ARKIV FÖR MATEMATIK Band 4 nr 44

Communicated 24 January 1962 by L. CARLESON

An iterative method for conformal mappings of multiply-connected domains

By INGEMAR LIND

Preliminary remarks

Let D be a connected domain in the z-plane containing the points $z=0$ and $z=\infty$. Suppose that there exists a function $w=F(z)$, analytic and univalent in D such that $F(\infty)=\infty$, $F'(\infty)=1$ and $F(0)=0$, conformally mapping D onto a domain of prescribed canonical type. There are four fundamental, harmonic functions in D in certain meanings measuring a point's deviation from its original position under the mapping:

$$\operatorname{Re}\{F(z)-z\},\quad \operatorname{Im}\{F(z)-z\},\quad \log\frac{F(z)}{z} \text{ and } \arg\frac{F(z)}{z}.$$

By studying the behaviour of these very simple types of functions under iterative mapping processes, described in detail below, existence proofs can be obtained for the simplest types of canonical domains — the parallel slit domain, the circular slit domain, the radial slit domain and the logarithmic spiral slit domain.

There are of course several simple proofs of these results. The iterative proof given here may have a certain value; it is in certain cases suitable for actual calculation of the mapping function, and it gives an explanation to the simplicity of these mappings.

The uniqueness proofs — as exemplified in [1], p. 57 (the second proof) for a parallel slit domain — are also based on simple properties of the named types of harmonic functions.

Preparations and denotations

The following paper demonstrates in detail an existence proof for conformal mapping onto a parallel slit domain where the slits are parallel to the imaginary axis. To be precise the following well-known theorem is reformulated:

Every domain D in the z-plane of connectivity k can be conformally mapped onto a parallel slit domain in the w-plane, where the k slits are parallel to the imaginary axis. Two arbitrary points α and β of D can be carried into the origin and the point of infinity, respectively. If $w=F(z)$ denotes the mapping function in question, $F(z)$ may be normalized by the requirement that its residue at $z=\beta$ be equal to 1. This normalization determines the function uniquely.

For the uniqueness proof one is referred to [1]. The existence proof is based on induction. The above theorem is true for $k=1$ (Riemann) and hereafter it is assumed true for $k-1$. With this assumption there is no loss of generality to assume that the given k-connected domain is bounded by $k-1$ rectilinear slits parallel to the imaginary axis and another non-rectilinear, analytic slit, and that α is chosen as the origin and β as the point of infinity. The following notations are used:

$D^{(n)}$: a domain of the above described type in the z_n-plane;

$L^{(n)} = \bigcup_{\nu=1}^{k} \Gamma_\nu^{(n)}$: the boundary of $D^{(n)}$; $\Gamma_\nu^{(n)}$, $\nu=1, 2 \dots k$, denote the separate boundary components one of which is non-rectilinear. Occasionally this component will be denoted by $\Gamma^{(n)}$ (lower index omitted) and its corresponding rectilinear boundary component in the z_{n-1}-plane (see below) by $\gamma^{(n-1)}$;

$D[\Gamma_j^{(n)}]$ or $D_j^{(n)}$: that domain of connectivity $k-1$ in the z_n-plane, which boundary can be written $L^{(n)} - \Gamma_j^{(n)}$ $(1 \leqslant j \leqslant k)$.

The iterative process

Assume i. e. that $\Gamma^{(1)} = \Gamma_2^{(1)}$. Map conformally $D_1^{(1)}$ in the z_1-plane onto $D_1^{(2)}$ in the z_2-plane where all the boundary slits $\Gamma_\nu^{(2)}$, $\nu = 2, 3 \dots k$ are rectilinear slits parallel to the imaginary axis. The possibility of this mapping is assured by the induction assumption. $\gamma^{(1)} = \Gamma_1^{(1)}$ in $D_1^{(1)}$ is hereby carried into a non-rectilinear, analytic slit $\Gamma_1^{(2)} = \Gamma^{(2)}$ in $D_1^{(2)}$. Next map in the same way $D_2^{(2)}$ in the z_2-plane onto $D_2^{(3)}$ in the z_3-plane, then $D_3^{(3)}$ onto $D_3^{(4)}$ and so on. The k:th step is to map $D_k^{(k)}$ onto $D_k^{(k+1)}$. Next, map $D_1^{(k+1)}$ onto $D_1^{(k+2)}$, then $D_2^{(k+2)}$ onto $D_2^{(k+3)}$ and so on. The mapping process is now iterated infinitely in the suggested cyclic manner. The analytic function performing the described mapping between the actual domain in the z_{n-1}-plane and that in the z_n-plane is denoted by $z_n = F_n(z_{n-1})$, and it is required that $F_n(0) = 0$, $F_n(\infty) = \infty$ and $F_n'(\infty) = 1$ $(n = 2, 3 \dots)$, the possibility of this again being assured by the induction assumption. Let $\psi_n(z_1) = F_n\{F_{n-1}(\,..\,(z_1)\,..)\}$ denote the composed function, analytic in $D^{(1)}$ and mapping $D^{(1)}$ conformally onto $D^{(n)}$.

Induction statement and proof

$\psi(z_1) = \lim_{n=\infty} \psi_n(z_1)$ exists, is analytic in $D^{(1)}$, maps $D^{(1)}$ conformally onto a parallel slit domain with k rectilinear slits parallel to the imaginary axis, and $\psi(0)=0$, $\psi(\infty) = \infty$, $\psi'(\infty)=1$.

Proof: The functions

$$u_n(z_n) = \mathrm{Re}\,\{F_{n+1}(z_n) - z_n\},\ n = 1, 2 \dots \tag{1}$$

are harmonic in $D[\gamma^{(n)}]$, $u_n(0)=0$, and since $F_{n+1}(z_n) - z_n$ is analytic and bounded in $D[\gamma^{(n)}]$, $u_n(z_n)$ attains its maximum and minimum values on $\Gamma^{(n)}$. With the notation

$$J_n = \sup_{z_n', z_n'' \in \Gamma^{(n)}} |\mathrm{Re}\,\{z_n' - z_n''\}| = \sup_{z_n', z_n'' \in \Gamma^{(n)}} |u_n(z_n') - u_n(z_n'')|$$

it thus follows for all z_n that

$$|u_n(z_n)| \leqslant J_n. \tag{2}$$

Since $u_n(z_n)$ is not constant, it follows from the maximum principle that there exists a number q_n, $0 < q_n < 1$, so that

$$\sup_{z_n^*, z_n^{**} \in \gamma^{(n)}} |u_n(z_n^*) - u_n(z_n^{**})| \leqslant J_n q_n.$$

Combining [4], lemma 1, p. 282 with [2], "Verzerrungssats V", p. 235 it most easily follows that it is possible to fix a number q, $0 < q < 1$, so that q_n can be chosen $\leqslant q$ for all n. Thus

$$\sup_{z_n^*, z_n^{**} \in \gamma^{(n)}} |u_n(z_n^*) - u_n(z_n^{**})| \leqslant J_n q.$$

But this is equivalent to

$$J_{n+1} = \sup_{z'_{n+1}, z''_{n+1} \in \Gamma^{(n+1)}} |\operatorname{Re}\{z'_{n+1} - z''_{n+1}\}| \leqslant J_n q.$$

Thus

$$J_n \leqslant J_1 q^{n-1} \text{ for all } n. \tag{3}$$

The functions $U_m(z_n) = \operatorname{Re}\{z_{n+m} - z_n\}$ are harmonic in $D^{(n)}$ and

$$U_m(z_n) = \sum_{\nu=0}^{m-1} u_{n+\nu}(z_{n+\nu}) \tag{4}$$

is valid. From (3) it follows for every point z_n that

$$|U_m(z_n)| \leqslant J_1 \frac{q^{n-1}}{1-q} < \varepsilon \text{ if } n > n_\varepsilon. \tag{5}$$

Thus $\operatorname{Re}\{\psi_n(z_1) - z_1\}$ converges uniformly to a harmonic function in $D^{(1)}$, and, since the conjugate functions are single-valued and well determined, the convergence is easily extended to the sequence of analytic functions $\{\psi_n(z_1) - z_1\}$. Further (3) states that the limit domain is of the desired type. Obviously $\psi(0) = 0$, $\psi(\infty) = \infty$ and $\psi'(\infty) = 1$ are valid. The induction statement is proved.

Complementary remarks

It has just been proved that $w = \psi(z_1)$ maps $D^{(1)}$ conformally onto a parallel slit domain where $u = c_\nu$, $\nu = 1, 2 \ldots k$, on the separate slits ($w = u + iv = Re^{i\theta}$; c_ν are certain real constants). Obvious modifications of the given proof yield the existence of the following mapping functions:

a) $w = \psi_\vartheta(z_1)$; $\alpha u + \beta v = c_\nu$ on the slits, $\operatorname{tg} \vartheta = -\dfrac{\alpha}{\beta}$;
b) $w = P(z_1)$; $R = c_\nu$ on the slits, (circular slit domain);
c) $w = Q(z_1)$; $\theta = c_\nu$ on the slits (radial slit domain);
d) $w = R(z_1)$; $\alpha \log R + \beta\theta = c_\nu$ on the slits (logathmic spiral slit domain), $\alpha, \beta \neq 0$.

Various kinds of iterative processes were a. o. studied by Koebe. An interesting example giving the existence of a conformal mapping onto the circle domain is found in [3].

REFERENCES

1. R. COURANT, Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces, Interscience Publishers, Inc., New York (1950).
2. P. KOEBE, Abhandlungen zur Theorie der konformen Abbildung V., Math. Z. 2. Band (1918), pp. 198–236.
3. ——, Abhandlungen zur Theorie der konformen Abbildung VI., Math. Z. 7. Band (1920), pp. 235–301.
4. L. SARIO, A linear operator method on arbitrary Riemann surfaces, Trans. Amer. Math. Soc., Vol. 72, No 2 (1952), pp. 281–295.

Tryckt den 23 november 1962

Uppsala 1962. Almqvist & Wiksells Boktryckeri AB