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On the extension of Lipschitz maps

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Introduction

Let X and Y be metric spaces. A map T from X into Y is called a *Lipschitz map* if there is a constant M, such that $d(Tx_1, Tx_2) \leqslant Md(x_1, x_2)$ for all $x_1, x_2 \in X$, and then the norm of T, $\|T\|$, is defined as the least such constant. T is called a *contraction* if $\|T\| \leqslant 1$. If D is a subset of X and $T: D \to Y$ a Lipschitz map, it is natural to ask whether T may be extended to a Lipschitz map $\bar{T}: X \to Y$ so that $\|\bar{T}\| = \|T\|$. This problem has been studied by several authors. A short survey of results in this connection is given in the paper of Danzer, Grünbaum and Klee [2]. In the present note we are concerned with some aspects of this problem in the case that X and Y are real Banach spaces. More precisely, we are interested in those pairs of Banach spaces X and Y, for which the extension problem formulated above always has a solution. It is obvious that, in the case of Banach spaces, it is sufficient to consider contractions, for if each contraction of a subset of X into Y may be extended to a contraction of X into Y, then also each Lipschitz map of a subset of X into Y may be extended to a Lipschitz map of X into Y, without increasing the norm. In order to abbreviate, we will say that (X, Y) has EPC (extension property for contractions) if for each subset $D \subset X$ and each contraction $T: D \to Y$, there exists a contraction $\bar{T}: X \to Y$, which extends T.

The fact that (X, Y) has EPC may also be formulated as an intersection property for cells in X and Y. A *cell* in a Banach space X is a set of the form $S_X(x_0; r) = \{x \in X : \|x - x_0\| \leqslant r\}$. The *unit cell of* X is the cell $S_X = S_X(0; 1)$. When $\mathfrak{F}$ and $\mathcal{G}$ are families of cells in X and Y respectively, we shall write $\mathfrak{F} \succ \mathcal{G}$ if there is a one-to-one correspondence between $\mathfrak{F}$ and $\mathcal{G}$, such that corresponding cells have equal radii and the distance between the centers of any two cells in $\mathcal{G}$ is less than or equal to the distance between the centers of the corresponding cells in $\mathfrak{F}$. If $\mathfrak{F}$ is a family of sets, then $\pi\mathfrak{F}$ denotes the intersection of the sets belonging to $\mathfrak{F}$. Then the following two statements are equivalent (see [2]):

(i) (X, Y) has EPC,

(ii) whenever $\mathfrak{F}$ and $\mathcal{G}$ are families of cells in X and Y respectively, such that $\mathfrak{F} \succ \mathcal{G}$ and $\pi\mathfrak{F} \neq \phi$, then $\pi\mathcal{G} \neq \phi$.

We will also use the following terminology, due to Aronszajn and Panitchpakdi [1] (see also Klee [10]). For a cardinal number $\gamma \geqslant 3$, a space X is said to be *γ-hyperconvex* if each collection $\{S_X^\alpha\}$, $\alpha \in A$, of mutually intersecting cells in X with card $A < \gamma$, has a nonempty intersection. X is called *weakly γ-hyperconvex* if this condition holds under the further assumption that all the S_X^α have a common radius. X is called (*weakly*) *hyperconvex* if it is (weakly) γ-hyperconvex for all γ. It was proved by Hanner [7, Theorem 6.1] that if γ is finite, then for finitedimensional spaces weak γ-hyperconvexity implies γ-hyperconvexity, and this result was extended to arbitrary Banach spaces by Lindenstrauss [11, Theorem 4.3]. Hanner also proved that, for finitedimensional spaces, 5-hyperconvexity implies $\aleph_0$-hyperconvexity, and this, too, was extended to arbitrary Banach spaces by Lindenstrauss [11, Theorem 4.1].

It is well-known (Aronszajn–Panitchpakdi [1], Goodner [5], Kelley [9], Nachbin [12]) that the following properties of a real Banach space are equivalent:

(*a*) X is hyperconvex,
(*b*) X is a $\mathcal{P}_1$ space, i.e. for any Banach space Z containing X as a subspace, there is a projection of norm 1 from Z onto X,
(*c*) X is isometric to a space $C(K)$, where K is an extremally disconnected compact Hausdorff space.

Returning now to our object in this paper, we note that the following facts are previously known (for the proper references we refer to [2]):

(1) (X, Y) has EPC for all X if and only if Y is a $\mathcal{P}_1$ space,
(2) (X, Y) has EPC if X and Y are Hilbert spaces,
(3) If dim $X=2$, then (X, X) has EPC if and only if X is a Hilbert space or a $\mathcal{P}_1$ space (Grünbaum [6]).

In section 1 we will prove that if Y is strictly convex and dim $Y>1$, then (X, Y) has EPC only if X and Y are Hilbert spaces (this result was announced in [13]). Section 2 contains a proof of the fact that 3) above remains true if dim $X=2$ is replaced by dim $X<\infty$. Finally, we give, in section 3, some results in the case that X is a $C(K)$ space; in particular, we prove that $(C(K), C(K))$ has EPC only if K is extremally disconnected.

1. Pairs (X, Y) with Y strictly convex

In this section we prove the following theorem:

Theorem 1.1. *If X and Y are Banach spaces such that Y is strictly convex, dim $Y>1$ and (X, Y) has EPC then X and Y are Hilbert spaces.*

Theorem 1.1 follows from the following two lemmas.

Lemma 1.2. *If X and Y are Banach spaces such that Y is strictly convex and (X, Y) has EPC, then X and Y satisfy the following condition:*

(A) *If $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $\|x_1\|=\|y_1\|$, $\|x_2\|=\|y_2\|$, $\|x_1-x_2\|=\|y_1-y_2\|$ then $\|\lambda x_1+\mu x_2\| \geqslant \|\lambda y_1+\mu y_2\|$ for all λ, μ.*

Proof. Let $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $\|x_1\|=\|y_1\|$, $\|x_2\|=\|y_2\|$, $\|x_1-x_2\|=\|y_1-y_2\|=d$. Suppose first that $\lambda \geqslant 0$, $\mu \geqslant 0$, $\lambda+\mu=1$. Consider the cells $S_X^1=S_X(x_1; \mu d)$, $S_X^2=S_X(x_2; \lambda d)$, $S_X^3=S_X(0; \|\lambda x_1+\mu x_2\|)$ in X and $S_Y^1=S_Y(y_1; \mu d)$, $S_Y^2=S_Y(y_2; \lambda d)$, $S_Y^3=S_Y(0; \|\lambda x_1+\mu x_2\|)$ in Y. If we put $\mathfrak{F}=\{S_X^1, S_X^2, S_X^3\}$ and $\mathcal{G}=\{S_Y^1, S_Y^2, S_Y^3\}$, we have $\mathfrak{F}\succ\mathcal{G}$ and $\pi\mathfrak{F}\neq\phi$, because $\lambda x_1+\mu x_2 \in \pi\mathfrak{F}$. Thus we get $\pi\mathcal{G}\neq\phi$ and, since $S_Y^1 \cap S_Y^2=\{\lambda y_1+\mu y_2\}$

(here we use the strict convexity of Y), we must have $\lambda y_1+\mu y_2 \in S^3_Y$, i.e. $\|\lambda y_1+\mu y_2\| \leqslant \|\lambda x_1+\mu x_2\|$. The condition $\lambda+\mu=1$ may now be removed and so we have proved that $\|\lambda x_1+\mu x_2\| \geqslant \|\lambda y_1+\mu y_2\|$ for $\lambda \geqslant 0$, $\mu \geqslant 0$. If we apply this result to x_1-x_2, $-x_2$ and y_1-y_2, $-y_2$ instead of x_1, x_2 and y_1, y_2 we get $\|\lambda(x_1-x_2)-\mu x_2\| \geqslant \|\lambda(y_1-y_2)-\mu y_2\|$ for $\lambda \geqslant 0, \mu \geqslant 0$, which is equivalent to $\|\lambda x_1+\mu x_2\| \geqslant \|\lambda y_1+\mu y_2\|$ for $\lambda \geqslant 0, \mu \leqslant 0, \lambda+\mu \leqslant 0$. Similarly, replacing x_1, x_2 and y_1, y_2 by x_2-x_1, $-x_1$ and y_2-y_1, $-y_1$, we get $\|\lambda x_1+\mu x_2\| \geqslant \|\lambda y_1+\mu y_2\|$ for $\lambda \geqslant 0, \mu \leqslant 0, \lambda+\mu \geqslant 0$. The inequalities now proved together imply that $\|\lambda x_1+\mu x_2\| \geqslant \|\lambda y_1+\mu y_2\|$ for all λ, μ, and the proof is complete.

Lemma 1.3. *Suppose X and Y are two normed linear spaces such that $\dim Y>1$. If X and Y satisfy the condition (A) of Lemma 1.2, then X and Y are euclidean (i.e. prehilbert) spaces.*

Proof. In the proof we will use the concept of normality defined as follows: Let L be a real normed linear space. An element $x \in L$ is said to be normal to an element $y \in L$ if $\|x+\lambda y\| \geqslant \|x\|$ for all λ. If x is normal to y we write xNy. We first prove that (A) implies:

(B) If $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $\|x_1\|=\|y_1\|$, $\|x_2\|=\|y_2\|$, x_1Nx_2, y_1Ny_2 then $\|\lambda x_1+\mu x_2\| \geqslant \|\lambda y_1+\mu y_2\|$ for all λ, μ.

It is sufficient to assume that $\|x_1\|=\|x_2\|=\|y_1\|=\|y_2\|=1$. There exist two sequences (x_n') and (x_n'') in the linear span of x_1 and x_2, such that

$$\|x_n'\|=\|x_n''\|=1,\ x_n''-x_n'=\|x_n''-x_n'\|x_2 \neq 0,\ \lim x_n'=\lim x_n''=x_1.$$

Then we can find two sequences (y_n') and (y_n'') in the linear span of y_1 and y_2, such that

$$\|y_n'\|=\|y_n''\|=1,\ y_n''-y_n'=\|x_n''-x_n'\|y_2,\ \lim y_n'=\lim y_n''=y_1.$$

From (A) it follows that $\|\lambda x_n'+\mu x_n''\| \geqslant \|\lambda y_n'+\mu y_n''\|$ for all λ, μ, which gives

$$\|\lambda x_n'+\mu(x_n''-x_n')/\|x_n''-x_n'\|\,\| \geqslant \|\lambda y_n'+\mu(y_n''-y_n'/\|x_n''-x_n'\|\,\| \text{ for all } \lambda, \mu,$$

or $\|\lambda x_n'+\mu x_2\| \geqslant \|\lambda y_n'+\mu y_2\|$ for all λ, μ. Letting n tend to infinity we get $\|\lambda x_1+\mu x_2\| \geqslant \|\lambda y_1+\mu y_2\|$ for all λ, μ, which was to be proved.

We can now show that normality is a symmetric relation in both X and Y. Suppose first that $x_1, x_2 \in X$ and x_1Nx_2. By a result of Day [3] we know that there exist two elements $y_1, y_2 \in Y$ such that $\|y_1\|=\|x_1\|$, $\|y_2\|=\|x_2\|$, y_1Ny_2 and y_2Ny_1. Hence we get, using (B), $\|\lambda x_1+x_2\| \geqslant \|\lambda y_1+y_2\| \geqslant \|y_2\|=\|x_2\|$, which means that x_2Nx_1. This proves that normality is symmetric in X. Now suppose that $y_1, y_2 \in Y$, $y_1 \neq 0$, $y_2 \neq 0$ and y_1Ny_2. Obviously, there exist two elements $x_1, x_2 \in X$, such that $\|x_1\|=\|y_1\|$, $\|x_2\|=\|y_2\|$, x_1Nx_2 and $\|x_1+\lambda x_2\|>\|x_1\|$ for $\lambda \neq 0$. Let u be an element in the linear span of y_1 and y_2 such that $\|u\|=\|y_1\|$ and y_2Nu. If $u=\alpha y_1+\beta y_2$ and if we put $z=\alpha x_1+\beta x_2$, then by (B), for each ν,

$$\|x_2+\nu z\|=\|\nu\alpha x_1+(1+\nu\beta)x_2\| \geqslant \|\nu\alpha y_1+(1+\nu\beta)y_2\|=\|y_2+\nu u\| \geqslant \|y_2\|=\|x_2\|.$$

Hence we get x_2Nz and, since normality is symmetric in X, zNx_2. Together with the condition $\|x_1+\lambda x_2\|>\|x_1\|$ for $\lambda \neq 0$, zNx_2 implies $\beta=0$. Thus we have $u=\alpha y_1$ and, since $\|u\|=\|y_1\|$, $u=\pm y_1$. Hence y_2Ny_1 and it follows that normality is symmetric in Y.

We now use the following result which follows from a construction of twodimensional spaces with symmetry of normality given by Day [4, p. 330]:

Let L and M be twodimensional normed linear spaces, in both of which normality is symmetric, and let $x_1, x_2 \in L$, $y_1, y_2 \in M$ be two pairs of elements such that $\|x_1\|=\|y_1\|$, $\|x_2\|=\|y_2\|$ and x_1Nx_2, y_1Ny_2. Then if $\|\lambda x_1+\mu x_2\| \geqslant \|\lambda y_1+\mu y_2\|$ for $\lambda\mu \geqslant 0$ we have $\|\lambda x_1+\mu x_2\| \leqslant \|\lambda y_1+\mu y_2\|$ for $\lambda\mu\leqslant 0$.

From this result and what we have proved above, it follows that if $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $\|x_1\|=\|y_1\|$, $\|x_2\|=\|y_2\|$, x_1Nx_2, y_1Ny_2, then we have $\|\lambda x_1+\mu x_2\|=\|\lambda y_1+\mu y_2\|$ for all λ, μ. In particular, we see that if Z is a twodimensional subspace of either X or Y and if $z, z' \in Z$, $\|z\|=\|z'\|=1$, then there is an isometry of Z onto itself which carries z into z'. However, it is well-known that this is possible only if the unit circle in Z is an ellipse, i.e. if Z is an euclidean space. Hence all twodimensional subspaces of X and Y are euclidean, which means that X and Y are euclidean spaces. Thus the lemma is proved.

Remark. It may be observed that Lemma 1.2 and therefore also Theorem 1.1, is still valid under the weaker hypothesis that (X, Y) has the following property: If $\mathfrak{F}$ and $\mathcal{G}$ are families of cells in X and Y respectively, such that $\mathfrak{F} \succ \mathcal{G}$, $\pi\mathfrak{F} \neq \phi$ and card $\mathfrak{F}=3$, then $\pi\mathcal{G} \neq \phi$. This is apparent from the proof of Lemma 1.3.

2. Pairs (X, X) with dim $X < \infty$

We first prove two lemmas without any restriction on dim X.

Lemma 2.1. *Let X be a Banach space such that (X, X) has EPC. Suppose that e_1, e_2 are extreme points of S_X and $x_1, x_2 \in X$ are such that $\|x_1+e_1\|=\|x_2+e_2\|$, $\|x_1-e_1\|=\|x_2-e_2\|$. Then it follows that $\|\lambda x_1+\mu e_1\|=\|\lambda x_2+\mu e_2\|$ for $|\lambda| \geqslant |\mu|$.*

Proof. It is clearly sufficient to prove that $\|x_1+\mu e_1\|=\|x_2+\mu e_2\|$ for $0\leqslant\mu\leqslant 1$. Consider the cells $S_1=S(x_1-e_1; 1+\mu)$, $S_2=S(x_1+e_1; 1-\mu)$, $S_3=S(0; \|x_1+\mu e_1\|)$ and $S_1'=S(x_2-e_2; 1+\mu)$, $S_2'=S(x_2+e_2; 1-\mu)$, $S_3'=S_3$. If we put $\mathfrak{F}=\{S_1, S_2, S_3\}$ and $\mathcal{G}=\{S_1', S_2', S_3'\}$ then we have $\mathfrak{F}\succ\mathcal{G}$ and $\pi\mathfrak{F}\neq\phi$ (for $x_1+\mu e_1 \in \pi\mathfrak{F}$). Since (X, X) has EPC it follows that $\pi\mathcal{G}\neq\phi$. Now, since e_2 is an extreme point of S_X the intersection $S_1' \cap S_2'$ contains exactly one point, namely $x_2+\mu e_2$. Thus we must have $x_2+\mu e_2 \in S_3' = S_3$, which means that $\|x_2+\mu e_2\| \leqslant \|x_1+\mu e_1\|$. By symmetry we must also have $\|x_1+\mu e_1\| \leqslant \|x_2+\mu e_2\|$ and hence $\|x_1+\mu e_1\|=\|x_2+\mu e_2\|$, which completes the proof.

Lemma 2.2. *Let X be a Banach space such that (X, X) has EPC. If Y is a twodimensional subspace of X containing an extreme point e of S_X, there exists $y \in Y$ such that $\|y\|=1$ and $\|\lambda y+\mu e\|=\|\lambda y-\mu e\|$ for all λ, μ. If Y' is another twodimensional subspace of X containing an extreme point e' of S_X and if $y' \in Y'$ is such that $\|y'\|=1$ and $\|\lambda y'+\mu e'\|=\|\lambda y'-\mu e'\|$ for all λ, μ, then the linear map T of Y onto Y' defined by $Te=e'$, $Ty=y'$ is an isometry.*

Proof. It is easily seen that for each $n=1, 2, \ldots$, we can find $y_n \in Y$ with $\|y_n\|=1$ such that $\|y_n+ne\|=\|y_n-ne\|$. Applying Lemma 2.1 with $e_1=e$, $e_2=-e$, $x_1=x_2=$

y_n/n we get $\|\lambda(y_n/n)+\mu e\| = \|\lambda(y_n/n)-\mu e\|$ for $|\lambda| \geqslant |\mu|$ or $\|\lambda y_n+\mu e\| = \|\lambda y_n - \mu e\|$ for $n|\lambda| \geqslant |\mu|$. If y is a limitpoint of the sequence (y_n), it follows that $\|y\|=1$ and $\|\lambda y+\mu e\| = \|\lambda y - \mu e\|$ for all λ, μ. Thus the first statement of the lemma is proved.

To prove the second statement, we observe that $1 = \|e\| \leqslant 1/2(\|\lambda y+e\| + \|\lambda y - e\|) = \|\lambda y+e\|$ for all λ. Thus for each $n=1, 2, \ldots$, there is a number $\varepsilon_n \geqslant 0$ such that $\|(y/n)+e\| = \|\varepsilon_n y' + e'\|$. Then we also have $\|(y/n)-e\| = \|\varepsilon_n y' - e'\|$. An application of Lemma 2.1 gives $\|\lambda(y/n)+\mu e\| = \|\lambda \varepsilon_n y' + \mu e'\|$ for $|\lambda| \geqslant |\mu|$. Putting $\mu=0$ we get $\varepsilon_n = 1/n$. Thus for each n we have $\|\lambda(y/n)+\mu e\| = \|\lambda(y'/n)+\mu e'\|$ for $|\lambda| \geqslant |\mu|$ or $\|\lambda y+\mu e\| = \|\lambda y' + \mu e'\|$ for $n|\lambda| \geqslant |\mu|$. Hence it follows that $\|\lambda y + \mu e\| = \|\lambda y' + \mu e'\|$ for all λ, μ, which means that T is an isometry.

Lemma 2.3. *Let X be a Banach space such that (X, X) has EPC, dim $X<\infty$ and X is not strictly convex. Then S_X is a polyhedron. Moreover, there is a number $a>0$, such that if e is an extreme point of S_X, if $x\in X$ and if the segment $[e, x]$ between e and x is a maximal line segment on the boundary of S_X, then $\|e-x\|=a$.*

Proof. Since X is not strictly convex, it is clear that the boundary of S_X must contain a proper line segment, one endpoint of which is an extreme point of S_X. In other words, there exists an extreme point e of S_X and a point x_0 with $\|x_0\|=1$, such that $x_0 \neq \pm e$ and $\|\lambda e+\mu x_0\| = \lambda+\mu$ for $\lambda \geqslant 0$, $\mu \geqslant 0$. Let Y be the twodimensional subspace of X determined by e and x_0, and let y be an element of Y such that $\|\lambda y+\mu e\| = \|\lambda y - \mu e\|$ for all λ, μ (Lemma 2.2). Then we may write $x_0 = \alpha e + \beta y$. If we put $x_0' = \alpha e - \beta y$, then $\|x_0'\|=1$ and $\|\lambda e + \mu x_0'\| = \lambda+\mu$ for $\lambda \geqslant 0$, $\mu \geqslant 0$. Geometrically this means that the two line segments $[e, x_0]$ and $[e, x_0']$ belong to the boundary of S_Y. The points x_0 and x_0' determine two closed arcs of the unit circle in Y with endpoints x_0 and x_0'. One of these arcs, which we shall call Γ, does not contain e. Let d be the distance from e to Γ. Then $d>0$.

Now, if e_1 and e_2 are two arbitrary extreme points of S_X, $e_1 \neq \pm e_2$, and if Z is the twodimensional subspace of X determined by e_1 and e_2, there is by Lemma 2.2 an isometry of Z onto Y, which carries e_1 into e. From the construction of Γ it is obvious that this isometry carries e_2 into a point belonging to Γ. Hence we have $\|e_1-e_2\| \geqslant d$. Since this holds for every pair of extreme points of S_X and since dim $X<\infty$, it follows that there are only a finite number of extreme points of S_X, which means that S_X is a polyhedron.

In order to prove the second statement of the lemma, let e_1, e_2 be extreme points of S_X and x_1, x_2 points with $\|x_1\| = \|x_2\| = 1$, such that $[e, x_1]$ and $[e_2, x_2]$ are maximal line segments belonging to the boundary of S_X. We then have to prove that $\|e_1-x_1\| = \|e_2-x_2\|$. Let Y_1 and Y_2 be the two-dimensional subspaces of X, determined by e_1, x_1 and e_2, x_2 respectively. According to Lemma 2.2, we may take $y_1 \in Y_1$ so that $\|y_1\|=1$, $\|\lambda y_1+\mu e_1\| = \|\lambda y_1 - \mu e_1\|$ for all λ, μ and so that y_1 and x_1 lie in the same halfplane in Y_1 determined by the line through 0 and e_1. We take $y_2 \in Y_2$ in the corresponding way. If T is the isometry of Y_1 onto Y_2 for which $Te_1=e_2$, $Ty_1=y_2$, then it is obvious that $Tx_1=x_2$. Thus $\|e_1-x_1\| = \|e_2-x_2\|$ and the proof is complete.

Theorem 2.4. *Let X be a finite-dimensional Banach space such that (X, X) has EPC. Then X is either a Hilbert space or a $\mathfrak{P}_1$ space (i.e. the unit cell of X is either a euclidean cell or a parallelotope).*

Proof. As we mentioned in the introduction, this theorem was proved by Grünbaum [6] in the case dim $X=2$. Thus we may assume that dim $X \geqslant 3$. If X is strictly convex, it follows from Theorem 1.1 that X is a Hilbert space. It remains to prove that if X is not strictly convex then X is a $\mathcal{D}_1$ space. In this case we know from Lemma 2.3 that S_X is a polyhedron. Let F be a twodimensional face of S_X. Then F is the convex hull of a certain set $\{e_1, ..., e_m\}$, $m \geqslant 3$, of extreme points of S_X. Let a be the number, whose existence was proved in Lemma 2.3. If e_i, e_j determine a side $[e_i, e_j]$ of F and if $e_k \notin [e_i, e_j]$ then $[e_k, x]$ is a maximal line segment on the boundary of S_X for each $x \in [e_i, e_j]$. Consequently, $\|e_k - x\| = a$ for $x \in [e_i, e_j]$. We now prove that every two disjoint sides of F are parallel. Suppose that e_1, e_2 and e_3, e_4 determine two disjoint sides of F. Without loss of generality, we may assume that e_1 and e_4 lie on different sides of the line joining e_2 and e_3. We know that $\|e_2 - x\| = a$ for all $x \in [e_3, e_4]$. From this it follows that $\|e_3 - x\| = a$ for all $x \in [e_2 + e_3 - e_4, e_2]$. Since we also have $\|e_3 - x\| = a$ for all $x \in [e_1, e_2]$, and since $e_2 + e_3 - e_4$ and e_1 lie on the same side of the line joining e_3 and e_2, it follows that $[e_2 + e_3 - e_4, e_2]$ and $[e_1, e_2]$ are parallel. Thus $[e_1, e_2]$ and $[e_3, e_4]$ are parallel.

The property of F just proved implies that the boundary of F is either a triangle with vertices e_1, e_2, e_3 or a parallelogram with vertices e_1, e_2, e_3, e_4. In the first case put $x_1 = e_1$, $x_2 = e_2 + e_3 - e_1$, $x_3 = 2e_2 - e_3$. Then $\|x_1 - x_2\| = \|x_2 - x_3\| = \|x_3 - x_1\| = 2a$ and $\|x_1 - e_2\| = \|x_2 - e_2\| = \|x_3 - e_2\| = a$. In the second case, assume that $[e_1, e_2]$ is a side of F and put $x_1 = 2e_1$, $x_2 = 2e_2$, $x_3 = 2e_3$. Then we have $\|x_1 - x_2\| = \|x_2 - x_3\| = \|x_3 - x_1\| = 2a$ and $\|x_1 - (e_1 + e_3)\| = \|x_2 - (e_1 + e_3)\| = \|x_3 - (e_1 + e_3)\| = a$. In any case this shows that X contains three cells $S_i = S(x_i; a)$, $i = 1, 2, 3$, such that $\|x_i - x_j\| = 2a$ for $i \neq j$ and $\bigcap_{i=1}^{3} S_i \neq \phi$. Since (X, X) has EPC, it follows that X is weakly 4-hyperconvex. Hanner [7, p. 75] has proved that if dim $X = n < \infty$ and X is weakly 4-hyperconvex, then the unit cell of X is affinely equivalent to the convex hull of some of the vertices of an n-dimensional cube. As is easily seen, this implies that for any pair of extreme points e_1, e_2 of S_X with $e_1 \neq \pm e_2$, we have $\|e_1 + e_2\| = \|e_1 - e_2\| = 2$. Hence, since the four cells $S(\pm e_i; 1)$, $i = 1, 2$, have a nonempty intersection and (X, X) has EPC it follows first that P is weakly 5-hyperconvex and from this that X is 5-hyperconvex. But this implies that X is $\aleph_0$-hyperconvex and therefore X is hyperconvex, because dim $X < \infty$. This completes the proof of Theorem 2.4.

Remark. From the proofs we have given it is evident that Theorem 2.4 remains true, if the hypothesis that (X, X) has EPC is replaced by the following weaker hypothesis: If $\mathfrak{F}$ and $\mathcal{G}$ are families of cells in X such that $\mathfrak{F} \succ \mathcal{G}$, $\pi\mathfrak{F} \neq \phi$ and card $\mathfrak{F} \leqslant 4$, then $\pi\mathcal{G} \neq \phi$.

3. Pairs (X, Y) with $X = C(K)$

In this section we give some results concerning pairs of Banach spaces X and Y, such that (X, Y) has EPC and X is a $C(K)$ space, i.e. the space of real continuous functions on a compact Hausdorff space K. We first prove a simple lemma.

Lemma 3.1. *If T is a topological space, there exists a dense subset $V \subset T$ and a mapping $\mu: V \to \mathcal{A}(T)$, where $\mathcal{A}(T)$ is the family of closed subsets of T, such that*

(*i*) *$v \notin \mu(v)$ for each $v \in V$,*
(*ii*) *If $v_1, v_2 \in V$, $v_1 \neq v_2$, we have either $v_1 \in \mu(v_2)$ or $v_2 \in \mu(v_1)$.*

Proof. Let Ω denote the set of all pairs (U, ν), where $U \subset T$ and ν is a mapping of U into $\mathcal{A}(T)$ satisfying (i) and (ii). Clearly Ω is not empty. We order Ω by a relation $\leqslant$ where $(U_1, \nu_1) \leqslant (U_2, \nu_2)$ means that $U_1 \subset U_2$ and $\nu_1(u_1) = \nu_2(u_1)$ for $u_1 \in U_1$. It is clear that each linearly ordered subset of Ω has an upper bound in Ω, and so by Zorn's lemma there exists a maximal element (V, μ) in Ω. We assert that V is dense in T. Indeed, if this were not the case, we could take a point $p \in T$ with $p \notin \overline{V}$, and then $(V', \mu') \in \Omega$, where $V' = V \cup \{p\}$ and $\mu'(v) = \mu(v)$ for $v \in V$, $\mu'(p) = \overline{V}$. Since $(V, \mu) < (V', \mu')$ this would contradict the maximality of (V, μ). Hence V is dense in T and the lemma is proved.

Definition 3.2. *If T is a topological space we denote by $d(T)$ the least cardinal number d, such that T contains a dense subset with d elements.*

Lemma 3.3. *Let K be a compact Hausdorff space and $\{r_\alpha\}$, $\alpha \in A$, a family of non-negative numbers with card $A < d(K)$. Then there exists a family of cells $S(x_\alpha; r_\alpha)$, $\alpha \in A$, in $C(K)$, such that $\|x_\alpha - x_\beta\| = r_\alpha + r_\beta$ for $\alpha \neq \beta$ and $\bigcap_{\alpha \in A} S(x_\alpha; r_\alpha) \neq \phi$.*

Proof. According to Lemma 3.1 there exists a family $\{k_\alpha\}$, $\alpha \in A$, of elements of K and a family $\{F_\alpha\}$, $\alpha \in A$, of closed subsets of K, such that $k_\alpha \notin F_\alpha$ and for $\alpha \neq \beta$ we have $k_\alpha \in F_\beta$ or $k_\beta \in F_\alpha$. Since K is completely regular, there is, for each $\alpha \in A$, a continuous realvalued function x_α on K, such that $x_\alpha(k_\alpha) = r_\alpha$, $x_\alpha(k) = -r_\alpha$ for $k \in F_\alpha$ and $|x_\alpha(k)| \leqslant r_\alpha$ for $k \in K$. Thus we have $\|x_\alpha\| = \sup_{k \in K} |x_\alpha(k)| = r_\alpha$ and from the properties of $\{k_\alpha\}$ and $\{F_\alpha\}$ it follows that $\|x_\alpha - x_\beta\| = r_\alpha + r_\beta$ for $\alpha \neq \beta$. Consequently, the family of cells $\{S(x_\alpha; r_\alpha)\}$, $\alpha \in A$, in $C(K)$ has the required properties.

Definition 3.4. *Let X be a Banach space. By dim X we understand the least cardinal number γ such that X is the closed linear hull of a subset with γ elements.*

Theorem 3.5. *Let K be a compact Hausdorff space and Y a Banach space such that $d(K) \geqslant \dim Y$. If $(C(K), Y)$ has EPC then Y is a $\mathfrak{P}_1$ space.*

Proof. If dim $Y < \infty$, it suffices to show that Y is 5-hyperconvex. Except in the trivial case dim $Y = 1$, we have $d(K) \geqslant 2$. Then it is easy to see that, for any $r_i > 0$, $i = 1, \ldots, 4$, $C(K)$ contains four cells $S(x_i; r_i)$, such that $\|x_i - x_j\| = r_i + r_j$ for $i \neq j$ and $\bigcap_{i=1}^{4} S(x_i; r_i) \neq \phi$. Since $(C(K), Y)$ has EPC it follows that Y is 5-hyperconvex.

Now assume that dim Y is infinite. Since $d(K) \geqslant \dim Y$ and $(C(K), Y)$ has EPC it follows, with the aid of Lemma 3.3, that any collection $\{S_\alpha\}$, $\alpha \in A$, of mutually intersecting cells in Y with card $A = \dim Y$ has a nonempty intersection. Aronszajn and Panitchpakdi (Theorem 1 of section 2 in [1]) have proved the following result: If a metric space E is γ-hyperconvex and at the same time γ-separable, then E is hyperconvex (E is γ-separable means that E contains a dense subset of cardinality $< \gamma$). Using this result we may now conclude that Y is hyperconvex, i.e. Y is a $\mathfrak{P}_1$ space. Hence Theorem 3.5 is proved.

It is well-known that there are no separable infinite-dimensional $\mathfrak{P}_1$ spaces. Thus we get

Corollary 3.6. *If K is an infinite compact Hausdorff space and Y a separable Banach space such that $(C(K), Y)$ has EPC, then Y is a finitedimensional $\mathfrak{P}_1$ space.*

The result of Aronszajn–Panitchpakdi used in the proof of Theorem 3.5 may be improved in the case of $C(K)$ spaces.

Lemma 3.7. *Let γ be a cardinal number and K a compact Hausdorff space containing a dense subset of cardinality $< \gamma$. If $C(K)$ is γ-hyperconvex then it is hyperconvex (and hence K is extremally disconnected).*

Proof. If $\gamma \leqslant \aleph_0$ there is nothing to prove, and so we may assume that $\gamma > \aleph_0$. Let $\{S_\alpha\} = \{S(x_\alpha; r_\alpha)\}$, $\alpha \in A$, be a collection of mutually intersecting cells in $C(K)$. We have to prove that $\bigcap_{\alpha\in A} S_\alpha \neq \phi$. Let D be a dense subset of K with card $D < \gamma$ and put $f(k) = \sup_{\alpha\in A}(x_\alpha(k) - r_\alpha)$, $g(k) = \inf_{\alpha\in A}(x_\alpha(k) + r_\alpha)$. Then $f(k) \leqslant g(k)$ for $k \in K$ and $\bigcap_{\alpha\in A} S_\alpha = \{x \in C(K) : f(k) \leqslant x(k) \leqslant g(k) \text{ for } k \in K\}$. To each $k \in D$ and each natural number n we choose $\alpha = \alpha(k, n) \in A$ so that $x_\alpha(k) + r_\alpha < g(k) + (1/n)$ and $\beta = \beta(k, n) \in A$ so that $x_\beta(k) - r_\beta > f(k) - (1/n)$. The cells $S_{\alpha(k,n)}$ and $S_{\beta(k,n)}$ for $k \in D$ and $n = 1, 2, \ldots$, are mutually intersecting and form a family of cardinality $\leqslant 2$ (card D) $\aleph_0 < \gamma$. Since $C(K)$ is γ-hyperconvex it follows that there exists $x \in C(K)$, such that $x \in S_{\alpha(k,n)} \cap S_{\beta(k,n)}$ for $k \in D$, $n = 1, 2, \ldots$. This means that $f(k) - (1/n) < x(k) < g(k) + (1/n)$ for $k \in D$, $n = 1, 2, \ldots$, from which follows that $f(k) \leqslant x(k) \leqslant g(k)$ for $k \in D$. Since f is lower semicontinuous and g is upper semicontinuous and D is dense in K, this implies that $f(k) \leqslant x(k) \leqslant g(k)$ for all $k \in K$, i.e. $x \in \bigcap_{\alpha\in K} S_\alpha$ and hence $\bigcap_{\alpha\in A} S_\alpha \neq \phi$, which was to be proved.

We can now prove an improved version of Theorem 3.5 for the case that Y is a $C(K)$ space.

Theorem 3.8. *Let K_1 and K_2 be compact Hausdorff spaces such that $d(K_1) \geqslant d(K_2)$. Then $(C(K_1), C(K_2))$ has EPC if and only if K_2 is extremally disconnected.*

Proof. The if part is clear. Assume then that $(C(K_1), C(K_2))$ has EPC. Let γ be the least cardinal number such that $\gamma > d(K_1)$. Then, using Lemma 3.3, we see that $C(K_2)$ is γ-hyperconvex, and since $d(K_2) < \gamma$ it follows from Lemma 3.7 that $C(K_2)$ is hyperconvex and hence that K_2 is extremally disconnected.

Corollary 3.9. *If K is a compact Hausdorff space, then $(C(K), C(K))$ has EPC if and only if K is extremally disconnected.*

Remarks. 1. To justify the statement that Lemma 3.7 and Theorem 3.8 are improvements of the earlier results, it should be noted that there are compact Hausdorff spaces for which $d(K) < \dim C(K)$. It is easy to verify that the Stone–Čech compactification of a countable discrete space has this property. Less trivial examples may also be given. As a matter of fact, it is not difficult to prove that for any compact Hausdorff space we have $\dim C(K) = b(K)$, where $b(K)$ is the least cardinal number b, such that the topology of K has a base with b elements. It is well-known that there are K for which $d(K) < b(K)$. (See, for instance, Example M on p. 164 of Kelley [8].)

2. If K is totally disconnected, it is easy to show that $\dim C(K) = \text{card } \mathcal{H}$, where $\mathcal{H}$ is the family of all open and closed subsets of K. This fact may be used to show that the conclusion of Lemma 3.3 still holds for K, if the hypothesis card $A \leqslant d(K)$ is weakened to card $A \leqslant \dim C(K)$. This means that, if K is totally disconnected, Theorem 3.5 remains true, when the hypothesis $d(K) \geqslant \dim Y$ is replaced by $\dim C(K) \geqslant \dim Y$. Similarly, if K_1 is totally disconnected, we may replace $d(K_1) \geqslant d(K_2)$ by $\dim C(K_1) \geqslant d(K_2)$ in Theorem 3.8. We do not know whether these improvements are actually valid for all compact spaces.

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