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Convergence and invariance questions for point systems in R_1 under random motion

By TORBJÖRN THEDÉEN

ABSTRACT

In section 2 we introduce and study the independence property for a sequence of two-dimensional random variables and by means of this property we define independent motion in section 3. Section 4 is mainly a survey of known results about the convergence of the spatial distribution of the point system as the time $t \to \infty$. In theorem 5.1 we show that the only distributions which are time-invariant under given reversible motion of non-degenerated type are the weighted Poisson ones. Lastly in section 6 we study a more general type of random motion where the position of a point after translation is a function f of its original position and its motion ability. We consider functions f which are monotone in the starting position. Limiting ourselves to the case when the point system initially is weighted Poisson distributed with independent motion abilities, we prove in theorem 6.1 that this is the case also after the translations, if and only if the function f is linear in the starting position. In the paper also some implications of our results to the theory of road traffic with free overtaking are given.

1. Introduction

In the study of road traffic the simpliest case is when the cars can overtake and meet each other without delay. The following so-called isoveloxic model for traffic (see F. Haight [8] pp. 114–123) has been proposed for this case.[1]

The cars are considered as points on an infinite road with no intersections. They can overtake and meet each other without delay and they will forever maintain their once chosen speeds. The trajectories in the road-time diagram will thus be lines. The initial speeds are independent and identically distributed random variables and they are also independent of the initial positions of the cars.

It has been shown that under rather weak conditions the spatial distribution of the cars will tend to a weighted Poisson distribution (as defined in section 2) as the time $t \to \infty$ (see ref. [1], [2], [4], [10] and [12]). Further if the initial positions are weighted Poisson distributed the spatial distribution and the independence conditions imposed at $t=0$ are conserved for all $t>0$ (see corollary 6.1). If now the model should be time-invariant, i.e. the spatial distribution and the independence conditions imposed at $t=0$ are conserved for all $t>0$ it will be shown that the spatial distribution must be a weighted Poisson one. It has also been possible to somewhat relax the constant speed assumption.

[1] The model description is taken from T. Thedéen [11].

After some introductory studies in sections 2–3, section 4, gives a short survey of the convergence of the point system and its set of motion as $t \to \infty$. In section 5 we shall give some results about the time-invariance of the spatial distribution of a point system under what we will call reversible independent motion. These results contain as a special case (corollary 5.1) the above mentioned one about time-invariant distributions in the constant speed case. Lastly in section 6 we shall study another type of independent motion conserving weighted Poisson distributions for the point system.

The investigations will be restricted to point systems in R_1 because of the traffic-theoretical background of the problems. However it is clear that with some modifications corresponding results can be proved for point systems in R_k.

Lastly it should be remarked that the results of sections 5 and 6 in the special case of constant speeds were presented by the author at The Third International Symposium on the Theory of Traffic Flow in New York 1965 (see T. Thedéen [11]).

2. Preliminaries

Let $\{Z_n\}$ be a sequence of random variables (r.v.'s). For any Borel set B let

$$N(B) = \text{number (no.) of } Z_n \in B.$$

We shall say that $\{Z_n\}$ has *no finite limit point* if for any finite interval I the r.v. $N(I)$ is proper, i.e.

$$P(N(I) < \infty) = 1.$$

Let us assume that this is the case. Then we can associate to $\{Z_n\}$ a *counting process* $N(x)$[1] defined by

$$N(x) = \begin{cases} \text{no. of } Z_n \in (0, x],\ x > 0 \\ 0,\ x = 0 \\ -\text{no. of } Z_n \in (x, 0],\ x < 0. \end{cases}$$

Then almost surely (a.s.) the sample functions of such a counting process are non-decreasing integer-valued stepfunctions with integer-valued jumps. Let now $\{Z_n\}$ be the sequence of positions for points in R_1. We shall say that one or more points form a *cluster* if they have the same positions. This in turn corresponds to a jump of the counting process $N(x)$. The *size* of the cluster is equal to the size of the corresponding jump of $N(x)$. To the sequence $\{Z_n\}$ then corresponds a sequence of clusters characterized by their positions and sizes. We can a.s. order the clusters after their positions thus getting the ordered cluster positions

$$\ldots < Z^{(-2)} < Z^{(-1)} \leq 0 < Z^{(1)} < Z^{(2)} < \ldots$$

The size of a cluster with position $Z^{(k)}$ will be called N_k. Thus to a sequence $\{Z_n\}$ with no finite limit point corresponds a counting process $N(x)$ and a sequence $\{(Z^{(k)}, N_k)\}$. The distribution of $N(x)$ and $\{(Z^{(k)}, N_k)\}$ is given by the distribution of $(N(I_1), \ldots, N(I_k))$ for any finite set of disjoint finite intervals $I_1, \ldots, I_k$ (open,

[1] To simplify the notation we shall use $N(\cdot)$ in two senses where the actual meaning will be clear from the argument used. Notice e.g. the difference between $N(x)$ and $N(\{x\})$.

semi-closed or closed). (This can be shown by the same method as in Doob [5] p. 403.) This distribution is in turn given by the generating function (g.f.).

$$\varphi(s_1, \ldots, s_k; I_1, \ldots, I_k) = E \prod_{j=1}^{k} s_j^{N(I_j)}$$

Now the indexing of a sequence $\{Z_n\}$ may depend on the sizes of the r.v.'s Z_n. Thus two sequences $\{Z_n\}$ and $\{Z'_n\}$ can have different distributions but their corresponding counting processes can nevertheless have the same distribution. We shall somewhat inadequately characterize the distribution of $\{Z_n\}$ by that of $N(x)$ (or $\{(Z^{(k)}, N_k)\}$). Let $I_1, \ldots, I_k$ be any disjoint finite intervals. The following distributions given by their g.f.'s will be of interest in the following.

The distribution corresponding to a Poisson process

$$\varphi(s_1, \ldots, s_k; I_1, \ldots, I_k) = \prod_{j=1}^{k} \exp\{\lambda |I_j| (s_j - 1)\}$$

where $|I_j|$ is the length of I_j and λ a positive constant. We shall then say that $\{Z_n\}$ *is Poisson distributed* (with the parameter λ).

The distribution corresponding to a weighted Poisson process

$$\varphi(s_1, \ldots, s_k; I_1, \ldots, I_k) = \int_0^\infty \prod_{j=1}^{k} \exp\{\lambda |I_j| (s_j - 1\} dW(\lambda)$$

where $W(\lambda)$ is a distribution function (d.f.) on $(0, \infty)$. We shall then say that $\{Z_n\}$ *is weighted Poisson distributed* (with the parameter d.f. $W(\lambda)$).

If $\{Z_n\}$ has any of these distributions then a.s. all the clusters have the size one. In the following case clusters of larger sizes are possible.

The distribution corresponding to a weighted compound Poisson process

$$\varphi(s_1, \ldots, s_k; I_1, \ldots, I_k) = \int_0^\infty \prod_{j=1}^{k} \exp\{\lambda |I_j| (\alpha(s_j) - 1) dW(\lambda)$$

where $\alpha(s)$ is the g.f. of a positive integer-valued r.v. and $W(\lambda)$ a d.f. on $(0, \infty)$. We shall then say that $\{Z_n\}$ *is weighted compound Poisson distributed.*

Let us now consider a sequence $\{(Z_n, V_n)\}$ where $\{Z_n\}$ has no finite limit point and $\{V_n\}$ is a sequence of r.v.'s. By ordering the clusters of $\{Z_n\}$ by position and the r.v.'s V_n in the clusters by size we get the sequence $\{(Z^{(k)}, N_k; V_1^{(k)}, \ldots, V_{N_k}^{(k)})\}$ where $V_1^{(k)} \leqslant \ldots \leqslant V_{N_k}^{(k)}$.

Definition 2.1. *Let $\{(Z_n, V_n)\}$ and $\{(Z'_n, V'_n)\}$ be such that $\{Z_n\}$ and $\{Z'_n\}$ have no finite limit points. $\{(Z_n, V_n)\}$ and $\{(Z'_n, V'_n)\}$ are said to have the same distribution but for indexing if the associated sequences $\{(Z^{(k)}, N_k; V_1^{(k)}, \ldots, V_{N_k}^{(k)})\}$ and $\{(Z'^{(k)}, N'_k; V_1'^{(k)}, \ldots, V'_{N'_k})\}$ have the same distributions.*

In the following sections we shall often consider sequences of the type $\{(Z_n, V_n)\}$. We shall see that two such sequences having the same distribution but for indexing can replace each other in the problems to be considered without changing the results. With this remark in mind we shall now introduce the independence property of $\{(Z_n, V_n)\}$.

Definition 2.2. *Let $\{(Z_n, V'_n)\}$ be a sequence of r.v.'s, where $\{Z_n\}$ has no finite limit point, such that*

(i) *$\{Z_n\}$ and $\{V'_n\}$ are independent and*

(ii) *$\{V'_n\}$ is a sequence of independent identically distributed* (i.i.d.) r.v.'s *with the common d.f. $F(v)$.*

The sequence $\{(Z_n, V_n)\}$ has the independence property with the d.f. $F(v)$ if $\{(Z_n, V_n)\}$ and $\{(Z_n, V'_n)\}$ has the same distribution but for indexing.

For any Borel set B in R_2 let

$$M(B) = \text{no. of } (Z_n, V_n) \in B$$

and denote the g.f. of $(M(B_1), \ldots, M(B_k))$ by $\psi(s_1, \ldots, s_k; B_1, \ldots, B_k)$ where $B_1, \ldots, B_k$ are Borel sets in R_2. The following lemma gives an equivalent characterization of the independence property. Let for any d.f. $F(x)$

$$F(A) = \int_A dF(x),$$

A Borel set.

Lemma 2.1. *$\{(Z_n, V_n)\}$ where $\{Z_n\}$ has no finite limit point has the independence property with the d.f. $F(v)$ if and only if for any disjoint finite intervals $I_1, \ldots, I_k$ and for any disjoint Borel sets $B_{j1}, \ldots, B_{jn_j}$ with $\bigcup_{\nu=1}^{n_j} B_{j\nu} = R_1$, $j = 1, \ldots, k$ the g.f.*

$$\psi(s_{11}, \ldots, s_{1n_1}, \ldots, s_{k1}, \ldots, s_{kn_k}; I_1 \times B_{11}, \ldots, I_1 \times B_{1n_1}, \ldots, I_k \times B_{k1}, \ldots, I_k \times B_{kn_k}) =$$

$$= \varphi(p_{11}s_{11} + \ldots + p_{1n_1}s_{1n_1}, \ldots, p_{k1}s_{k1} + \ldots + p_{kn_k}s_{kn_k}; I_1, \ldots, I_k) \tag{2.1}$$

where

$$p_{j\nu} = F(B_{j\nu}), \nu = 1, \ldots, n_j, j = 1, \ldots, k$$

Proof. Necessity. The sequence $\{(Z_n, V'_n)\}$ of definition 2.2 determines g.f.'s ψ and φ which fulfil (2.1). Further $\{(Z_n, V_n)\}$ and $\{(Z_n, V'_n)\}$ have the same distribution but for indexing and thus determine equal g.f.'s ψ and φ. Then the g.f.'s given by $\{(Z_n, V_n)\}$ also fulfil (2.1).

Sufficiency. In the proof we shall use an idea from Doob [5] p. 403. Now $\{Z_n\}$ has no finite limit point. Then to the sequence $\{(Z_n, V_n)\}$ corresponds another sequence $\{(Z^{(j)}, N_j; V_1^{(j)}, \ldots, V_{N_j}^{(j)})\}$. We have to prove that this last sequence has the same distribution as the sequence $\{(Z^{(j)}, N_j; V_1'^{(j)}, \ldots, V_{N_j}'^{(j)})\}$ of definition 2.2. This is the case if and only if any finite set of r.v.'s from the sequence $\{(Z^{(j)}, N_j; V_1^{(j)}, \ldots, V_{N_j}^{(j)})\}$ has the same distribution as the corresponding set from the sequence $\{(Z^{(j)}, N_j; V_1'^{(j)}, \ldots, V_{N_j}'^{(j)})\}$. It is easily seen that it is no restriction to choose the r.v.'s from $\{(Z^{(j)}, N_j; V_1^{(j)}, \ldots, V_{N_j}^{(j)})\}$ with consecutive indexes (j). In order to avoid notational complications we shall here consider only positive indexes and choose the indexes 1, ..., k. (The case with some negative indexes can be treated in the same way.) Let for any Borel set B in R_1

$$\chi_j(B) = \text{no. of } V_\nu^{(j)} \in B$$

$$\chi_j'(B) = \text{no. of } V_\nu'^{(j)} \in B$$

The distribution of $(Z^{(j)}, N_j;\ V_1^{(j)}, \dots, V_{N_j}^{(j)}, j=1, \dots, k)$ is determined by that of $(Z^{(j)}, N_j; \chi_j(B_{j1}), \dots, \chi_j(B_{jn_j}),\ j=1, \dots, k)$ for all disjoint Borel sets B_{jn_j}, $j=1, \dots, k$, $\bigcup_{\nu=1}^{n_j} B_{j\nu} = R_1$. Thus we have to prove that $(Z^{(j)}, N_j; \chi_j(B_{j1}), \dots, \chi_j(B_{jn_j}), j=1, \dots, k)$ has the same distribution as $(Z^{(j)}, N_j;\ \chi_j'(B_{j1}), \dots, \chi_j'(B_{jn_j}),\ j=1, \dots, k)$. Let now $0 = z_0 < z_1 < \dots < z_k$ and let $\alpha_{j\nu}$, $\nu = 1, \dots, n_j$, $j=1, \dots, k$ be non-negative integers with $\sum_{\nu=1}^{n_j} \alpha_{j\nu} = n_j$, $j=1, \dots, k$. Put

$$p(\alpha) = \prod_{j=1}^{k} \frac{n_j!}{\alpha_{j1}! \dots \alpha_{jn_j}!} p_{j1}^{\alpha_{j1}} \dots p_{jn_j}^{\alpha_{jn_j}}$$

It is easily seen from the definition 2.2 that $P(Z^{(j)} \leqslant z_j,\ N_j = n_j,\ \chi_j'(B_{j\nu}) = \alpha_{j\nu},\ \nu = 1, \dots, n_j,\ j=1, \dots, k) = P(Z^{(j)} \leqslant z_j,\ N_j = n_j,\ j=1, \dots, k) \cdot p(\alpha)$.

The sufficiency of (2.1) is proved if

$$P = P(Z^{(j)} \leqslant z_j, N_j = n_j, \chi_j(B_{j\nu}), \nu = 1, \dots, n_j, j=1, \dots, k)$$
$$= P(Z^{(j)} \leqslant z_j, N_j = n_j, j=1, \dots, k).p(\alpha) \qquad (2.2)$$

Let us for $j=1, \dots, k$ divide the interval (z_{j-1}, z_j) into n intervals of equal length $I_\nu = (a_\nu, b_\nu)$, $\nu = n(j-1)+1, \dots, nj$, where the intervals are numbered from left to right. Note that

$$\max_{1 \leqslant \nu \leqslant kn} |I_\nu| < z_k/n \qquad (2.3)$$

Put further

$$A = \{\chi_j(B_{j\nu}) = \alpha_{j\nu},\ \nu = 1, \dots, n_j,\ j=1, \dots, k\}$$

Then approximating P by the probability in the case when no $Z^{(j)}$'s fall in the same I_ν we get

$$\Big|P - \sum_{\substack{\nu_1 < \dots < \nu_k \\ \nu_j \leqslant jn, j=1, \dots, k}} P(Z^{(j)} \in I_{\nu_j},\ N_j = n_j,\ j=1, \dots, k,\ Z^{(k+1)} \in {}'I_{\nu_k};\ A)\Big|$$
$$\leqslant \sum_{j=1}^{k} \sum_{\nu=1}^{kn} P(Z^{(j)} \in I_\nu,\ Z^{(j+1)} \in I_\nu)$$

From (2.3) and the fact that $0 < Z^{(1)} < \dots < Z^{(k+1)}$ we get

$$\sum_{j=1}^{k} \sum_{\nu=1}^{kn} P(Z^{(j)} \in I_\nu, Z^{(j+1)} \in I_\nu) \leqslant \sum_{j=1}^{k} P(|Z^{(k+1)} - Z^{(j)}| < z_k/n) \to 0,\ n \to \infty \qquad (2.4)$$

Thus

$$P = \lim_{n \to \infty} \sum_{\substack{\nu_1 < \dots < \nu_k \\ \nu_j \leqslant jn, j=1, \dots, k}} P(Z^{(j)} \in I_{\nu_j},\ N_j = n_j,\ j=1, \dots, k;\ Z^{(k+1)} \in {}'I_{\nu_k};\ A) \qquad (2.5)$$

and in the same way we find that

$$P(Z^{(j)} \leqslant z_j, N_j = n_j, j = 1, \ldots, k)$$
$$= \lim_{n\to\infty} \sum_{\substack{\nu_1 < \ldots < \nu_k \\ \nu_j \leqslant jn,\, j=1,\ldots,k}} P(Z^{(j)} \in I_{\nu_j}, N_j = n_j, j = 1, \ldots, k;\ Z^{(k+1)} \in{}' I_{\nu_k} \tag{2.6}$$

We shall now estimate the summands of (2.5) and (2.6). Put $b_{\nu_0} = 0$. Let us approximate

$$P(Z^{(j)} \in I_{\nu_j}, N_j = n_j, j = 1, \ldots, k;\ Z^{(k+1)} \in{}' I_{\nu_k};\ A) \quad \text{by}$$

$$P(N((b_{\nu_{j-1}}, a_{\nu_j}]) = 0,\ M(I_{\nu_j} \times B_{j\mu}) = \alpha_{j\mu},\ \mu = 1, \ldots, n_j,\ j = 1, \ldots, k) \tag{2.7}$$

From (2.1) we get

$$P(N((b_{\nu_{j-1}}, a_{\nu_j}]) = 0,\ M(I_{\nu_j} \times B_{j\mu}) = \alpha_{j\mu},\ \mu = 1, \ldots, n_j,\ j = 1, \ldots, k) =$$
$$P(N((b_{\nu_{j-1}}, a_{\nu_j}]) = 0,\ N(I_{\nu_j}) = n_j,\ j = 1, \ldots, k).p(\alpha) \tag{2.8}$$

Combining (2.7), (2.8) with (2.5) we get

$$P = p(\alpha) \lim_{n\to\infty} \sum_{\substack{\nu_1 < \ldots < \nu_k \\ \nu_j \leqslant jn,\, j=1,\ldots,k}} P(N((b_{\nu_{j-1}}, a_{\nu_j}]) = 0,\ N(I_{\nu_j}) = n_j,\ j = 1, \ldots, k) \tag{2.9}$$

in the same way as we got (2.4).
Let us further approximate

$$P(Z^{(j)} \in I_{\nu_j}, N_j = n_j, j = 1, \ldots, k;\ Z^{(k+1)} \in{}' I_{\nu_k}) \quad \text{by}$$

$$P(N((b_{\nu_{j-1}}, a_{\nu_j}]) = 0,\ N(I_{\nu_j}) = n_j,\ j = 1, \ldots, k) \tag{2.10}$$

Using (2.4) we get from (2.10) and (2.6) that

$$P(Z^{(j)} \leqslant z_j, N_j = n_j,\ j = 1, \ldots, k)$$
$$= \lim_{n\to\infty} \sum_{\substack{\nu_1 < \ldots < \nu_k \\ \nu_j \leqslant jn,\, j=1,\ldots,k}} P(N((b_{\nu_{j-1}}, a_{\nu_j}]) = 0,\ N(I_{\nu_j}) = n_j,\ j = 1, \ldots, k) \tag{2.11}$$

(2.9) and (2.11) proves (2.2) and thus the sufficiency of (2.1) is shown.
A weaker type of independence property is given by the following.

Definition 2.3. *A sequence* $\{(Z_n, V_n)\}$ *where* $\{Z_n\}$ *has no finite limit point has the weak independence property with the d.f.* $F(v)$ *if for any Borel sets* $B_1, \ldots, B_k$ *and any disjoint finite intervals* $I_1, \ldots, I_k$

$$\psi(s_{11}, s_{12}, \ldots, s_{k1}, s_{k2};\ I_1 \times B_1,\ I_1 \times \bar{B}_1, \ldots, I_k \times B_k,\ I_k \times \bar{B}_k)$$
$$= \varphi(p_1 s_{11} + q_1 s_{12}, \ldots, p_k s_{k1} + q_k s_{k2};\ I_1, \ldots, I_k)$$

where $p_j = F(B_j)$, $q_j = 1 - p_j$, $j = 1, \ldots, k$.

The relation between the independence property and the weak independence property is given by the following

Lemma 2.2. *Let $\{(Z_n, V_n)\}$ be a sequence with $\{Z_n\}$ having no finite limit point. Then*

(i) *if $\{(Z_n, V_n)\}$ has the independence property with the d.f. $F(v)$ it also has the weak independence property with the d.f. $F(v)$*

(ii) *if no two of the Z_n's are equal with positive probability the weak independence property with the d.f. $F(v)$ implies the independence property with the d.f. $F(v)$.*

Proof. (i) follows at once from lemma 2.1. Let now $\{(Z_n, V_n)\}$ have the weak independence property with the d.f. $F(v)$ and suppose that no two of the Z_n's are equal with positive probability. Then

$$P(N_k=1, k=\pm 1, \pm 2, \dots)=1$$

and using the same technique as in the sufficiency part of lemma 2.1, (ii) can be proved.

Denote by $\varphi(s_1, \dots, s_k; B_1, \dots, B_k)$ the g.f. of $N(B_1), \dots, N(B_k)$, where $B_1, \dots, B_k$ are Borel sets in R_1.

The case when $\{Z_n\}$ is weighted Poisson distributed and $\{(Z_n, V_n)\}$ has the independence property will be of particular interest in the sequel. Lemma 2.4 will give a characterization of this case. In the proof of that lemma we shall need the following

Lemma 2.3. *Let $\{Z_n\}$ be weighted Poisson distributed with the parameter d.f. $W(\lambda)$. Let further $B_1, \dots, B_k$ be disjoint Borel sets in R_1 with finite Lebesgue measures $\mu(B_1), \dots, \mu(B_k)$. Then*

$$\varphi(s_1, \dots, s_k; B_1, \dots, B_k)=\int_0^\infty \exp\left\{\lambda \sum_{j=1}^{k} \mu(B_j)\,(s_j-1)\right\} dW(\lambda)$$

Remark. This simple lemma may be found in the literature but since the same technique will be used in the proof of lemma 2.4 the proof will be given below.

Proof. Let us first consider the case $k=1$. Let B be a Borel set with $\mu(B)<\infty$. We shall prove that $N(B)$ has the g.f.

$$\varphi(s; B)=\varphi_P(s; B)=\int_0^\infty \exp\{\lambda\mu(B)\,(s-1)\}\,dW(\lambda). \tag{2.12}$$

(2.12) is easily seen to hold for $B=\bigcup_{j=1}^{\infty} I_j$ where I_j, $j=1, 2 \dots$ are disjoint intervals with $\sum_{j=1}^{\infty}\mu(I_j)<\infty$. Let $\mathcal{J}$ be the class of all intervals (open, semiclosed and closed). We know that

$$\mu(B)=\inf\left\{\sum_{j=1}^{\infty}\mu(I_j);\ B\subset\bigcup_{j=1}^{\infty} I_j,\ I_j\cap I_k=\phi,\ j\neq k,\ I_j\in\mathcal{J},\quad j=1,2,\dots\right\}.$$

Thus given any $\delta>0$ there is a sequence of disjoint intervals $\{I_j\}$ such that

$$B\subset\bigcup_{j=1}^{\infty} I_j,\ B_\delta=\bigcup_{j=1}^{\infty} I_j-B \quad \text{where} \quad \mu(B_\delta)<\delta.$$

In the same way we see that there is a sequence of disjoint intervals $\{I'_j\}$ such that

$$B_\delta \subset \bigcup_{j=1}^{\infty} I'_j;\ \mu(B_\delta) \leqslant \mu\left(\bigcup_{j=1}^{\infty} I'_j\right) < 2\delta.$$

Note that

$$H(x) = \int_0^\infty (1 - e^{-\lambda x}) dW(\lambda)$$

is a continuous d.f. with $H(0) = 0$, since $W(0) = 0$. Thus given $\varepsilon > 0$ there is a $\delta > 0$ such that $H(x) < \varepsilon$, $0 \leqslant x < 2\delta$. Choose δ in accordance to this requirement. Now

$$|\varphi(s; B) - \varphi_P(s; B)| \leqslant \left|\varphi\left(s; \bigcup_{j=1}^{\infty} I_j\right) - \varphi_P(s; B)\right| + \left|\varphi(s; B) - \varphi\left(s; \bigcup_{j=1}^{\infty} I_j\right)\right|$$

But for $0 \leqslant s \leqslant 1$

$$\left|\varphi\left(s; \bigcup_{j=1}^{\infty} I_j\right) - \varphi_P(s; B)\right|$$

$$= \left|\int_0^\infty \exp\left\{\lambda \sum_{j=1}^{\infty} \mu(I_j)\,(s-1)\right\} dW(\lambda) - \int_0^\infty \exp\{\lambda\mu(B)\,(s-1)\}\, dW(\lambda)\right|$$

$$\leqslant \int_0^\infty |1 - \exp\{-\lambda\mu(B_\delta)\}|\, dW(\lambda) = H(\mu(B_\delta)) < \varepsilon$$

Further for $0 \leqslant s \leqslant 1$

$$\left|\varphi(s; B) - \varphi\left(s; \bigcup_{j=1}^{\infty} I_j\right)\right| = |Es^{N(B)} - Es^{N(B)+N(B_\delta)}|$$

$$\leqslant E|1 - s^{N(B_\delta)}| \leqslant P\left(N\left(\bigcup_{j=1}^{\infty} I'_j\right) > 0\right) < H(2\delta) < \varepsilon$$

Thus $|\varphi(s; B) - \varphi_P(s; B)| < 2\varepsilon$, $0 \leqslant s \leqslant 1$ which proves the case $k = 1$. In the general case we use the same approximation procedure for each of the Borel sets $B_1, \ldots, B_k$.

Lemma 2.4. *The sequence $\{(Z_n, V_n)\}$ has the independence property with the d.f. $F(v)$ and $\{Z_n\}$ is weighted Poisson distributed with the parameter d.f. $W(\lambda)$ if and only if for any disjoint Borel sets $B_1, \ldots, B_k$ in R_2 such that*

$$\varkappa(B_i) < \infty \text{ where } \varkappa(B_i) = \int_{B_i} dx\, dF(v),\ i = 1, \ldots, k$$

the r.v. $(M(B_1), \ldots, M(B_k))$ has the g.f.

$$\psi(s_1, \ldots, s_k;\ B_1, \ldots, B_k) = \int_0^\infty \exp\left\{\lambda \sum_{i=1}^{k} \varkappa(B_i)\,(s_i - 1)\right\} dW(\lambda). \tag{2.13}$$

Remark. In the sufficiency part we need only (2.13) to hold for the B_j's being products of intervals.

Proof. Sufficiency. Put $B_j = I_j \times R_1$ where the I_j's are disjoint finite intervals. Then it follows at once from (2.13) that $\{Z_n\}$ is weighted Poisson distributed with the parameter d.f. $W(\lambda)$. Taking the B_i's as products of intervals it follows from definition 2.3 that $\{(Z_n, V_n)\}$ has the weak independence property with the d.f. $F(v)$. Since $\{Z_n\}$ is weighted Poisson distributed no two of the Z_n's are equal with positive probability. The independence property then follows from lemma 2.2 (ii).

Necessity.

1. From lemma 2.3 it follows that (2.13) holds for $B_i = A_i \times R_1$, $i = 1, ..., k$, where the A_i's are disjoint Borel sets in R_1 with $\mu(A_i) < \infty$, $i = 1, ..., k$.
2. The independence property implies (2.13) to hold for disjoint products of Borel sets of finite $\varkappa$-measure.
3. (2.13) is easily seen to hold also for disjoint sets of finite $\varkappa$-measure in the algebra generated by all finite unions of measurable rectangles.
4. Let B be a Borel set in R_2 with finite $\varkappa$-measure. In a similar way as that used in the proof of lemma 2.3 we can approximate this set in $\varkappa$-measure by a union of disjoint products of measurable rectangles and prove (2.13) for $k = 1$.
5. The proof of (2.13) for any k is done in the same way.

3. The point system and its set of motion

We shall consider a countable number of points distributed on R_1 and performing a random motion in time. The positions of the points at $t = 0$ are given by the sequence of r.v.'s $\{X_n\}$ the points being arbitrarily enumerated. In the following we shall always assume that $\{X_n\}$ has no finite limit point. If the position of point n at t ($t > 0$) is denoted by $X_n(t)$ the positions at t ($t > 0$) are given by the sequence of r.v.'s $\{X_n(t)\}$. Using the notation

$$Y_n(t) = X_n(t) - X_n$$

we shall call $\{Y_n(t)\}$, following J. Goldman [7], *the set of motion* for the point system. The special case when for all $t > 0$.

$$Y_n(t) = U_n \cdot t$$

will be called *the constant speed case* (in this case the trajectories will be straight lines). We shall here deal with the case when the points do not interact with each other in their motions and we will thus introduce the following definitions.

Definition 3.1. *$\{X_n(t)\}$ has (or $\{Y_n(t)\}$ is) an independent set of motion at t with the d.f. $F_t(y)$ if $\{(X_n, Y_n(t))\}$ has the independence property with the d.f. $F_t(y)$.*

Definition 3.2. *$\{X_n(t)\}$ has (or $\{Y_n(t)\}$ is) an independent set of motion with the family $\{F_t(y)\}$ if it has an independent set of motion at t with the d.f. $F_t(y)$ for all $t > 0$.*

In the constant speed case we replace the family $\{F_t(y)\}$ in this definition by the d.f. of the speed $G(u)$.

4. The asymptotic distribution of the point system

We shall here study the asymptotic distribution of $\{X_n(t)\}$ when $\{Y_n(t)\}$ is an independent set of motion for all $t > 0$. Let $\{X_n^0\}$ be a sequence of r.v.'s with no finite limit point. Similarly to section 2 we introduce the following r.v.'s and associated g.f.'s

$$N(I) = \text{no. of } X_n \in I, \text{ g.f. } \varphi$$

$$N_t(I) = \text{no. of } X_n(t) \in I, \text{ g.f. } \varphi_t, \ N^0(I) = \text{no. of } X_n^0 \in I,$$

where I is a finite interval.

Definition 4.1. *$\{X_n(t)\}$ is said to converge in distribution to $\{X_n^0\}$ if for any disjoint finite intervals $I_1, ..., I_k$ the distribution of $(N_t(I_1), ..., N_t(I_k))$ converges to that of $(N^0(I_1), ..., N^0(I_k))$, as $t \to \infty$.*

Such a convergence will take place if and only if we have convergence of the corresponding g.f.'s. The convergence problem was first studied by R. Dobrushin [4] and G. Maruyama [10]. Later similar results were obtained by L. Breiman [1] and [2] and T. Thedéen [12]. These results were summarized and completed by J. Goldman [7]. We shall in this section somewhat generalize the results of Goldman. Our treatment will also serve as a motivation and introduction to the invariance problems dealt with in section 5.

Let us assume that $\{X_n(t)\}$ has an independent set of motion with the family $\{F_t(y)\}$. It follows from definition 2.2 that when we consider the distribution of $N_t(I_j)$, $j = 1, ..., k$, we will get the same distribution if we replace the last assumption by the following assumptions for $\{(X_n, Y_n(t))\}$

(i) $\{X_n\}$ and $\{Y_n(t)\}$ are independent
(ii) $\{Y_n(t)\}$ is a sequence of i.i.d. r.v.'s with $P(Y_n(t) \leq y) = F_t(y)$.

With these assumptions it is easily seen that

$$\varphi_t(s_1, \dots, s_k; I_1, \dots, I_k) = E \prod_n \left\{ \sum_{j=1}^k s_j F_t(I_j - X_n) + 1 - \sum_{j=1}^k F_t(I_j - X_n) \right\}. \tag{4.1}$$

It should be noted that this g.f. does not necessarily have the value one for

$$s_1 = \dots = s_k = 1.$$

In order to get any general results about the convergence of φ_t it seems natural to study the g.f.

$$\alpha_t = \prod_n \left\{ \sum_{j=1}^k s_j F_t(I_j - x_n) + 1 - \sum_{j=1}^k F_j(I_j - x_n) \right\}$$

where $\{x_n\}$ is an infinite sequence of real numbers. This is the g.f. of a sum of independent r.v.'s. The equivalent in this case to the so called 'uan'-condition (see Loève [9] p. 290) is

$$\lim_{t \to \infty} \sup_n F_t(I - x_n) = 0 \tag{4.2}$$

for all finite intervals I. If (4.2) should hold for all sequences $\{x_n\}$ then we will require

$$\lim_{t \to \infty} \sup_{x \in R_1} F_t(I - x) = 0 \tag{4.3}$$

for all finite intervals I. Following J. Goldman [7] we shall then say that we have *a spread out set of motion*.

In the study of the convergence of α_t under (4.3) we shall use a slight generalization of the fundamental lemma by J. Goldman [7] p. 23. First we shall give the following notations and definitions. Let us consider arrays $\{X_{tn}\}$ of k-dimensional r.v.'s, $n=1, 2, \ldots$ and t indexed on the positive integers or positive real numbers where $X_{tn}=(X_{tn}^{(1)}, \ldots, X_{tn}^{(k)})$. The row sums are denoted by

$$Y_t=\sum_{n=1}^{\infty} X_{tn}$$

Definition 4.2. *An array $\{X_{tn}\}$ will be called a null array if*

$$\lim_{t\to\infty} \sup_{j,n} P(X_{tn}^{(j)}>0)=0$$

Definition 4.3. *A Bernoulli sequence is a sequence of independent k-dimensional r.v.'s*

$$X_1=(X_1^{(1)}, \ldots, X_1^{(k)}),\ X_2=(X_2^{(1)}, \ldots, X_2^{(k)}), \ldots$$

assuming only the values

$$(0, 0, \ldots, 0),\ (1, 0, 0, \ldots, 0),\ (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1).$$

Definition 4.4. *A Bernoulli array $\{X_{tn}\}$ is an array such that for any t $\{X_{tn}\}$ is a Bernoulli sequence.*

Definition 4.5. *A r.v. X has a k-dimensional Poisson distribution with parameter $(\lambda_1, \ldots, \lambda_k)$ if*

$$P(X=(n_1, \ldots, n_k))=\prod_{j=1}^{k} \frac{\lambda_j^{n_j}}{n_j!} e^{-\lambda_j}$$

for all non-negative integers n_j.

Lemma 4.1. *Let $\{X_{tn}\}$ be a Bernoulli null array. Then*

(i) *the only possible limit distributions for Y_t are the Poisson ones (including those with some λ_i equal to zero).*

(ii) *the distribution of Y_t converges to a Poisson distribution with parameter $(\lambda_1, \ldots, \lambda_k)$ if and only if*

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} P(X_{tn}^{(j)}=1)=\lambda_j\ j=1, \ldots, k. \tag{4.4}$$

Proof. The sufficiency of (4.4) was proved for $k=1$ by L. Breiman [2] and for any k by T. Thedéen [12]. (ii) was proved by J. Goldman [7] p. 23. Thus we only need to prove (i). Let

$$Y_t^{(j)}=\sum_{n=1}^{\infty} X_{tn}^{(j)}$$

and fix j for a while. $Y_t^{(j)}$ may be an improper r.v. More precisely $Y_t^{(j)}$ being a sum of independent r.v.'s must be a.s. finite or infinite. With the notation

$$P(X_{tn}^{(j)}=1)=p_{tn}^{(j)}$$

the g.f. of $Y_t^{(j)}$ is

$$\beta_t^{(j)}(s)=\prod_{n=1}^{\infty}(1-p_{tn}^{(j)}(1-s)).$$

In order that $\beta_t^{(j)}(s)$ should converge to a g.f. of a proper r.v. it is necessary that $\lim_{t\to\infty}\sum_{n=1}^{\infty}p_{tn}^{(j)}$ exists and is equal to a constant e.g. λ_j. Using the same argument for $j=1, ..., k$ we see that (4.4) (for some $(\lambda_1, ..., \lambda_k)$) is necessary for the complete convergence of the d.f. of Y_t. The sufficiency of (4.4) then proves (i).

Following J. Goldman [7] we now give the following definitions.

In the following two definitions let $\{I_n\}$ be any sequence of finite intervals,

$$I_1\subset I_2\subset ..., \lim_{n\to\infty}|I_n|=\infty.$$

Definition 4.6. *A point system $\{X_n\}$ is well-distributed with a parameter d.f. $W(\lambda)$ if*

$$\lim_{n\to\infty}N(I_n)/|I_n|=\Lambda \text{ a.s.}$$

where Λ is a r.v. with d.f. $W(\lambda)$.

Definition 4.7. *A set of motion $\{Y_n(t)\}$ is well-distributed if for any finite λ and for any set of numbers $\{x_n\}$ such that*

$$\lim_{n\to\infty}(\text{no. of } x_k \varepsilon I_n)/|I_n|=\lambda$$

we have

$$\lim_{t\to\infty}\sum_n F_t(I-x_n)=\lambda|I|$$

for any finite interval I.

J. Goldman proved using his equivalent to lemma 4.1 (ii) the following theorem (this is his theorem 6.2 where we have just somewhat changed the formulation).

Theorem 4.1. *Let $\{Y_n(t)\}$ be a spread out set of motion. Then a necessary and sufficient condition for every initially well-distributed point system under an independent set of motion for all time $u>0$ to be in the limit $t\to\infty$ weighted Poisson distributed is that $\{Y_n(t)\}$ is well-distributed.*

Remark. The case when the weighted Poisson distribution has a parameter d.f. $W(\lambda)$ with $W(0)>0$ is not excluded. Using lemma 4.1 (i) and (ii) we get

Theorem 4.2. *Let $\{Y_n(t)\}$ be a spread out set of motion.*

Then a necessarg condition for every initially well-distributed point system under an independent set of motion for all time $u>0$ to converge in distribution as $t\to\infty$ is that $\{Y_n(t)\}$ is well-distributed.

The *proof* is omitted.

In the following example the set of motion is well-distributed and spread out.

Let $Y_n(t)=U_n\cdot t$ (the constant speed case) and let U_n have an absolutely continuous d.f. $G(u)$ with the density $g(u)$. Assume $g(u)$ bounded, almost everywhere continuous and with compact support. (Cf. L. Breiman [2] and T. Thedéen [12].) Let now $\{X_n\}$ be an initially well-distributed point system with the parameter d.f. $W(\lambda)$ and let $\{Y_n(t)\}$ be the independent set of motion above. Then by theorem 4.1 we have convergence to a weighted Poisson distribution with the parameter d.f. $W(\lambda)$. Further it can be shown that $\{(X_n(t), U_n)\}$ has the independence property with the d.f. $G(u)$ in the limit $t\to\infty$.

More precisely: Let for any Borel set B_i in R_2, $i=1, \ldots, k$, $M(B_i)=$ no. of $(X_n(t), U_n)\in B_i$ with the g.f. $\psi_t^u(s_1, \ldots, s_k; B_1, \ldots, B_k)$.

Definition 4.8. $\{(X_n(t), U_n)\}$ *has asymptotically the independence property with the d.f.* $G(u)$ *and* $\{X_n(t)\}$ *is asymptotically weighted Poisson distributed with the parameter d.f.* $W(\lambda)$ *if for any disjoint finite intervals* $I_1, \ldots, I_k$ *and intervals* $J_1, \ldots, J_k$

$$\lim \psi_t^u(s_{11}, s_{12}, \ldots, s_{k1}, s_{k2}; I_1\times J_1, I_1\times \bar{J}_1, \ldots, I_k\times J_k, I_k\times \bar{J}_k)$$
$$= \int_0^\infty \exp\left\{\lambda \sum_{i=1}^k |I_i|(G(J_i)s_{i1}+G(\bar{J}_i)s_{i2}-1)\right\} dW(\lambda)$$

Remark. Compare with definition 2.3, lemma 2.2 and lemma 2.4 and the remark following that lemma.

By the same method of proof as that used by L. Breiman [2] we get

Theorem 4.3. *Let* $\{X_n\}$ *be a well-distributed point system with the parameter d.f.* $W(\lambda)$ *and let* $\{Y_n(t)=U_nt\}$ *be an independent set of motion for all* $t>0$ *(the constant speed case). The d.f. of* U_n, $G(u)$, *is absolutely continuous with the density* $g(u)$ *being bounded, almost everywhere continuous and with compact support.*

Then $\{(X_n(t), U_n)\}$ *has asymptotically the independence property with the d.f.* $G(u)$ *and* $\{X_n(t)\}$ *is asymptotically weighted Poisson distributed with the parameter d.f.* $W(\lambda)$.

Remark. It should be possible to weaken the conditions on $g(u)$. This we have not done since the theorem is included mainly as a motivation for the condition about reversible independent motion used in section 5.

5. Time-invariant distributions for point systems under reversible independent motion

The important role of weighted Poisson distributions as limit distributions for point systems with independent sets of motion stands out clearly from section 4.

Doob [5] p. 404 showed that if a point system is Poisson distributed at $t=0$ and has an independent set of motion its spatial distribution will be conserved for all $t>0$. The same result for weighted Poisson distributions was shown by J. Goldman [7]. From these results it is rather easily seen that a point system with $EN(I)<\infty$ for any finite interval I has the same distribution for all $t\geq 0$ for *all* independent sets of motion if and only if it is weighted Poisson distributed (see R. L. Dobrushin [4] and J. Goldman [7]). Here we shall try to characterize those distributions for point systems which are time-invariant for a given independent set of motion. Let us first note that in the special case treated in theorem 4.3 the sequence $\{(X_n(t), U_n)\}$ has the independence property in the limit $t \to \infty$. This means that in the limit we have a kind of backwards independent set of motion. In the general case we have the following definition of this concept.

Definition 5.1. *$\{X_n(t)\}$ has (or $\{Y_n(t)\}$ is) a backwards independent set of motion at t with the d.f. $F_t(y)$ if $\{X_n(t)\}$ has no finite limit point and $\{(X_n(t), Y_n(t))\}$ has the independence property with the d.f. $F_t(y)$.*

Definition 5.2. *$\{X_n(t)\}$ has (or $\{Y_n(t)\}$ is) a backwards independent set of motion with the family $\{F_t(y)\}$ if it has a backwards independent set of motion at t with the d.f. $F_t(y)$ for all $t > 0$.*

If in these definitions we have the weak independence property in place of the independence property we shall say that we have a *backwards weak independent set of motion*. We can now introduce the concept of time-invariance under a reversible set of motion.

Definition 5.3. *$\{X_n\}$ has an invariant distribution at t for a d.f. $F_t(y)$ under reversible independent motion if $\{X_n(t)\}$ has an independent set of motion at t and a backwards independent set of motion at t both with the d.f. $F_t(y)$ and $\{X_n\}$ and $\{X_n(t)\}$ have the same distribution.*

Definition 5.4. *$\{X_n\}$ has a time-invariant distribution under reversible independent motion for the family $\{F_t(y)\}$ if the distribution of $\{X_n\}$ is invariant at t under reversible independent motion for the d.f. $F_t(y)$ for all $t > 0$.*

If in definition 5.3 we instead of a backwards independent set of motion have a weak one we shall say that $\{X_n\}$ has a invariant distribution under weak reversible independent motion. We shall in this section show that the distributions of $\{X_n\}$ which are time-invariant under reversible independent motion for a non-degenerated family $\{F_t(y)\}$ (see definition 5.5) are the weighted Poisson ones. We shall need the following two simple lemmas about g.f.'s for random vectors. Let $Y = (Y_1, ..., Y_k)$ and $Z = (Z_1, ..., Z_k)$ be two random vectors the components of which are non-negative and integervalued and let their g.f.'s be $\gamma_1(s_1, ..., s_k)$ and $\gamma_2(s_1, ..., s_k)$ respectively.

Lemma 5.1. *If*

$$\gamma_1(s_1, ..., s_k) = \gamma_2(s_1, ..., s_k)$$

for $s_i \in (a_i, b_i)$ where $-1 \leq a_i < b_i \leq 1$, $i = 1, ..., k$, then Y and Z have the same distribution.

The *proof* follows at once from the generalization of the identity principle for holomorphic functions of a complex variable to the case with several complex variables.

Lemma 5.2. *If $Y_i \leq Z_i$, $i = 1, ..., k$ a.s. then*

$$\gamma_1(s_1, ..., s_k) \geq \gamma_2(s_1, ..., s_k),\ s_i \in [0, 1],\ i = 1, ..., k.$$

Proof. The given inequalities imply that

$$s_1^{Y_1} \dots s_k^{Y_k} \geqslant s_1^{Z_1} \dots s_k^{Z_k} \text{ a.s. for } s_i \in [0, 1],\ i=1, \dots, k$$

from which inequality the lemma follows at once.

Let for any Borel set B in R_2

$$M_{f,t}(B) = \text{no. of } (X_n, Y_n(t)) \in B,\ M_{b,t}(B) = \text{no. of } (X_n(t), Y_n(t)) \in B$$

and denote the corresponding g.f.'s by $\psi_{f,t}$ and $\psi_{b,t}$ respectively (cf. section 2). Let further for any d.f. $F(y)$ and a Borel set B in R_1 with $F(B) > 0$

$$F_B(y) = F(B \cap (-\infty, y])/F(B)$$

be the conditional d.f. given B.

Lemma 5.3. *Let the distribution of $\{X_n\}$ be invariant at t under reversible independent motion for a d.f. $F_t(y)$. Then it is also invariant at t under weak reversible independent motion for the d.f. $F_{t,B}(y)$.*

Remark. Actually the lemma is true also when weak is omitted but this we do not need in the following.

Proof. Put $F_t(B) = p$, $q = 1-p$. Let $I_1, \dots, I_k$ be any disjoint finite intervals and $B_1, \dots, B_k$ any Borel sets in R_1. Put further

$$A_j = B_j \cap B,\ C_j = \bar{B}_j \cap B,\ p_{j1} = F_t(A_j),\ p_{j2} = F_t(C_j),\ j=1, \dots, k$$

Note that for the distributions considered in this section $\{(X_n, Y_n(t))\}$ can be assumed to fulfil the conditions (i) and (ii) just above (4.1). Then

$$\begin{aligned}
&\psi_{b,t}(s_{11}, s_{12}, \dots, s_{k1}, s_{k2}; I_1 \times A_1, I_1 \times C_1, \dots, I_k \times A_k, I_k \times C_k) \\
&= E \prod_n \left\{ \sum_{j=1}^k s_{j1} F_t((I_j - X_n) \cap A_j) + \sum_{j=1}^k s_{j2} F_t((I_j - X_n) \cap C_j) \right. \\
&\qquad \left. + 1 - \sum_{j=1}^k F_t((I_j - X_n) \cap B) \right\} \\
&= \varphi(p_{11}s_{11} + p_{12}s_{12} + q, \dots, p_{k1}s_{k1} + p_{k2}s_{k2} + q; I_1, \dots, I_k) \qquad (5.1)
\end{aligned}$$

where the last equality follows from the assumed invariance under reversible independent motion together with lemma 2.1. Putting

$$s'_{ij} = ps_{ij} + q,\ i=1, \dots, k,\ j=1, 2$$

we get from (5.1)

$$\begin{aligned}
&E \prod_n \left\{ \sum_{j=1}^k s'_{j1} F_{t,B}((I_j - X_n) \cap A_j) + \sum_{j=1}^k s'_{j2} F_{t,B}((I_j - X_n) \cap C_j) + 1 - \sum_{j=1}^k F_{t,B}(I_j - X_n) \right\} \\
&= \varphi(p_{11}s'_{11}/p + p_{12}s'_{12}/p, \dots, p_{k1}s'_{k1}/p + p_{k2}s'_{k2}/p; I_1, \dots, I_k) \quad \text{for} \\
&\quad s'_{ij} \in [q-p, 1],\ i=1, \dots, k,\ j=1, 2. \qquad (5.2)
\end{aligned}$$

Under an independent set of motion at t with the d.f. $F_{t,B}(y)$ we have for any finite interval I and any Borel set A in R_1

$$M_{b,t}(I\times A)=M_{b,t}(I\times(A\cap B))\text{ a.s.}$$

Using the index B for the g.f. of $M_{b,t}$ in this case we get

$$\begin{aligned}\psi_{b,t,B}&(s_{11},s_{12},\dots,s_{k1},s_{k2};I_1\times B_1,I_1\times\bar{B},\dots,I_k\times B_k,I_k\times\bar{B}_k)\\&=\psi_{b,t,B}(s_{11},s_{12},\dots,s_{k1},s_{k2};I_1\times A_1,I_1\times C_1,\dots,I_k\times A_k,I_k\times C_k)\\&=E\prod_n\left\{\sum_{j=1}^k s_{j1}F_{t,B}((I_j-X_n)\cap A_j)+\sum_{j=1}^k s_{j2}F_{t,B}((I_j-X_n)\cap C_i)\right.\\&\qquad\left.+1-\sum_{j=1}^k F_{t,B}(I_j-X_n)\right\}\end{aligned}\tag{5.3}$$

But for $s_{ij}=s'_{ij}$, $i=1,\dots,k$, $j=1,2$, the right member of (5.3) equals the left member of (5.2). Thus

$$\begin{aligned}\psi_{b,t,B}&(s_{11},s_{12},\dots,s_{k1},s_{k2};I_1\times B_1,I_1\times\bar{B}_1,\dots,I_k\times B_k,I_k\times\bar{B}_k)\\&=\varphi(p_{11}s_{11}/p+p_{12}s_{12}/p,\dots,p_{k1}s_{k1}/p+p_{k2}s_{k2}/p;I_1,\dots,I_k)\end{aligned}\tag{5.4}$$

for $s_{ij}\in[q-p,1]$ and by lemma 5.1 for $s_{ij}\in[-1,1]$, $i=1,\dots,k$, $j=1,2$. Now

$$F_{t,B}(B_j)=p_{j1}/p,\quad F_{t,B}(\bar{B}_j)=p_{j2}/p,\quad j=1,\dots,k$$

which put into (5.4) proves the lemma. Note that the distribution of $\{X_n(t)\}$ is the same as that of $\{X_n\}$ under an independent set of motion with the d.f. $F_{t,B}(y)$. This is seen by putting $B_1=\dots=B_k=R_1$ in (5.4). The lemma is proved.

In order to get our results about time-invariant distributions for point systems we shall have to require that arbitrarily small displacements and unequal displacements arbitrarily close to each other are possible. These requirements are made precise in the following definition 5.5. Let the *support set* S_F of a d.f. $F(x)$ be

$$S_F=\{x;\ F(x+h)-F(x-h)>0,\quad\text{all}\quad h>0\}$$

Definition 5.5. *A family of d.f.'s $\{F_t(y),0<t<\infty\}$ is said to be non-degenerated if*

(i) *for any given $\varepsilon>0$ there is a t and a y such that $y\in S_{F_t}$ and $0<|y|<\varepsilon$ and*
(ii) *for any given $\varepsilon>0$ there is a t and $y_1\neq y_2$ such that $y_1,y_2\in S_{F_t}$ and $|y_1-y_2|<\varepsilon$.*

Remark. In the constant speed case when $Y_n(t)=U_n\cdot t$ and $G(u)=P(U_n\leqslant u)$ the non-degeneracy of $\{F_t(y)\}$ is equivalent to the d.f. $G(u)$ being non-degenerated.

In deciding whether (ii) holds or not the following lemma may be useful.

Lemma 5.4. *If at least one of the $F_t(y)$'s is not purely discontinuous the condition* (ii) *of definition* 5.5 *is fulfilled.*

The *proof* follows at once from the following

Lemma 5.5. *Let $F(x)$ be a d.f. which is not purely discontinuous. Then for any given $\varepsilon>0$ there are numbers $x_1\neq x_2$ such that $x_1,x_2\in S_F$ and $|x_1-x_2|<\varepsilon$.*

Proof. Let D be the discontinuity set of $F(x)$ and put $C=\bar{D}\cap S_F$. Let us for a moment assume that there is a $\varepsilon_0>0$ such that for all $x_1 \neq x_2$ with $x_1, x_2 \in C$ we have $|x_1-x_2|>\varepsilon_0$. This implies that C is countable which contradicts $F(x)$ being not purely discontinuous. This proves the lemma.

Lemma 5.6. *Let the distribution of $\{X_n\}$ be time-invariant under reversible independent motion for a non-degenerated family $\{F_t(y)\}$. Then*

(i) *the counting process corresponding to $\{X_n\}$ has a.s. no fixed discontinuity points,*

(ii) *for a monotonely decreasing (or increasing) sequence of intervals $\{I_n\}$ such that $\lim_{n\to\infty} I_n = I$ where I is a finite interval we have*

$$\lim_{n\to\infty} \varphi(s; I_n) = \varphi(s; I) \text{ uniformly for } s\in[-1, 1];$$

Proof. (i). Let us assume that (i) does not hold. Then there is an x_1 such that

$$P(N(\{x_1\})>0) = p>0.$$

By the non-degeneracy assumption given $\varepsilon>0$ there is a t and a y such that $y\in S_{F_t}$ and $0<|y|<\varepsilon$. Put $B=(-\varepsilon, 0)\cup(0, \varepsilon)$. Then $F_t(B)>0$ and by lemma 5.3 it follows that the distribution of $\{X_n\}$ is invariant at t under weak reversible independent motion for the d.f. $F_{t,B}(y)$. This implies that with $A=(x_1-\varepsilon, x_1)\cup(x_1, x_1+\varepsilon)$ and an independent set of motion at t with the d.f. $F_{t,B}(y)$

$$N_t(A)\geqslant N(\{x_1\} \text{ a.s.}$$

But $N(A)$ and $N_t(A)$ have the same distribution. Thus

$$P(N(A)>0)\geqslant P(N(\{x_1\})>0)$$

and with a probability no less than p there is a point arbitrarily near but separate from the position x_1. This contradicts $\{X_n\}$ having no finite limit point and (i) is proved.

(ii) follows at once from (i) and the fact that g.f.'s are bounded and continuous in $[-1, 1]$.

Theorem 5.1.

(i) $\{X_n\}$ *is time-invariant under reversible independent motion for the family* $\{F_t(y)\}$, *if* $\{X_n\}$ *is weighted Poisson distributed.*

(ii) *Let* $\{F_t(y)\}$ *be a non-degenerated family of d.f.'s.*

Then the distribution of $\{X_n\}$ *is time-invariant under reversibe independent motion for the family* $\{F_t(y)\}$ *only if* $\{X_n\}$ *is weighted Poisson distributed.*

Remark. In comparison with the invariance theorems by R. L. Dobrushin [4] and J. Goldman [7], (ii) of our theorem on the one hand demands *reversible* independent motion but on the other hand characterize distributions for point systems which are time-invariant under reversible independent motion for an arbitrary *fixed* non-degenerated family $\{F_t(y)\}$.

Proof. (i). The proof follows from the more general statement of theorem 6.1 if in that theorem we put $V_n = Y_n(t)$, $F_1(y) = F_t(y)$ and $f(x, y) = x + y$.

(ii) In the proof we shall rely on the results by H. Bühlmann [3] 4. Kap. concerning processes on [0, 1] with exchangeable increments. It is however easy to see that his results still hold also for processes on $(-\infty, +\infty)$ in which case his proofs can be done in the same way. He considers processes which are separable with the set of the diadic numbers as a separating set. In our case it is seen from the definition of the counting process $N(x)$ defined by $\{X_n\}$ that $N(x)$ is separable with any countable set which is dense in R_1 as separating set. Let us divide R_1 into disjoint intervals I_n of equal length $|I|$. The counting process $N(x)$ is a process with exchangeable increments if for any such division the distribution of $N(I_{j_k}), ..., N(I_{j_k})$ where $j_1 \neq ... \neq j_k$ only depends on k and the length $|I|$. Let now $\mathfrak{J}$ be the class of distributions for random processes with stationary independent increments having infinitely divisible distributions. Then Bühlmann proved that any process which is separable with the set of diadic numbers as a separating set and which has exchangeable increments has a distribution weighted over $\mathfrak{J}$. In our case the increments are non-negative and integer-valued. The only infinitely divisible distributions of non-negative integer-valued r.v.'s are the compound Poisson ones (see e.g. Feller [6] p. 271). We shall show that $N(x)$ has exchangeable increments (point 1–4 below). Then it is seen that $\{X_n\}$ must have a weighted compound Poisson distribution. Lastly in point 5 we shall prove that this distribution cannot be compound. Now $\{X_n\}$ was assumed to be a countable set of r.v.'s, which set we tacitly assume to be non-empty. By this the case when the parameter d.f. $W(\lambda)$ has $W(0) > 0$ is excluded.

1. The distribution of $N(I)$ where I is a finite interval is independent of the position of I.

Let $I_1 = I + y$ where $y \in S_{F_t}$ for some t. For any intervals $I^+ \supset I \supset I^-$ we have by lemma 5.2

$$\varphi(s;\, I^+) \leqslant \varphi(s;\, I) \leqslant \varphi(s;\, I^-),\ s \in [0, 1] \tag{5.5}$$

By lemma 5.6 (ii) we can choose I^+ and I^- such that given $\varepsilon > 0$

$$|\varphi(s;\, I^-) - \varphi(s;\, I^+)| < \varepsilon,\ s \in [0, 1] \tag{5.6}$$

and such that $I^+ - I$ and $I - I^-$ are unions of intervals with positive lengths. There is an interval B with $y \in B$ such that

$$M_{f,t}(I^- \times B) \leqslant M_{b,t}(I_1 \times B) \leqslant M_{f,t}(I^+ \times B). \tag{5.7}$$

For an independent set of motion at t with the d.f. $F_{t,B}(y)$ we get from lemma 5.2, lemma 5.3 and (5.7)

$$\varphi(s;\, I^+) \leqslant \varphi(s;\, I_1) \leqslant \varphi(s;\, I^-),\ s \in [0, 1]. \tag{5.8}$$

(5.5), (5.6) and (5.8) together with lemma 5.1 give that

$$\varphi(s;\, I) = \varphi(s;\, I_1),\ s \in [-1, 1]$$

and thus $N(I)$ and $N(I_1)$ have the same distribution. Let n be a positive integer. Then it follows that for any intervals I and J such that $J = I + ny$ or $J = I - ny$

with $y \in S_{F_t}$ for some t the r.v.'s $N(I)$ and $N(J)$ have the same distribution. Let now y_0 be any positive number and put

$$I_{01} = I + y_0, \; I_{02} = I - y_0.$$

In the same way as in lemma 5.6 (ii) it can be shown that given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|\varphi(s;\, I_{0j}+h) - \varphi(s;\, I_{0j})| < \varepsilon,\; |\varphi(s;\, I_{0j}-h) - \varphi(s;\, I_{0j})| < \varepsilon,\; s \in [0,1],\; j=1,2,\; 0 \leqslant h < \delta. \tag{5.9}$$

By the non-degeneracy assumption there is a t, an integer $n > y_0/\delta$ and a number y such that

$$y \in S_{F_t} \quad \text{and} \quad y_0/(n+1) \leqslant |y| \leqslant y_0/n$$

Let us first assume that $y > 0$. Then

$$I_{01} = I + ny + h_1 = I + (n+1)y - h_2,\; 0 \leqslant h_i < \delta,\; i=1,2. \tag{5.10}$$

We have already proved that $N(I)$, $N(I+ny)$ and $N(I+(n+1)y)$ have the same distribution. Then this fact together with (5.9), (5.10) and lemma 5.1 gives that $N(I)$ and $N(I+y_0)$ have the same distribution. If on the other hand $y < 0$

$$I_{02} = I + ny - h_1 = I + (n+1)y + h_2,\; 0 \leqslant h_i < \delta,\; i=1,2$$

from which relation we by the same reasoning prove that $N(I)$ and $N(I-y_0)$ have the same distribution.

Thus the distribution of $N(I)$ does not depend on the position of the interval I.

2. Let D_F stand for the discontinuity set of a d.f. F. Choose for any $\delta > 0$ a t and $y_1 > y_2$ such that $y_1, y_2 \in S_{F_t}$, $y_1 - y_2 < \delta$ (see definition 5.5 (ii)). It is easily seen that y_1 and y_2 can be chosen such that either

(a) $y_1, y_2 \in D_{F_t}$ or

(b) $y_1, y_2 \in {}'D_{F_t}$ and neither y_1 nor y_2 are accumulation points of D_{F_t}.

By consecutive intervals we shall in the following mean disjoint finite intervals numbered from the left to the right and with their union also being an interval. Let now J_1 and J_2 be two consecutive intervals with $|J_1| = |J_2| = y_1 - y_2$. In this point we shall show that $(N(J_1), N(J_2))$ has the same distribution as $(N(J_2), N(J_1))$ and that this distribution is independent of the position of J_1. Put $J = J_1 + y_1 = J_2 + y_2$ and choose similarly to point 1 of the proof for given $\varepsilon > 0$ two intervals J^+ and J^- such that (1) $J^+ \supset J \supset J^-$, (2) $J^+ - J$ and $J - J^-$ are unions of non-degenerated intervals, (3) $\varphi(s;\, J^-) - \varphi(s;\, J^+) < \varepsilon$, $s \in [0,1]$.

There exist disjoint intervals B_1 and B_2 with $y_1 \in B_1$ and $y_2 \in B_2$ for which

$$M_{b,t}(J^- \times B_i) \leqslant M_{f,t}(J_i \times B_i) \leqslant M_{b,t}(J^+ \times B_i),\; i=1,2. \tag{5.11}$$

Further

$$M_{b,t}(J^- \times B_i) \leqslant M_{b,t}(J \times B_i) \leqslant M_{b,t}(J^+ \times B_i),\; i=1,2. \tag{5.12}$$

Let $B = B_1 \cup B_2$ and put $F_{t,B}(B_i) = p_i$, $0 < p_i < 1$, $i=1,2$. In case (a) we choose B_1 and B_2 degenerated at y_1 and y_2 respectively and in this case p_i, $i=1,2$ are independent of the chosen ε, J^+ and J^-. Consider now the case (b) and fix two numbers

p_i, $0<p_i<1$, $i=1, 2$, $p_1+p_2=1$. We shall prove that we can choose B_1 and B_2 such that $F_{t,B}(B_i)=p_i$, $i=1, 2$ and (5.11) (and (5.12)) is fulfilled. Consider the intervals $B_i(h_i)=(y_i-h_i, y_i+h_i]$, $h_i>0$, and put $p_i(h_i)=F_t(B_i(h_i))$, $i=1, 2$. There exist numbers h_1', h_2', such that for $h_i\leq h_i'$, $i=1, 2$, $B_1(h_1)$ and $B_2(h_2)$ are disjoint and (5.11) holds (with $B_i=B_i(h_i)$, $i=1, 2$). For $h_i>0$ the functions $p_i(h_i)$ are non-decreasing and positive and $\lim_{h_i\to 0} p_i(h_i)=0$, $i=1, 2$. Now y_1 and y_2 are not accumulation points of D_{F_t}. Thus there exists h_i'' such that $0<h_i''\leq h_i'$ and $B_i(h_i'')\cap D_{F_t}=\emptyset$, $i=1, 2$. Hence

$$\{y;\, p_i(h_i)=y \text{ for some } 0<h_i\leq h_i''\} = (0, p_i(h_i'')],\; i=1, 2 \tag{5.13}$$

From (5.13) it is easily seen that we can choose h_i with $0<h_i\leq h_i''$, $i=1, 2$ such that

$$p_1(h_1)/p_2(h_2) = p_1/p_2$$

Note that given a $h>0$ we can in the last relation choose both h_1 and h_2 less than h. Then $B_i=B_i(h_i)$, $i=1, 2$ fulfil (5.11) (and (5.12)) and $F_{t,B}(B_i)=p_i$, $i=1, 2$.

For any interval I we get from lemma 5.3

$$\psi_{b,t,B}(s_1, s_2;\, I\times B_1, I\times B_2) = \varphi(p_1s_1+p_2s_2;\, I) \tag{5.14}$$

Using (5.14) and lemma 5.2 we get from (5.11)

$$\varphi(p_1s_1+p_2s_2;\, J^+)\leq\varphi(p_1s_1+p_2,\, p_1+p_2s_2;\, J_1, J_2)\leq\varphi(p_1s_1+p_2s_2;\, J^-),\; s_1, s_2\in[0, 1] \tag{5.15}$$

and from (5.12)

$$\varphi(p_1s_1+p_2s_2;\, J^+)\leq\varphi(p_1s_1+p_2s_2;\, J)\leq\varphi(p_1s_1+p_2s_2;\, J^-),\; s_1s_2\in[0, 1] \tag{5.16}$$

Let us now put $s_1'=p_1s_1+p_2$ and $s_2'=p_1+p_2s_2$ in (5.15) and (5.16). Then

$$\varphi(s_1'+s_2'-1;\, J^+)\leq\varphi(s_1', s_2';\, J_1, J_2)\leq\varphi(s_1'+s_2'-1;\, J^-)$$

$$\varphi(s_1'+s_2'-1;\, J^+)\leq\varphi(s_1'+s_2'-1;\, J)\leq\varphi(s_1'+s_2'-1;\, J^-)$$

for $s_1'\in[p_2, 1]$, $s_2'\in[p_1, 1]$.
Observing that p_1 and p_2 are independent of ε we get by lemma 5.1

$$\varphi(s_1, s_2;\, J_1, J_2) = \varphi(s_1+s_2-1;\, J),\; s_1, s_2\in[-1, 1] \tag{5.17}$$

Since by point 1 of the proof the right member of (5.17) does not depend on the position of J point 2 is proved.

3. Let $J_1 \ldots, J_{k+1}$ be consecutive intervals with $|J_i|=|y_2-y_1|$, $i=1, \ldots, k+1$ where y_1 and y_2 are the same as in point 2 of the proof. We shall show by induction that

$$\varphi(s_1, \ldots, s_{k+1};\, J_1, \ldots, J_{k+1}) = \varphi(s_1+\ldots+s_{k+1}-k;\, J) \tag{5.18}$$

where J is any interval with $|J|=|y_2-y_1|$. Let us assume that

$$\varphi(s_1, \ldots, s_k;\, J_1, \ldots, J_k) = \varphi(s_1+\ldots+s_k-k+1;\, J) \tag{5.19}$$

Let the distance between two intervals A_1 and A_2 be

$$d(A_1, A_2) = \inf\{|x-y|;\, x\in A_1,\, y\in A_2\}$$

Choose the intervals $J_{i,h}$, $i=1, \dots, k+1$, such that

$$J_{i,h}\subset J_i,\ d(J_{i,h}, J_{i+1,h})=h,\ i=1, \dots, k \quad \text{and} \quad J_{k,h}+y_1=J_{k+1,h}+y_2$$

Let $J'_{i,h}=J_{i,h}+y_1$, $i=1, \dots, k$. Choose similarly to point 2 intervals $J'^{+}_{i,h}$ and $J'^{-}_{i,h}$ such that

(1) $J'^{+}_{i,h}\supset J'_{i,h}\supset J'^{-}_{i,h}$, $i=1, \dots, h$,

(2) $J'^{+}_{i,h}-J'_{i,h}$ and $J'_{i,h}-J'^{-}_{i,h}$ are unions of non-degenerated intervals, $i=1, \dots, k$,

(3) $J'^{+}_{i,h}$, $i=1, \dots, k$ are disjoint intervals,

(4) $\varphi(s_1, \dots, s_k;\, J'^{-}_{1,h}, \dots, J'^{-}_{k,h})-\varphi(s_1, \dots, s_k;\, J'^{+}_{1,h}, \dots, J'^{+}_{k,h})<\varepsilon$
for $s_i\in[0,1]$, $i=1, \dots, k$ (5.20)

In the case when $y_1, y_2\in D_{F_t}$ let us choose $B_1=\{y_1\}$ and $B_2=\{y_2\}$. If $y_1, y_2\in \bar{D}_{F_t}$ we choose $B_i=B_i(h_i)$ with $h_i<h$ (see the remark just above (5.14)) as in point 2 of the proof. Then

$$M_{b,t}(J'^{-}_{i,h}\times B_1)\leqslant M_{f,t}(J_{i,h}\times B_1)\leqslant M_{b,t}(J'^{+}_{i,h}\times B_1),\ i=1, \dots, k$$

$$M_{b,t}(J'^{-}_{k,h}\times B_2)\leqslant M_{f,t}(J_{k+1,h}\times B_2)\leqslant M_{b,t}(J'^{+}_{k,h}\times B_2) \tag{5.21}$$

$$N(J'^{-}_{i,h})\leqslant N(J'_{i,h})\leqslant N(J'^{+}_{i,h}),\ i=1, \dots, k \tag{5.22}$$

We get by lemma 5.3 with $B=B_1\cup B_2$ in the same way as in point 2 from (5.21)

$$\begin{aligned}&\varphi(p_1s_1+p_2, \dots, p_1s_{k-1}+p_2, p_1s_k+p_2s_{k+1};\, J'^{+}_{1,h}, \dots, J'^{+}_{k-1,h}, J'^{+}_{k,h})\\ &\leqslant\varphi(p_1s_1+p_2, \dots, p_1s_k+p_2,\, p_1+p_2s_{k+1};\, J_{1,h}, \dots, J_{k+h}, J_{k+1,h})\\ &\leqslant\varphi(p_1s_1+p_2, \dots, p_1s_{k-1}+p_2, p_1s_k+p_2s_{k+1};\, J'^{-}_{1,h}, \dots, J'^{-}_{k,h})\\ &\text{for}\quad s_i\in[0,1],\ i=1, \dots, k+1\end{aligned} \tag{5.23}$$

Let us put $s'_i=p_1s_i+p_2$, $i=1, \dots, k$ and $s'_{k+1}=p_1+p_2s_{k+1}$ in (5.23). Then

$$\begin{aligned}&\varphi(s'_1, \dots, s'_{k-1}, s'_k+s'_{k+1}-1;\, J'^{+}_{1,h}, \dots, J'^{+}_{k-1,h}, J'^{+}_{k,h})\leqslant\varphi(s'_1, \dots, s'_{k+1};\, J_{1,h}, \dots, J_{k+1,h})\\ &\leqslant\varphi(s'_1, \dots, s'_{k-1}, s'_k+s'_{k+1}-1;\, J'^{-}_{1,h}, \dots, J'^{-}_{k-1,h}, J'^{-}_{k,h}),\ s'_i\in[p_2, 1],\\ &i=1, \dots, k,\ s'_{k+1}\in[p_1, 1]\end{aligned} \tag{5.24}$$

From (5.22), lemma 5.2 and lemma 5.3 we get

$$\begin{aligned}&\varphi(s_1, \dots, s_k;\, J'^{+}_{1,h}, \dots, J'^{+}_{k,h})\leqslant\varphi(s_1, \dots, s_k;\, J'_{1,h}, \dots, J'_{k,h})\\ &\leqslant\varphi(s_1, \dots, s_k;\, J'^{-}_{1,h}, \dots, J'^{-}_{k,h}),\ s_i\in[0,1],\ i=1, \dots, k\end{aligned} \tag{5.25}$$

We get from (5.20), (5.24) and (5.25) that

$$\varphi(s_1, \dots, s_{k+1};\, J_{1,h}, \dots, J_{k+1,h})=\varphi(s_1, \dots, s_{k-1}, s_k+s_{k+1}-1;\, J'_{1,h}, \dots, I'_{k-1,h}, J'_{k,h})$$

If we let $h \downarrow 0$ in this relation we get using lemma 5.6

$$\varphi(s_1, \ldots, s_{k+1}; J_1, \ldots, J_{k+1}) = \varphi(s_1, \ldots, s_{k-1}, s_k + s_{k+1} - 1; J_1', \ldots, J_{k-1}', J_k')$$

where $$J_i' = J_i + y_1, i = 1, \ldots, k \quad \text{and} \quad J_k' = J_{k+1} + y_2$$

From the induction assumption (5.19) we then get

$$\varphi(s_1, \ldots, s_{k+1}; J_1, \ldots, J_{k+1}) = \varphi(s_1 + \ldots + s_{k+1} - k; J)$$

But (5.19) holds for $k=2$ by (5.17). This proves point 3.

4. Let $I_1, \ldots, I_k$ be consecutive intervals all with the length $|I|$. We wish to show that the g.f.

$$\varphi(s_1, \ldots, s_k; I_1, \ldots, I_k) = \varphi(s_1 + \ldots + s_k - k + 1; I_1) \tag{5.26}$$

From lemma 5.6 it is easily seen that given $\varepsilon > 0$ there is a $\delta_0 > 0$ such that

$$|\varphi(s_1, \ldots, s_k; I_1, \ldots, I_k) - \varphi(s_1, \ldots, s_k; I_1', \ldots, I_k')| < \varepsilon, s_i \in [-1, 1], \quad i = 1, \ldots, k \tag{5.27}$$

where $I_1', \ldots, I_k'$ are disjoint intervals with

$$I_j' \subset I_j, |I_j'| = |I_j| - \delta_0, j = 1, \ldots, k; \ 0 < d(I_j', I_{j+1}') \leqslant \delta_0, \quad j = 1, \ldots, k-1$$

Let now $J_1^{(n)}, \ldots, J_{m_n}^{(n)}$ be consecutive intervals all with the length $|J^{(n)}|$ and with the left endpoint of $J_1^{(n)}$ coinciding with that of I_1' and the right endpoint of $J_{m_n}^{(n)}$ being the first to lie to the right of the right endpoint of I_k'. Let for any n the numbers y_1 and y_2 be such that they fulfil the conditions in the first part of point 2 with $\delta = |I_j'|/n$. There are arbitrarily large n for which y_1 and y_2 can be chosen such that $|y_1 - y_2| = |J^{(n)}|$ with $|I_j'|/(n+1) < |J^{(n)}| \leqslant |I_j'|/n$. Then I_j' is included in an interval $I_{n,j}^+$ which is a union of $(n+2)$ consecutive intervals from $\{J_{n,j}^+\}$ and includes an interval $I_{n,j}^-$ which is a union of $(n-1)$ consecutive intervals from $\{J_i^{(n)}\}$. Choose n so large that the intervals $I_{n,j}^+$ are disjoint.

From point 3 it follows that

$$\varphi(s_1, \ldots, s_{m_n}; J_1^{(n)}, \ldots, J_{m_n}^{(n)}) = \varphi(s_1 + \ldots + s_{m_n} - m_n + 1; J_1^{(n)})$$

and from this

$$\varphi(s_1, \ldots, s_k; I_{n,1}^+, \ldots, I_{n,k}^+) = \varphi((n+2)(s_1 + \ldots + s_k) + 1 - k(n+2); J_1^{(n)}) \tag{5.28}$$

$$\varphi(s_1, \ldots, s_k; I_{n,1}^-, \ldots, I_{n,k}^-) = \varphi((n-1)(s_1 + \ldots + s_k) + 1 - k(n-1); J_1^{(n)}) \tag{5.29}$$

Further by point 3

$$\varphi(s; I_{n,1}^+) = \varphi(n+2)s - n - 1; J_1^{(n)}) \tag{5.30}$$

$$\varphi(s; I_{n,1}^-) = \varphi((n-1)s - n + 2; J_1^{(n)}) \tag{5.30}$$

From (5.28) and (5,30) we get using lemma 5.1

$$\varphi(s_1, \ldots, s_k; I_{n,1}^+, \ldots, I_{n,k}^+) = \varphi(s_1 + \ldots + s_k - k + 1; I_{n,1}^+) \tag{5.32}$$

and from (5.29) and (5.31)

$$\varphi(s_1, \ldots, s_k; I_{n,1}^-, \ldots, I_{n,k}^-) = \varphi(s_1 + \ldots + s_k - k + 1; I_{n,1}^-) \tag{5.33}$$

Using lemma 5.6 and (5.27), (5.32) and (5.33) it is easily shown that (5.26) holds. This proves the exchangeability of $N(x)$.

5. From point 4 and the beginning of the proof it follows that $\{X_n\}$ must have a weighted compound Poisson distribution. We shall now show that this distribution cannot be compound. This is equivalent with no two of the $X_n(t)$'s being equal with positive probability. We shall deal with two cases separately.

(a) $F_t(y)$ has a continuous part for some $t>0$. By lemma 5.3 we can without any loss of generality assume that $F_t(y)$ is purely continuous for this t. Let us consider the point system at this t. Now

$$P(X_i(t)=X_j(t))=P(X_i+Y_i(t)=X_j+Y_j(t))=P(Y_i(t)-Y_j(t)=X_j-X_i).$$

But $Y_i(t)-Y_j(t)$ has a continuous distribution. Thus by the independence property

$$P(X_i(t)=X_j(t))=0,\ i\neq j$$

which proves the proposition in case (a).

(b) All $F_t(y)$ are purely discontinuous. Let I_1 and I_2 be disjoint intervals with the same length such that $I_1+y_1=I_2+y_2=I$ where $y_1, y_2 \in S_{F_t}$ for some t. Put $B=\{y_1\}\cup\{y_2\}$ and $F_{t,B}(\{y_i\})=p_i,\ i=1, 2$. Now

$$\varphi(s_1, s_2; I_1, I_2)=\int_0^\infty \exp\{\lambda|I|(\alpha(s_1)-1+\alpha(s_2)-1)\}dW(\lambda) \tag{5,34}$$

where $\alpha(s)$ is a g.f. and $W(\lambda)$ a d.f. on $(0,\infty)$. By lemma 5.3 we have for an independent set of motion at t with the d.f. $F_{t,B}(y)$

$$M_{f,t}(I_1\times\{y_1\})+M_{f,t}(I_2\times\{y_2\})=N_t(I). \tag{5.35}$$

From (5.34) and (5.35) we get

$$\int_0^\infty \exp\{\lambda|I|(\alpha(p_1s+p_2)+\alpha(p_1+p_2s)-2)\}dW(\lambda)$$
$$=\int_0^\infty \exp\{\lambda|I|(\alpha(s)-1)\}dW(\lambda). \tag{5.36}$$

From the uniqueness of the Laplace-Stieltjes transform of $W(\lambda)$ we get from (5.36)

$$\alpha(p_1)+\alpha(p_2)=1.$$

Since $\alpha(0)=0$, $\alpha(1)=1$ and $\alpha(s)$ is convex and continuous it follows that $\alpha(s)=s$ which proves the impossibility of a compound distribution in case (b).

The theorem is proved.

In the constant speed case we have $Y_n(t)=U_n\cdot t$ with $P(U_n\leqslant u)=G(u)$. In this case time-invariance under reversible motion with a non-degenerated family $\{F_t(y)\}$ is equivalent to that for all $t>0\{(X_n, U_n)\}$ and $\{(X_n(t), U_n)\}$ have the independence property with the same non-degenerated d.f. $G(u)$ and $\{X_n\}$ and $\{X_n(t)\}$ have the same distribution.

Corollary 5.1. *The distribution of $\{X_n\}$ is time-invariant under reversible independent motion in the constant speed case for a non-degenerated d.f. $G(u)$ if and only if $\{X_n\}$ is weighted Poisson distributed.*

Proof. From the remark following definition 5.5 we see that a non-degenerated d.f. in the constant speed case corresponds to a non-degenerated family in the general case. Then the corollary follows at once from theorem 5.1.

6. Sets of motion conserving weighted Poisson distributed point systems

For point systems with reversible independent sets of motion the role of weighted Poisson distributions was clarified by the results of section 5. In this section we shall give a result indicating in which cases weighted Poisson distributions are conserved for more general types of motion than those considered earlier in this paper. Let us think of $Y_n(t)$ as measuring the motion ability of point no n. The actual motion can, however, also be dependent of the starting position of the point. This leads us to consider the position at t of point n, $X_n(t)$, as a function of the starting position X_n and the motion ability $Y_n(t)$, i.e.

$$X_n(t) = f(X_n, Y_n(t)) \tag{6.1}$$

which with $f(x, y) = x + y$ includes the case studied in the earlier sections. Similarly to that case we shall here require the independence property for the positions of the point system and its set of motion abilities.

Definition 6.1. *The distribution class $\mathcal{C}$ consists of the distributions for sequences $\{(Z_n, V_n)\}$ where $\{Z_n\}$ is weighted Poisson distributed with some parameter d.f. $W(\lambda)$ (with $W(0) = 0$) and $\{(Z_n, V_n)\}$ has the independence property with some d.f. $F(v)$.*

If the distribution of $\{(Z_n, V_n)\}$ belongs to $\mathcal{C}$ it is characterized by the parameter d.f. $W(\lambda)$ and the d.f. $F(v)$ and we shall in the following just write: $\{(Z_n, V_n)\}$ has the distribution $(W, F) \in \mathcal{C}$. Suppose that the motion is given by (6.1). We shall try to describe those functions $f(x, y)$ for which both $\{(X_n, Y_n(t))\}$ and $\{(X_n(t), Y_n(t))\}$ have distributions in $\mathcal{C}$. We have to place two main restrictions on the functions f.

(i) $f(x, y)$ should be a function from R_2 onto R_1. This means that all positions are possible at time t.

(ii) $f(x, y)$ should be a monotone function in x for any fixed y. This will be the case e.g. if points with the same motion ability have the same internal order at t as initially. Since t is fixed in the problem studied in this section we shall in the following theorem use another notation than that used above.

Theorem 6.1. *Let $\{(Z_n, V_n)\}$ have the distribution $(W_1, F_1) \in \mathcal{C}$. Let further $f(z, v)$ be a Borel-measurable function from R_2 onto R_1 such that $f(z, v)$ is a monotone function of z for a.s.[1] all v. Let*

$$Z_n' = f(Z_n, V_n) \ a.s.$$

[1] Almost surely in this theorem is with respect to the probability measure ν_1 on the Borel sets in R_1 induced by $F_1(v)$.

Then the distribution of $\{(Z_n', V_n)\}$ *belongs to* $\mathcal{C}$ *if and only if*

$$f(z, v) = a(v)\cdot z + b(v) \ a.s. \tag{6.2}$$

where

$$0 < \int_{-\infty}^{+\infty} dF_1(v)/|a(v)| < \infty. \tag{6.3}$$

The distribution of $\{(Z_n', V_n)\}$ *is then given by* (W_2, F_2) *which is obtained from* $W_1(\lambda)$, $F_1(y)$ *and* $a(v)$ *by*

$$W_2(\lambda) = W_1(\lambda/c) \tag{6.4}$$

where

$$c = \int_{-\infty}^{+\infty} dF_1(v)/|a(v)| \tag{6.5}$$

and

$$F_2(v) = 1/c \int_{-\infty}^{v} dF_1(u)/|a(u)|. \tag{6.6}$$

Given (W_1, F_1) *and* (W_2, F_2), $|a(v)|$ *is determined a.s. from (6.4) and (6.6), whereas* $b(v)$ *is arbitrary.*

Remark 1. If for a.s. all v, $f(z, v)$ is one-to-one and continuous then it is also monotone for a.s. all v.

Remark 2. Let us drop the assumption $\{(Z_n', V_n)\} \in \mathcal{C}$ and replace it by the assumption that $\{Z_n'\}$ is weighted Poisson distributed. Then (6.2) does not necessarily hold. This is seen from the following example:

Let $F_1(z)$ have the jump $\frac{1}{2}$ at $z=0$ and at $z=1$. Put

$$f(z, 0) = \begin{cases} z, & z < 0 \\ z+1, & z \geq 0 \end{cases} \qquad f(z, 1) = \begin{cases} z, & z < 0 \\ \frac{1}{2}z, & 0 \leq z < 2 \\ z-1, & z \geq 2 \end{cases}$$

Then it is easily seen that $\{Z_n'\}$ is weighted Poisson distributed with the same parameter d.f. $W_1(\lambda)$ as $\{Z_n\}$.

Proof. Sufficiency. Let us introduce the following notation (cf. section 2).

$M_1(B) =$ no. of $(Z_n, V_n) \in B$, g.f. $\psi_1(s; B)$

$M_2(B) =$ no. of $(Z_n', V_n) \in B$, g.f. $\psi_2(s; B)$, B a Borel set in R_2.

Let further ν_1 and ν_2 denote the probability measures on the Borel sets in R_1 induced by $F_1(v)$ and $F_2(v)$ respectively and μ the Lebesgue measure in R_1. These measures induce in R_2 the product measures $\varkappa_i = \mu \times \nu_i$, $i = 1, 2$. In the proof we shall use lemma 2.4 and the remark following this lemma. Let us choose finite intervals I_i and J_i, $i = 1, \ldots, k$ such that the sets $B_i = I_i \times J_i$ are disjoint. We have to show that the g.f. of $(M_2(B_1), \ldots, M_2(B_k))$

$$\psi_2(s_1, \ldots, s_k; B_1, \ldots, B_k) = \int_0^\infty \exp\left\{\lambda \sum_{i=1}^k \mu(I_i) \nu_2(J_i)(s_i - 1)\right\} dW_2(\lambda) \tag{6.7}$$

where $W_2(\lambda)$ and $F_2(v)$ (which induces ν_2) are given by (6.4), (6.5) and (6.6). Denote by T the transformation from R_2 onto R_2 given by

$$T(z, v) = (f(z, v), v)$$

Now by (6.2)

$$\nu_1(\{v; f(z, v) \neq a(v)\cdot z + b(v)\}) = 0$$

and by (6.3) the set $\{(z, v); a(v) = 0\}$ has $\varkappa_1$-measure zero. Then

$$\varkappa_1(T^{-1}B_i) = \mu(I_i) \int_{J_i} dF_1(v)/|a(v)| < \infty \tag{6.8}$$

The event

$$\{M_2(B_i) = n_i\} = \{M_1(T^{-1}B_i) = n_i\},\ i = 1, \ldots, k$$

where $T^{-1}B_i$ are disjoint Borel sets. Lemma 2.4 gives that the r.v.'s $M_2(B_i)$, $i = 1, \ldots, k$ have the g.f.

$$\psi_2(s_1, \ldots, s_k; B_1, \ldots, B_k) = \int_0^\infty \exp\left\{\lambda \sum_{i=1}^k \varkappa_1(T^{-1}B_i)(s_i - 1)\right\} dW_1(\lambda) \tag{6.9}$$

Using (6.8) it is seen that (6.9) can be written in the form

$$\psi_2(s_1, \ldots, s_k; B_1, \ldots, B_k) = \int_0^\infty \exp\left\{\lambda c \sum_{i=1}^k \mu(I_i) \nu_2(J_i)(s_i - 1)\right\} dW_1(\lambda) \tag{6.10}$$

where c is given by (6.5). Putting $W_2(\lambda) = W_1(\lambda/c)$ in (6.10) we obtain (6.7). We have thus proved the sufficiency of (6.2) and (6.3) and the validity of (6.4), (6.5) and (6.6).

Necessity. Suppose that $\{(Z_n, V_n)\}$ has the distribution $(W_1, F_1) \in \mathfrak{C}$ and that $\{(Z_n', V_n)\}$ has the distribution $(W_2, F_2) \in \mathfrak{C}$. The sequences are related through the transformation T defined above. From lemma 2.4 we get that for any disjoint Borel sets B_i, $i = 1, \ldots, k$ with finite $\varkappa_2$-measures

$$\int_0^\infty \exp\left\{\lambda \sum_{i=1}^k \varkappa_2(B_i)(s_i - 1)\right\} dW_2(\lambda) = \int_0^\infty \exp\left\{\lambda \sum_{i=1}^k \varkappa_1(T^{-1}B_i)(s_i - 1)\right\} dW_1(\lambda). \tag{6.11}$$

We shall first show that $W_2(\lambda) = W_1(\lambda/c)$, where c is a constant. Let

$$B_i = (i, i+1] \times R_1,\ i = 1, \ldots$$

Then $\varkappa_2(B_i) = 1$, $i = 1, \ldots$ and from (6.11)

$$\int_0^\infty \exp\{\lambda(s-1)\} dW_2(\lambda) = \int_0^\infty \exp\{\lambda \varkappa_1(T^{-1}B_i)(s-1)\} dW_1(\lambda) \quad i = 1, \ldots \tag{6.12}$$

If $F(x)$ is a d.f. on $(0, \infty)$ and $\tilde{F}(u)$ its Laplace-Stieltjes transform then

(a) $F(x)$ and $\tilde{F}(u)$ are in a one-to-one correspondence with each other
(b) $\tilde{F}(u)$ is a decreasing function of u, $0 \leqslant u < \infty$.

Using these properties on (6.12) we get

$$\varkappa_1(T^{-1}B_i) = \text{constant} = c \tag{6.13}$$

and from (6.11)

$$W_2(\lambda) = W_1(\lambda/c). \tag{6.14}$$

(6.14) together with (6.11) give that for any Borel set with finite $\varkappa_2$-measure

$$\varkappa_2(B) = 1/c \cdot \varkappa_1(T^{-1}B). \tag{6.15}$$

Let us now choose $B=[z_1, z_2]\times C$ where C is a Borel set in R_1 and let $B_0=[z_1, z_2]\times(-\infty, +\infty)$. Let further for any set A in the (z, v)-plane A_v be the section at v. Then we get from (6.15)

$$(z_2-z_1)\nu_2(C)=1/c\int_C \mu((T^{-1}B)_v d\nu_1 = 1/c\int_C \mu((T^{-1}B_0)_v)d\nu_1.$$

Thus ν_2 is absolutely continuous with respect to ν_1 and by the Radon–Nikodym theorem

$$\nu_2(C)=\int_C g(v)\,d\nu_1$$

where for fixed z_1 and z_2

$$g(v) = 1/c\cdot\mu((T^{-1}B_0)_v)/(z_2-z_1) \text{ a.s.} \tag{6.16}$$

Let $\mathcal{D}$ be a dense countable set in R_1. Then

$$\nu_1\{v;\, g(v) \neq 1/c\cdot\mu((T^{-1}B_0)_v)/(z_2-z_1) \quad \text{for some} \quad z_1, z_2\in\mathcal{D},\, z_1<z_2\} = 0. \tag{6.17}$$

We shall need the following lemma

Lemma 6.1. *Let $h(x)$ be a monotone function from R_1 onto R_1 and suppose that*

(i) *$\mu\{x;\, y_1\leqslant h(x)\leqslant y_2\}/(y_2-y_1)=\text{constant}=c$ for all $y_1<y_2$ such that $y_1, y_2\in\mathcal{D}$ where $\mathcal{D}$ is a dense countable set in R_1. Then*

(ii) *$h(x)$ is a linear function of x, i.e. $h(x)=ax+b$ where $|a|=1/c>0$.*

Remark 1. (ii) implies (i) without the restriction of $y_1, y_2\in\mathcal{D}$.

Remark 2. If $h(x)$ is one-to-one and continuous then it is also monotone.

Proof. Let us first note that a monotone function is measurable so that (i) has a meaning. We shall first show that every monotone function which satisfies (i) is strictly monotone. Let us now for simplicity suppose that $h(x)$ is non-decreasing. In order to prove the strict monotony let us suppose the opposite, i.e. there exist two numbers $x_1<x_2$ such that $h(x_1)=h(x_2)=y_1<h(x)$, $x>x_2$. Let us first consider the case when $y_1\in\mathcal{D}$. Then for any $\delta>0$ such that $y_1+\delta\in\mathcal{D}$ we have

$$c=\mu\{x;\, y_1\leqslant h(x)\leqslant y_1+\delta\}/\delta>(x_2-x_1)/\delta$$

For δ sufficiently small and with $y_1+\delta\in\mathcal{D}$ this cannot be true and thus

$$h(x+\varepsilon)-h(x)>0 \quad \text{for} \quad \varepsilon>0 \quad \text{and} \quad x\in h^{-1}(\mathcal{D})$$

Let us now consider the case when $y_1 \in {}'\mathcal{D}$. Choose an ε such that $0<\varepsilon<(x_2-x_1)/c$ and let y_1' be a number such that $y_1' \in \mathcal{D}$, $y_1'<y_1$ and $y_1-y_1'<\varepsilon$. Then for any $\delta>0$ such that $y_1+\delta \in \mathcal{D}$ we have

$$c \geqslant \mu\{x;\, y_1' \leqslant h(x) \leqslant y_1+\delta\}/(\delta+\varepsilon) > (x_2-x_1)/(\delta+\varepsilon)$$

Let now $\delta \to 0$ such that $y_1+\delta \in \mathcal{D}$. Then

$$c \geqslant (x_2-x_1)/\varepsilon$$

But $\varepsilon<(x_2-x_1)/c$ means a contradiction and thus $h(x)$ is strictly increasing and in the general case strictly monotone. From this we see that the inverse function h^{-1} is well defined and also strictly monotone. Then (i) can be written in the form

$$\mu\{x;\, y_1 \leqslant h(x) \leqslant y_2\}/(y_2-y_1) = |h^{-1}(y_2)-h^{-1}(y_1)|/(y_2-y_1) = c$$

or all $y_1<y_2$ such that $y_1, y_2 \in \mathcal{D}$. This means that for $x \in h^{-1}(\mathcal{D})$, $h(x)$ is of the form

$$h(x) = ax+b \quad \text{where} \quad |a| = 1/c>0 \tag{6.18}$$

Now $h^{-1}(\mathcal{D})$ is also a dense countable set in R_1 and this fact together with the monotony of $h(x)$ implies that $h(x)=ax+b$ for all x. $|a|$ is determined by (6.18) and b is arbitrary. The validity of the two remarks follows at once. This ends the proof of the lemma.

By applying lemma 6.1 to (6.17) the necessity of (6.2) and (6.3) is easily shown. The function $|a(v)|$ is determined a.s. from (6.4) and (6.6).

Corollary 6.1. *In order that in the theorem* 6.1

$$W_2(\lambda) = W_1(\lambda) \quad \textit{and} \quad F_2(v) = F_1(v)$$

it is necessary and sufficient that

$$|a(v)| = 1 \textit{ a.s. with respect to } \nu_1.$$

The *proof* follows at once from theorem 6.1.

Remark. For the case $a(v)=1$ the sufficiency of theorem 6.1 implies the sufficiency part of theorem 5.1.

An application to road traffic flow. Let the car positions at $t=0$, $\{X_n\}$, be weighted Poisson distributed and suppose that the cars can overtake (and meet) each other without any delay. The trajectories in the road-time diagram are assumed to be lines, i.e. we have the constant speed case. Further the sequence of car positions and speeds at $t=0$ $\{(X_n, U_n)\}$ has the independence property. Let L be a fixed line in the road-time diagram and denote the intersections between the trajectories and L and the corresponding speeds by $\{(X_n^L, U_n^L)\}$. From theorem 6.1 it then follows that $\{X_n^L\}$ is weighted Poisson distributed and that $\{(X_n^L, U_n^L)\}$ has the independence property (if (6.3) holds with the relevant $a(.)$ and d.f. $F_1(.)$). In the case when L is the t-axis this has been treated by F. Haight [8] p. 121 for $\{X_n\}$ Poisson distributed.

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Department of Mathematical Statistics, Royal Institute of Technology, Stockholm 70, Sweden

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