad \text{(ii) } |v_{\phi,s}-c_{si}\xi_1^{\delta_{si}}|<\varepsilon\xi_1^{\delta_{si}}.$$

(Cf. (2.13).) In other words, $V^0_{\phi,s}$ and all the sets $V^0_{\phi_i,s+1}$ together cover the entire set $\{\xi\in R^2;\ \xi_1>0\}$, without overlapping. We are therefore led to introduce the new set

$$W_{\phi,s}:\quad \xi_1>A;\quad |v_{\phi,s}|<\varepsilon\xi_1^{\delta_{s-1}};\quad |v_{\phi,s}-c_{si}\xi_1^{\delta_{si}}|\geqslant\varepsilon\xi_1^{\delta_{si}}\text{ for all } i \text{ with } c_{si}\text{ real},\quad \delta_{si}\neq\delta_s, \tag{2.15}$$

which contains $V_{\phi,s}$ and all the $V_{\phi_i,s+1}$ defined by (2.14). Recalling (2.9), it is then easy to see that, as $t\to+0$,

$$t^\theta\int_{W_{\phi,s}}\xi^\gamma\exp\{-tP(\xi)\}\,d\xi\to\int_{\xi_1>0}\xi_1^{\gamma_\phi}\exp\{-P_{\phi,s}(\xi_1,v)\}\,d\xi_1\,dv, \tag{2.16}$$

where
$$P_{\phi,s}(\xi_1,v)=\lim_{\lambda\to0}\lambda^{m_{\phi,s}}P\Big(\lambda^{-1}\xi_1,\lambda^{-\delta_s}v+\sum_0^{s-1}c_j(\lambda^{-1}\xi_1)^{\delta_j}\Big). \tag{2.17}$$

Similarly, let
$$m_0=m+\sum_i\delta_{0i} \tag{2.18}$$

and suppose that $\theta=(\gamma_1+\delta_{0j}\gamma_2+1+\delta_{0j})/m_0=\langle\nu^j,\gamma+e\rangle$ for some $j$. Then we may introduce the set

$$W_0:\quad \xi_1>A,\quad |\xi_2-c_{0i}\xi_1^{\delta_{0i}}|\geqslant\varepsilon\xi_1^{\delta_{0i}}\quad\text{for } c_{0i}\text{ real},\quad \delta_{0i}\neq\delta_{0j},$$

and prove that

$$t^\theta\int_{W_0}\xi^\gamma\exp\{-tP(\xi)\}\,d\xi\to\int_{\xi_1>0}\xi^\gamma\exp\{-P^j_F(\xi)\}\,d\xi, \tag{2.19}$$

with $P^j_F$ defined as in section 1.

We have been able to show so far that the leading term of the singularity of the integral

$$I_{\gamma,A}(t;P)=\int_{\xi_1>A}\xi^\gamma\exp\{-tP(\xi)\}\,d\xi$$

may be referred to the behavior of the integrand in one or more "domains of slow growth" for $P$, the sets $W_{\phi,s}$ and $W_0$. The same arguments will work if we study the integral of $\xi^\gamma \exp\{-tP(\xi)\}$ over the set $\{\xi_1 < -A\}$. Then we have to start, of course, not with (2.1), but with a factorization

$$P(\xi) = p_1(\xi_1) \prod_{i=1}^{m_2} (\xi_2 - \psi_i(-\xi_1)).$$

In this way we are able to determine all the contributions to the leading term of $I_\gamma(t;P)$ from domains of slow growth for $P(\xi)$ corresponding to real truncated factors $\xi_2 - \sum_0^{s-1} c_j \xi_j^{\delta_j}$ or $\xi_2 - \sum_0^{s-1} c_j(-\xi_j)^{\delta_j}$ with $\delta_0 > 0$. The contributions due to the remaining domains of slow growth, which are parallell to or converging towards the $\xi_2$-axis, can be determined in the same way, simply by interchanging the roles of $\xi_1$ and $\xi_2$.

We are now ready to collect our results as follows:

**Theorem 2.1.** *Let $P(\xi)$, $\xi \in R^2$, be a real polynomial satisfying* (2.3), *and let $\gamma$ be an even multi-index. Then*

$$I_\gamma(t) = \int \xi^\gamma \exp\{-tP(\xi)\}\, d\xi = t^{-\theta} |\log t|^r (K_\gamma(P) + o(1)), \quad \text{as} \quad t \to +0,$$

*where $\theta$ and $r$, $r = 0$ or 1, can be explicitly computed by the methods of Lemma 2,2. and where*

$$A^{\theta+1}\Gamma(\theta) \leqslant K_\gamma(P) \leqslant A_1^{\theta+1}\Gamma(\theta), \tag{2.20}$$

*for some constants $A$, $A_1 > 0$ depending only on $P$.*

Most of the details of the proof have already been given, at least for the case $r = 0$, and the case $r = 1$ does not offer any additional difficulties. It remains only to recall that $K_\gamma(P)$ has been found to be a sum of integrals determined by limits such as (2.16) and (2.19), from which the estimate (2.20) easily follows.

*Remark.* If $m_0$ and $m_{\phi,s}$ are given by (2.18) and (2.12), respectively, then it follows that

$$m_{\phi,s} = m_0 - \sum_{k=1}^{s} \sum_i (\delta_{k-1} - \delta_{k,i}).$$

This means that $m_{\phi,s}$ is a decreasing function of $s$, for fixed $\phi$. However, $m_{\phi,s}$ is always positive, because it is never smaller than the exponent of the highest power of $\xi_1$ in $M_{\phi,s}(\xi_1, 0)$, and $M_{\phi,s}(\xi_1, 0) \to \infty$ as $\xi_1 \to \infty$. Now, let $\phi$ vary over all truncated factors for $P(\xi)$ of all the four types $\xi_2 - \sum_0^{s-1} c_j(\pm\xi_1)^{\delta_j}$, $\xi_1 - \sum_0^{s-1} c_j(\pm\xi_2)^{\delta_j}$, with $0 \leqslant s \leqslant J(\phi)$. Then

$$\theta = \max_{\phi,s} \theta(\gamma_\phi + e; M_{\phi,s}),$$

with an appropriate definition of $\gamma_\phi$ and $M_{\phi,s}$. But if $\theta(\gamma_\phi + e; M_{\phi,s})$ is given by (2.11), for instance, then, at least for big values of $\gamma$,

$$\max_s \theta(\gamma_\phi + e; M_{\phi,s}) = \theta(\gamma_\phi + e; M_{\phi,J(\phi)}).$$

This means that, for big values of $\gamma$,

$$\theta(\gamma+e;P)=\max_{\phi}\theta(\gamma_\phi+e;M_{\phi,J(\phi)}).$$

Under all circumstances we have the estimate

$$\theta(\gamma+e;P)\leqslant\max_{\phi}(\gamma_\phi+1+\delta_0)/m_{\phi,J(\phi)},$$

which follows from (2.11), because $\delta_0\geqslant\delta_s$ for all $s$. This (non-sharp) estimate could also have been obtained directly from a lower estimate for $P(\xi)$, of the type that was discussed in the paper [4] on principal parts of hypoelliptic polynomials. If we extend the definition of a principal part given in [4] to the case of a real polynomial satisfying (2.1), we get the obvious result that $\theta$ and $r$ depend only on the principal part of $P(\xi)$.

## 3. Examples

Let
$$P(\xi)=|\xi|^{2m}+(\xi_1\dots\xi_n)^{2p},\quad \frac{1}{p}<\frac{n}{m}.$$

Then $F(P)$ has exactly $n$ faces of dimension $n-1$, all passing through the point $(2p, \dots, 2p)$. Using the results of section 1 it is easy to check that, for instance,

$$I_0(t)=\int\exp\{-tP(\xi)\}\,d\xi=\frac{1}{p}\Gamma\left(\frac{1}{2p}\right)\left(\frac{n}{m}-\frac{1}{p}\right)^{n-1}t^{-(1/2p)}|\log t|^{n-1}(1+o(1))$$

as $t\to+0$, which confirms the example given in the introduction.

As a second example, consider the real polynomial $P=|P_1|^2$, where

$$P_1(\xi)=\xi_2^3-\xi_1^4+i\xi_1^2\xi_2.$$

(The polynomial $P_1$, which is hypoelliptic but not multi-quasielliptic, has been studied in other connections by Pini [10] and Friberg [4].) Let us first use the factorization

$$P_1(\xi)=(\xi_2-\xi_1^{4/3}-(i/3)\,\xi_1^{2/3}+\dots)\,(\xi_2-\omega\xi_1^{4/3}+\dots)\,(\xi_2-\omega^2\xi_1^{4/3}+\dots),$$

for $\xi_1>A$, with $\omega^3=1$, $\omega\neq1$. Here the only real truncated factors are $v_0=\xi_2$, and $v_{\phi,1}=\xi_2-\xi_1^{4/3}$, with

$$M_{\phi,1}(\xi_1,v)=|v-(i/3)\,\xi_1^{2/3}|^2\,|v-(\omega-1)\,\xi_1^{4/3}|^4.$$

Hence we find, using the results of section 2, that $\theta(\gamma+e;P)=\langle\gamma_\phi+e,\nu\rangle$, with $\gamma_\phi=(\gamma_1+4/3\,\gamma_2,0)$, and $\nu=(1/8,1/6)$ if $3\gamma_1+4\gamma_2<5$, $\nu=(3/20,1/10)$ if $3\gamma_1+4\gamma_2>5$, i.e. for all large $\gamma$. The degenerate case $r=1$ would appear, with $\theta=1/2$, for $3\gamma_1+4\gamma_2=5$, but there is no solution to this equation because $\gamma_1,\gamma_2$ must be non-negative integers. Therefore $r=0$ for all $\gamma$. Finally, the coefficient $K_\gamma(P)$ is, in the case $3\gamma_1+4\gamma_2>5$ for instance,

$$K_\gamma(P)=\int_{\xi_1>0}\xi_1^{\gamma_1+4/3\gamma_2}\exp\{-3\xi_1^{8/3}((\xi_2-\xi_1^{4/3})^2+(1/9)\,\xi_1^{4/3})\}\,d\xi.$$

In order to check the result we may use instead the factorization, for $\xi_2 > A$,

$$P_1(\xi) = (\xi_1 - i\xi_2^{3/4} + \dots)(\xi_1 + i\xi_1^{3/4} + \dots)$$
$$\times (\xi_1 - \xi_2^{3/4} + (i/4)\,\xi_2^{1/4} + \dots)(\xi_1 + \xi_2^{3/4} - (i/4)\,\xi_2^{1/4} + \dots),$$

with $v_0 = \xi_1$, $v_{\phi,1} = \xi_1 - \xi_2^{3/4}$, $v_{\phi',1} = \xi_1 + \xi_2^{3/4}$, and for instance

$$M_{\phi,1}(\xi_2, v) = |v - (i-1)\,\xi_2^{3/4}|^4 (v + 2\xi_2^{3/4})^2 (v^2 + (1/16)\,\xi_2^{1/2}).$$

For $\xi_2 < -A$, the corresponding factorization shows that $v_0 = \xi_1$ is the only real truncated factor. The values for $\theta$ and $r$ computed by use of the new factorizations are easily seen to be the same as the values we already know. However, the formula for $K_\gamma(P)$ will not be the same, since it is now given by the sum of two integrals over the half-plane $\xi_2 > 0$, instead of by one integral over the half-plane $\xi_1 > 0$.


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