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# Weighted mean square approximation in plane regions, and generators of an algebra of analytic functions

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## 1. Introduction

If $D$ is a region in the complex plane, and $a(z)$ is a continuous, positive function in $D$, we denote by $H^2(a; D)$ the set of all analytic functions, $h(z)$, in $D$, which have the property that

$$\|h\|_a^2 = \int_D |h(z)|^2 a(z)\, dA < \infty,$$

where $dA$ denotes plane Lebesgue measure.

$D$ is called a Carathéodory region if it is simply connected, bounded, and its boundary, $\partial D$, coincides with the boundary of the infinite component, $D_\infty$, of the complement of the closure of $D$.

In 1934 Markuševič and Farrell proved independently that for any Carathéodory region, $D$, the polynomials are complete in $H^2(1; D)$. It is well known that this property need not hold for non-Carathéodory regions. (See e.g. [3]). The result has been generalized to spaces with weight functions other than the identity by various, notably Soviet, mathematicians. A survey of this theory is given in Mergeljan's paper [3]. Most of the results, however, deal with non-Carathéodory regions, and because of this the weight function $a(z)$ is required to tend to zero, when $z$ approaches the boundary.

For Carathéodory regions much more can be said, and the first part of this paper is devoted to this problem. The result is stated in Theorem 1.

In the second part we shall study the related problem of finding generators of the algebra, $A$, of all analytic functions, $g(w) = \sum_0^\infty g_n w^n$, in the unit disc, such that the norm, $\|g\| = \sum_0^\infty |g_n|$, is finite. By a generator of $A$ we mean a function, $\varphi$, in the algebra $A$, such that the polynomials (with constant term) $P(\varphi)$ are dense in $A$. For a function to be a generator of $A$ it is obviously necessary that it is univalent in the closed unit disc, but whether this condition is also sufficient is an open problem. D. J. Newman proved [5] that a univalent function which maps the unit disc onto a region with rectifiable boundary is a generator of $A$, and a simpler proof of this was given by H. S. Shapiro [7]. See also [6]. As a corollary to Theorem 1 we get another sufficient condition which we state as Theorem 2, and then we show by means of examples (Theorems 3 and 4) that our result neither includes, nor is included in Newman's.

I wish to acknowledge my great indebtedness to Professor Lennart Carleson, who has contributed important ideas to this work.

## 2. Polynomial approximation

If $D$ is a simply connected region, we denote by $f(z)$ a function which maps $D$ conformally onto the unit disc, and we denote the inverse function to $f(z)$ by $\varphi(w)$. We denote by $\delta(z)$ the distance from $z$ to $\partial D$. Then we have the following theorem.

**Theorem 1:** *Let $D$ be a Carathéodory region and $a(z)$ a continuous, positive function in $D$. Then the polynomials are complete in $H^2(a; D)$ if the weight, $a(z)$, satisfies the following two conditions:*

(a) $\int_D a(z)^4 \left(\log \frac{1}{\delta(z)}\right)^8 dA < \infty$

(b) *the polynomials are complete in* $H^2(a(\varphi(w)); |w| < 1)$.

*Remark:* Little seems to be known about when condition (b) holds, except the easily proved fact that it holds when $a(\varphi(w))$ depends only on $|w|$, i.e. when $a(z)$ is constant on every level curve $|f(z)| = K$. (See [3]).

As for condition (a) it would be an interesting task to try to replace it by the clearly necessary condition $\int_D a(z)\, dA < \infty$.

We need the following lemma, which it of course well known, but since there seems to be no convenient reference, we include its proof.

**Lemma:** *For every bounded, simply connected region, $D$, there is a constant $K$ such that the mapping function $f(z)$ satisfies*

$$1 - |f(z)| \leqslant K \{\delta(z)\}^{\frac{1}{2}},$$

*for all $z$ in $D$.*

*Proof:* The proof is a simple application of the Beurling–Nevanlinna estimates of harmonic measures.

Let the diameter of $D$ be $d$, and choose a positive number $\varrho < d/6$. Let $z_0$ be a point on a level curve $|f(z)| = 1 - \eta$ and let $\sigma(z_0)$ be the disc with radius $\varrho$ and centre $z_0$. Then, if $\omega(z)$ is the harmonic measure of $\partial D \cap \sigma(z_0)$ with respect to $D$ at the point $z$,

$$\omega(z_0) \geqslant \frac{2}{\pi} \arcsin \frac{\varrho - \delta(z_0)}{\varrho + \delta(z_0)},$$

by the Beurling–Nevanlinna theorem ([4] p. 104 ff.). It follows that

$$1 - \omega(z_0) \leqslant \frac{4}{\pi \varrho^{\frac{1}{2}}} \{\delta(z_0)\}^{\frac{1}{2}}.$$

When $D$ is mapped onto $|w| < 1, \partial D \cap \sigma(z_0)$ corresponds to a set, $S_{z_0}$, on $|w| = 1$, and because of the invariance of the harmonic measure $\omega(z) = \omega_1(f(z); S_{z_0})$, where $\omega_1$ is the harmonic measure with respect to the unit circle.

Now $\partial D$ can be covered by a finite number of discs, $\sigma_1, \sigma_2, \ldots, \sigma_N$, with radius $\varrho$. $\partial D$ always contains a point, $z_1$, with $|z_1 - z_0| \geqslant d/2$. $z_1$ is contained in a disc $\sigma_i$ with centre $a_i$, and then $|a_i - z_0| \geqslant d/2 - \varrho > 2\varrho$. Thus, for every $z_0$ in $D$ there is a $\sigma_i$ such that $\sigma(z_0)$ and $\sigma_i$ are disjoint. But all the sets $\partial D \cap \sigma_i$ correspond to sets of positive measure on $|w|=1$, otherwise their harmonic measures with respect to $D$ would be identically zero. It follows from the Poisson representation that there is a constant $K$, independent of $z_0$, such that

$$1-\omega_1(f(z_0), S_{z_0}) \geqslant K\eta.$$

Hence there is a constant $K$ such that $\eta \leqslant K\{\delta(z_0)\}^{\frac{1}{2}}$.

*Proof of Theorem 1:* The proof depends mainly on methods of Bers and Carleson.

We assume $D$ to be Carathéodory, and start by observing, with Mergeljan, [3] p. 136, that it is enough to show that for every $n>0$ and every $\varepsilon>0$ there is a polynomial, $P(z)$, such that

$$\int_D |f^n(z)\, f'(z)-P(z)|^2\, a(z)\, dA<\varepsilon.$$

For if $h(z)$ is arbitrary in $H^2(a; D)$, $h(\varphi(w))\,\varphi'(w)$ is clearly in $H^2(a(\varphi(w)); |w|<1)$, and thus, by condition (b), there is a polynomial, $Q(w)$, such that

$$\int_D |h(z)-Q(f(z))\, f'(z)|^2\, a(z)\, dA=\int_{|w|<1} |h(\varphi(w))\,\varphi'(w)-Q(w)|^2\, a(\varphi(w))\, dA<\varepsilon.$$

But $Q(f(z))\, f'(z)$ is a linear combination of functions $f^n(z)\, f'(z)$, and thus there is a polynomial, $P(z)$, such that

$$\int_D |Q(f(z))\, f'(z)-P(z)|^2\, a(z)\, dA<\varepsilon.$$

It follows that for every $\varepsilon>0$ there is a $P(z)$ such that

$$\int_D |h(z)-P(z)|^2\, a(z)\, dA<\varepsilon,$$

which proves this first assertion.

Any bounded linear functional, $L$, on $H^2(a; D)$ can be expressed in the form

$$L(h)=\int_D h(z)\,\mu(z)\, dA,$$

where $\mu(z)$ is a function satisfying

$$\int_D |\mu(z)|^2\, a(z)^{-1}\, dA<\infty. \tag{1}$$

We are thus required to prove that if the function

$$m(z)=\int_D \frac{\mu(\zeta)}{\zeta-z}\, dA=0$$

for all $z$ in $D_\infty$, then (1) implies that

$$\int_D f^n(z) f'(z) \mu(z)\, dA = 0, \quad n \geqslant 0.$$

Now, for $q > 0$ we define a $C^\infty$ function, $\omega_q(z)$, in $D$ with the following properties:

$$\begin{aligned}
&0 \leqslant \omega_q(z) \leqslant 1,\\
&\omega_q(z) = 0 \quad \text{for} \quad \delta(z) \leqslant q,\\
&\omega_q(z) = 1 \quad \text{for} \quad \delta(z) \geqslant 2q,\\
&|\operatorname{grad} \omega_q(z)| \leqslant K/q \text{ for some constant } K.
\end{aligned}$$

Such a function obviously exists, for the function $\delta(z)$ itself satisfies $|\delta(z_1) - \delta(z_2)| \leqslant |z_1 - z_2|$. We denote by $D_q$ the set $\{z; q \leqslant \delta(z) \leqslant 2q\}$.

We assume for the moment that $\mu(z) \in C^\infty$ and is zero outside a compact subset of $D$. Then the function $m(z)$ is continuous in the whole plane and

$$\frac{\partial m(z)}{\partial \bar{z}} = -\pi \cdot \mu(z)$$

in $D$ (see e.g. [9], p. 29). Now, following L. Bers [1], we apply Green's formula to a region $D' \subset D$, such that $\partial D'$ is smooth and contained in the set where $\delta(z) < q$. By the analyticity of $f^n(z) f'(z)$ we find

$$\begin{aligned}
-\pi \int_{D'} \omega_q(z) f^n(z) f'(z) \mu(z)\, dA &= \int_{D'} \omega_q(z) \frac{\partial}{\partial \bar{z}} (f^n(z) f'(z) m(z))\, dA \\
&= -\int_{D'} \frac{\partial \omega_q(z)}{\partial \bar{z}} f^n(z) f'(z) m(z)\, dA - \int_{\partial D'} \omega_q(z) f^n(z) f'(z) m(z)\, dz.
\end{aligned}$$

From the definition of $\omega_q(z)$ it follows that the boundary integral is zero, and that we can replace $D'$ by $D$ in the other integrals. Thus

$$\pi \int_D \omega_q(z) f^n(z) f'(z) \mu(z)\, dA = \int_{D_q} \frac{\partial \omega_q(z)}{\partial \bar{z}} f^n(z) f'(z) m(z)\, dA. \tag{2}$$

Letting $q \to 0$ we find that the integral on the left tends to $\int_D f^n(z) f'(z) \mu(z)\, dA$, and hence we have to prove that the integral on the right tends to zero.

By the definition of $\omega_q(z)$ and the Schwarz inequality

$$\begin{aligned}
&\left| \int_{D_q} \frac{\partial \omega_q(z)}{\partial \bar{z}} f^n(z) f'(z) m(z)\, dA \right|^2 \\
&\qquad \leqslant \frac{K}{q^2} \int_{D_q} |f^n(z) f'(z)|^2 \delta(z)^{-\frac{1}{2}}\, dA \cdot \int_{D_q} |m(z)|^2 \delta(z)^{\frac{1}{2}}\, dA.
\end{aligned} \tag{3}$$

Here

$$\begin{aligned}
\int_{D_q} |f^n(z) f'(z)|^2 \delta(z)^{-\frac{1}{2}}\, dA &\leqslant K q^{-\frac{1}{2}} \int_{\delta(z) \leq 2q} |f^n(z) f'(z)|^2\, dA \\
&\leqslant K q^{-\frac{1}{2}} \int_{1 - |f(z)| \leq K q^{\frac{1}{2}}} |f'(z)|^2\, dA = K q^{-\frac{1}{2}} \int_{0 < 1 - |w| < K q^{\frac{1}{2}}} dA \leqslant K,
\end{aligned}$$

where the second inequality follows from the lemma. (Throughout the paper we use $K$ to denote different constants.) Because of this and the definition of $D_q$ the left-hand side in (3) is majorized by $q^{-\frac{3}{2}}\int_{D_q}|m(z)|^2 dA$, and we have to prove that $\int_{D_q}|m(z)|^2 dA = o(q^{\frac{3}{2}})$.

By a theorem of Sobolev [8] a function

$$m_\lambda(z) = \int_D \frac{\mu(\zeta)}{|\zeta - z|^\lambda} dA$$

belongs to $L^2(D)$ if $\mu(z) \in L^p(D)$ for some $p > 1$ such that $\frac{1}{2} \geq 1/p + \lambda/2 - 1$, and then

$$\left\{\int_D |m_\lambda(z)|^2 dA\right\}^{\frac{1}{2}} \leq K\left\{\int_D |\mu(z)|^p dA\right\}^{1/p}, \tag{4}$$

where $K$ depends on $p, \lambda$ and $D$.

By Hölder's inequality, for $p < 2$

$$\int_D |\mu(z)|^p dA \leq \left\{\int_D |\mu(z)|^2 a(z)^{-1} dA\right\}^{p/2} \left\{\int_D a(z)^{p/(2-p)}\right\}^{1-p/2}. \tag{5}$$

The second integral on the right is finite as soon as $p/(2-p) \leq 4$, i.e. $p \leq 8/5$, by assumption (a), and so $\mu(z) \in L^p(D)$ for $p \leq 8/5$, by (1).

Now we can remove the regularity hypothesis on $\mu(z)$ and prove that (2) holds for all $\mu(z)$ satisfying (1). For if $\lambda = 1$, (4) holds for all $p > 1$, and thus by (3), (4), and (5)

$$\left|\int_{D_q} \frac{\partial \omega_q(z)}{\partial \bar{z}} f^n(z) f'(z) m(z) dA\right|^2 \leq K q^{-\frac{3}{2}} \int_D |\mu(z)|^2 a(z)^{-1} dA.$$

If we apply the Schwarz inequality to the left-hand side in (2) we find

$$\left|\int_D \omega_q(z) f^n(z) f'(z) \mu(z) dA\right|^2 \leq \int_D |f^n(z) f'(z)|^2 a(z) dA \int_D |\mu(z)|^2 a(z)^{-1} dA.$$

Because for any $\mu(z)$ satisfying (1) there is a sequence $\{\mu_n(z)\}_1^\infty$ of functions in $C^\infty$ such that $\int_D |\mu_n(z) - \mu(z)|^2 a(z)^{-1} dA \to 0$, these two inequalities show that (2) holds for all such $\mu(z)$.

Assuming that $m(z) = 0$ for $z \in D_\infty$ we shall now estimate $m(z)$ in $D$ by means of a device due to Carleson, [2], and show that $\int_{D_q}|m(z)|^2 dA = o(q^{\frac{3}{2}})$.

Fix a $z \in D$ and let $z_0 \in \partial D$ be such that $|z - z_0| = \delta(z)$. Let $C_z$ be the disc with centre $z_0$ and radius $\delta(z)$. Every circle $|s - z_0| = \varrho$ intersects the open set $D_\infty$ because of the Carathéodory property, and hence there is a measure $d\sigma(s)$ which is supported by linear segments in $D_\infty \cap C_z$, such that

$$\int_{|s - z_0| \leq \varrho} d\sigma(s) = \varrho$$

for all $\varrho \leq \delta(z)$. As $m(z) = 0$ in $D_\infty$ we find

$$m(z) = \frac{1}{\delta(z)} \int (m(z) - m(s)) d\sigma(s) = \frac{1}{\delta(z)} \int_D \frac{\mu(\zeta)}{\zeta - z} dA_\zeta \int \frac{z - s}{\zeta - s} d\sigma(s), \tag{6}$$

where the change in the order of integration is permissible, because it follows from the estimates below that

$$\int_D |\mu(\zeta)|\,dA \int \frac{d\sigma(s)}{|\zeta-s|} < \infty.$$

In (6) we always have $|z-s| \leqslant 2\delta(z)$. For $\delta(\zeta) \geqslant \frac{1}{2}\delta(z)$ (and $|\zeta-z| \leqslant 4\delta(z)$) we have $|\zeta-s| \geqslant \delta(\zeta) \geqslant \frac{1}{2}\delta(z)$, and hence, in this case

$$\frac{1}{\delta(z)}\left|\int \frac{z-s}{\zeta-s}\,d\sigma(s)\right| \leqslant 4. \tag{7}$$

If $|\zeta-z| \geqslant 4\delta(z)$ we have $|\zeta-s| \geqslant |\zeta-z| - |z-s| \geqslant \frac{1}{2}|\zeta-z|$, and thus

$$\frac{1}{\delta(z)}\left|\int \frac{z-s}{\zeta-s}\,d\sigma(s)\right| \leqslant \frac{4\delta(z)}{|\zeta-z|}. \tag{8}$$

Finally we assume that $\delta(\zeta) \leqslant \frac{1}{2}\delta(z)$ and $|\zeta-z| \leqslant 4\delta(z)$. We let $\zeta_0$ be a point on $\partial D$ such that $|\zeta-\zeta_0| = \delta(\zeta)$. For every $s$ in $D_\infty$, $|\zeta-s| \geqslant \delta(\zeta)$, and if $s$ also satisfies $|\zeta_0-s| \geqslant 2\delta(\zeta)$, we have $|\zeta-s| \geqslant |\zeta_0-s| - \delta(\zeta) \geqslant \frac{1}{2}|\zeta_0-s|$. If we put $|s-z_0| = r$ and $|\zeta_0-z_0| = r_0$, then $|\zeta_0-s| \geqslant |r-r_0|$, and it follows that $|\zeta-s| \geqslant \frac{1}{2}|r-r_0|$ if $|r-r_0| \geqslant 2\delta(\zeta)$. Hence, by the definition of $d\sigma$,

$$\int \frac{d\sigma(s)}{|\zeta-s|} \leqslant \int \frac{dr}{\delta(\zeta)} + 2\int \frac{dr}{|r-r_0|},$$

where the first integral is taken over all $r$ with $|r-r_0| \leqslant 2\delta(\zeta)$, and the second over all $r$ with $|r-r_0| \geqslant 2\delta(\zeta)$ and $0 \leqslant r \leqslant \delta(z)$. The second integral is clearly greatest when $r_0 = \frac{1}{2}\delta(z)$, and it follows that

$$\frac{1}{\delta(z)}\left|\int \frac{z-s}{\zeta-s}\,d\sigma(s)\right| \leqslant K_1 \log\frac{\delta(z)}{\delta(\zeta)} + K_2 \leqslant K_1 \log\frac{1}{\delta(\zeta)} + K_2, \tag{9}$$

since $\delta(z)$ is bounded by a constant.

Now $m(z)$ can be written as the sum of three integrals $m_i(z)$, $i=1,2,3$, corresponding to the domains $A_i$ where (7), (8), and (9) hold respectively. It is thus sufficient to show that for $i=1,2,3$

$$\int_{D_q} |m_i(z)|^2\,dA = o(q^{\frac{3}{2}}).$$

By (7) we find

$$\begin{aligned}\int_{D_q} |m_1(z)|^2\,dA &\leqslant K\int_{D_q}\left\{\int_{A_1}\frac{|\mu(\zeta)|}{|\zeta-z|}\,dA_\zeta\right\}^2 dA_z\\ &\leqslant K\int_{D_q}\left\{\delta(z)^{\frac{3}{4}}\int_{A_1}\frac{|\mu(\zeta)|}{|\zeta-z|^{7/4}}\,dA_\zeta\right\}^2 dA_z\\ &\leqslant Kq^{\frac{3}{2}}\int_{D_q}\left\{\int_D\frac{|\mu(\zeta)|}{|\zeta-z|^{7/4}}\,dA_\zeta\right\}^2 dA_z.\end{aligned}$$

But by (4) and (5)

$$\int_D \left\{ \int_D \frac{|\mu(\zeta)|}{|\zeta - z|^{7/4}} dA_\zeta \right\}^2 dA_z \leqslant K \left\{ \int_D |\mu(\zeta)|^{8/5} dA \right\}^{\frac{5}{4}} < \infty,$$

and it follows that if we put $q = 2^{-\nu}$, $\nu = 1, 2, \ldots,$

$$\lim_{\nu\to\infty} \int_{D_q} \left\{ \int_D \frac{|\mu(\zeta)|}{|\zeta - z|^{7/4}} dA_\zeta \right\}^2 dA_z = 0$$

which proves our assertion for $m_1(z)$.

For $m_2(z)$ we find by (8) that

$$\int_{D_q} |m_2(z)|^2 dA \leqslant K \int_{D_q} \left\{ \delta(z) \int_{A_2} \frac{|\mu(\zeta)|}{|\zeta - z|^2} dA_\zeta \right\}^2 dA_z$$

$$\leqslant K q^{\frac{3}{2}} \int_{D_q} \left\{ \int_D \frac{|\mu(\zeta)|}{|\zeta - z|^{7/4}} dA_\zeta \right\}^2 dA_z = o(q^{\frac{3}{2}}),$$

as above.

Similary, in the third case it suffices to prove that

$$\int_{D_q} \left\{ \int_D \frac{|\log \delta(\zeta)|\,|\mu(\zeta)|}{|\zeta - z|^{7/4}} dA_\zeta \right\}^2 dA_z = o(q^{\frac{3}{2}}).$$

If we replace the inequality (5) by

$$\int_D |\mu(z)|^p |\log \delta(z)|^p dA$$

$$\leqslant \left\{ \int_D |\mu(z)|^2 a(z)^{-1} dA \right\}^{p/2} \left\{ \int_D a(z)^{p/(2-p)} |\log \delta(z)|^{2p/(2-p)} dA \right\}^{1-p/2},$$

this case follows as before, by assumption (a), and the proof of Theorem 1 is complete.

*Remark:* It is easily seen from the proof that if $\partial D$ is so regular that $1 - |f(z)| \leqslant K\{\delta(z)\}^\alpha$ for some $\alpha$ in $\frac{1}{2} < \alpha \leqslant 1$, assumption (a) can be replaced by

$$\int_D (a(z)\,|\log \delta(z)|^2)^{2/\alpha} dA < \infty.$$

## 3. Application to a generator problem, and examples

Applied to the generator problem stated in the introduction, Theorem 1 gives the following result. For notation see the introduction.

**Theorem 2:** *A function $\varphi(w) = \sum_0^\infty \varphi_n w^n$ is a generator for $A$ if it is univalent in $|w| \leqslant 1$, and if, for some $\alpha > 12$,*

$$\sum_2^\infty n(\log n)^\alpha |\varphi_n|^2 < \infty.$$

*Remark 1:* Note that the condition that $\varphi$ is in $A$ and is univalent implies that $\sum_1^\infty n|\varphi_n|^2 < \infty$.

*Remark 2:* In the case when $\sum_1^\infty n^{1+\alpha}|\varphi_n|^2 < \infty$ for some $\alpha > 0$, H. S. Shapiro has recently obtained a simple direct proof of the above theorem (private communication).

*Proof:* $\varphi$ maps the unit disc onto a region $D$ which is bounded by a Jordan curve. Let the inverse of $\varphi$ be $f$.

It is easy to prove by means of Parseval's relation and elementary estimates, that for any $g(w) = \sum_0^\infty g_n w^n$ the condition $\sum_2^\infty n(\log n)^\alpha |g_n|^2 < \infty$ is equivalent to

$$\int_{|w|<1} |g'(w)|^2 \left(\log \frac{1}{1-|w|}\right)^\alpha dA < \infty. \tag{10}$$

Thus, if we apply this to $\varphi$, and pass to $D$, we find

$$\int_D \left(\log \frac{1}{1-|f(z)|}\right)^\alpha dA < \infty,$$

for some $\alpha > 12$. It follows, by the lemma and by the remark following Theorem 1, that the weight function

$$a_\beta(z) = 1 + \left(\log \frac{1}{1-|f(z)|}\right)^{1+\beta}, \; z \in D,$$

satisfies all the conditions in Theorem 1, whenever $\beta \leqslant \alpha/4 - 3$.

Furthermore, by Cauchy's inequality,

$$\|g\| = \sum_0^\infty |g_n| \leqslant |g_0| + \left\{\sum_1^\infty \frac{1}{n + n(\log n)^{1+\beta}}\right\}^{\frac{1}{2}} \left\{\sum_1^\infty (n + n(\log n)^{1+\beta})|g_n|^2\right\}^{\frac{1}{2}}. \tag{11}$$

It is enough to show that for every $\varepsilon > 0$ there is a polynomial $P$ such that $\|w - P(\varphi(w))\| < \varepsilon$. But by (10) and (11), for $\beta > 0$,

$$\|w - P(\varphi(w))\| \leqslant |P(\varphi(0))| + K\left\{\int_{|w|<1} |1 - P'(\varphi(w))\varphi'(w)|^2 a_\beta(\varphi(w))\, dA\right\}^{\frac{1}{2}}$$
$$= |P(\varphi(0))| + K\left\{\int_D |f'(z) - P'(z)|^2 a_\beta(z)\, dA\right\}^{\frac{1}{2}}.$$

Here the last integral can be made less than $\varepsilon$ (if $\beta \leqslant \alpha/4 - 3$) by Theorem 1, for

$$\int_D |f'(z)|^2 a_\beta(z)\, dA = \int_{|w|<1} \left(1 + \left(\log \frac{1}{1-|w|}\right)^{1+\beta}\right) dA,$$

which is certainly finite. Then we can choose $P(\varphi(0)) = 0$, which proves the theorem.

Our result is neither included in, nor does it include, Newman's theorem. This is a consequence of the following two constructions.

**Theorem 3:** *There is a region $D$, bounded by a non rectifiable Jordan curve, such that the Riemann mapping function $f(z)$ satisfies*

$$\int_D (1-|f(z)|)^{-\alpha}\, dA < \infty$$

*for every $\alpha < 1$.*

*Proof:* We shall construct inductively a sequence of regions, $\{D_n\}_0^\infty$, such that $D_n \subset D_{n+1}$, and then define $D = \bigcup_{n=0}^{\infty} D_n$.

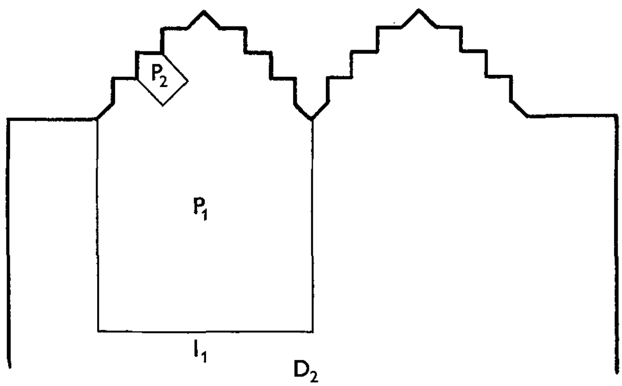


Fig. 1

See Fig. 1. We let $D_0$ be a square with sides of unit length. We divide one of the sides in three parts so that the length of the middle part is $1/\sqrt{2}$, and the lengths of the other parts are $\frac{1}{2} - 1/2\sqrt{2}$. We choose a number, $N_1$, and divide the middle part in $N_1$ equal parts, and then we let each of these parts be the base of an isosceles right triangle, which lies outside $D_0$. The union of $D_0$ and these $N_1$ triangles is $D_1$.

To construct $D_2$ we first divide each of the $2N_1$ legs of the isosceles triangles of $D_1 - D_0$ in three part in the same proportions as above. Then we choose a number, $N_2$, (to be determined later) which is a multiple ($\geqslant 2N_1$) of $N_1$, and divide each of the middle parts in $N_2/N_1$ equal parts, and add isosceles right triangles lying outside $D_1$ as above. The length of the legs of one of these triangles is clearly $1/4N_2$. The union of $D_1$ and these $2N_2$ triangles is $D_2$.

Now assume that $D_n$ is constructed and that $D_n - D_{n-1}$ consists of $2^{n-1}N_n$ isosceles right triangles with legs $1/2^n N_n$. To construct $D_{n+1}$ we choose a multiple, $N_{n+1}$ ($\geqslant 2N_n$), of $N_n$ and divide the $2^n N_n$ legs of the triangles constituting $D_n - D_{n-1}$ in three parts as before. Then we divide the middle parts in $N_{n+1}/N_n$ equal parts and add isosceles right triangles lying outside $D_n$ as above. We evidently have $2^n N_{n+1}$ such triangles whose legs are $1/2^{n+1}N_{n+1}$.

It is easy to see that the $\partial D$ so constructed is a Jordan curve. The difference in length between $\partial D_{n+1}$ and $\partial D_n$ is $1-1/\sqrt{2}$ for all $n$, and thus $\partial D$ is not rectifiable. We shall show that the numbers $N_n$ can be chosen so that $D$ satisfies the requirement in the theorem.

It is enough to show that the Green's function, $g(z)$, of $D$, with the centre of $D_0$ as pole, satisfies $\int_D g(z)^{-\alpha}dA<\infty$ for all $\alpha<1$.

We shall determine the sequence $\{N_n\}_1^\infty$ inductively. We assume that the numbers $N_i$, $1\leqslant i\leqslant n$, are already chosen. Let the Green's function of $D_n$ with pole at the centre of $D_0$ be $g_n(z)$. Then, if $I_n$ is the subset of $\partial D_n$ to which the triangles of $D_{n+1}$ are to be joined, the inner normal derivative, $\partial g_n(z)/\partial n$, of $g_n(z)$ is continuous on and near $I_n$, and

$$\operatorname*{Min}_{z\in I_n}\frac{\partial g_n(z)}{\partial n}=2\eta_n>0.$$

Thus there exists an $\varepsilon_n>0$, such that if $z_0\in I_n$, if $z\in D_n$ lies on the normal to $\partial D_n$ through $z_0$, and if $|z-z_0|<\varepsilon_n$, then

$$g_n(z)>\eta_n|z-z_0|.$$

We choose
$$N_{n+1}\geqslant \operatorname{Max}(\sqrt{2}/2^{n+1}\varepsilon_n, \exp(1/\eta_n)),$$

and complete the construction of $D$ by choosing $N_1$ arbitrarily.

For every $n$, $0<g_n(z)<g(z)$ in $D_n$. Thus, for $\alpha>0$,

$$\int_D g(z)^{-\alpha}dA<\int_{D_1}g_1(z)^{-\alpha}dA+\sum_2^\infty\int_{D_n-D_{n-1}}g_n(z)^{-\alpha}dA.$$

If one of the triangles in $D_n-D_{n-1}$ is extended into $D_{n-1}$ by a square (which then has the side $\sqrt{2}/2^nN_n$), we have on the side, $l_n$, of the resulting pentagon, $P_n$, which faces the triangle,

$$g_n(z)>g_{n-1}(z)>\eta_{n-1}\sqrt{2}/2^nN_n,$$

for by the choice of $N_n$, $\sqrt{2}/2^nN_n\leqslant\varepsilon_{n-1}$. Hence in $P_n$,

$$g_n(z)>(\eta_{n-1}\sqrt{2}/2^nN_n)\,\omega_n(z),$$

where $\omega_n(z)$ is the harmonic measure of $l_n$ with respect to $P_n$. But

$$\int_{P_n}\omega_n(z)^{-\alpha}dA=K(1/2^nN_n)^2\int_{P_1}\omega_1(z)^{-\alpha}dA,$$

by the invariance of the harmonic measure, and the last integral is finite for all $\alpha<1$, because the angles in $P_1$ are all greater or equal to $\pi/2$. It follows that

$$\int_{D_n-D_{n-1}}g_n(z)^{-\alpha}dA\leqslant K\,2^{n(\alpha-1)}N_n^{\alpha-1}\eta_{n-1}^{-\alpha}\leqslant K\,2^{n(\alpha-1)}N_n^{\alpha-1}(\log N_n)^\alpha,$$

by the choice of $N_n$. But $N_n^{1-\alpha}(\log N_n)^\alpha$ is bounded as $n\to\infty$. Hence $\int_D g(z)^{-\alpha}dA<\infty$ for all $\alpha<1$.

**Theorem 4:** *Let $k(t)$ be a decreasing function for $0<t\leqslant 1$, such that $\lim_{t\to 0}k(t)=\infty$. Then there is a region $D$, bounded by a rectifiable Jordan curve, which is such that the mapping function $f(z)$ satisfies*

$$\int_D k(1-|f(z)|)\,dA=\infty.$$

*Proof:* Let $\{\sigma_i\}_0^\infty$ be a sequence of discs with centres at the points $a_i$ on the $x$-axis, and radii $r_i$ with

$$r_i<a_{i+1}-a_i<r_i+r_{i+1}$$

for all $i$. Let $D_n=\bigcup_0^n\sigma_i$ and $D=\bigcup_0^\infty\sigma_i$. See Fig. 2. We shall prove that the sequences $\{a_i\}$ and $\{r_i\}$ can be chosen so that $D$ fulfils the requirements.

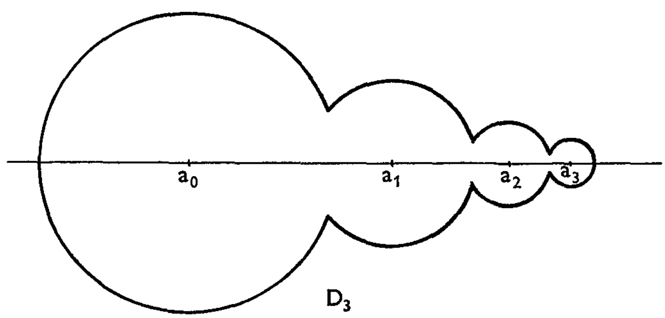


Fig. 2

We first choose $\{r_i\}$ so that $\sum_0^\infty r_i<\infty$. Then $D$ is clearly bounded by a rectifiable Jordan curve.

For given $\{a_i\}$ we let $\omega(z)$ be the harmonic measure with respect to $D-\sigma_0$ of $\partial\sigma_1\cap\sigma_0$. If $g(z)$ is the Green's function of $D$ with pole at $a_0, g(z)$ is bounded on $\partial\sigma_1\cap\sigma_0$ by a constant $C$, which is independent of the choice of $a_i, i>1$. This follows from the fact that there exists a region which has $\partial\sigma_1\cap\sigma_0$ as a part of its boundary, and which contains $D-\sigma_0$ for every choice of $a_i, i>1$. Then $g(z)\leqslant C\omega(z)$ in $D-\sigma_0$. Also, if we assume that $f(a_0)=0, 1-|f(z)|\leqslant\log 1/|f(z)|=g(z)$. It is therefore enough to show that we can make $\int_{D-\sigma_0}k(C\omega(z))\,dA=\infty$.

In a disc $\sigma_i, i\geqslant 1, \omega(z)$ is always less than the harmonic measure with respect to $\sigma_i$ of the part of $\partial\sigma_i$ which is contained in $\sigma_{i-1}\cup\sigma_{i+1}$. That is, $\omega(a_i)$ is majorized by the sum of the central angles corresponding to the arcs $\partial\sigma_i\cap\sigma_{i-1}$ and $\partial\sigma_i\cap\sigma_{i+1}$. Thus, for any given positive sequence, $\{t_i\}_2^\infty$, we can clearly choose the $a_i$ inductively in such a way that $2C\omega(a_i)\leqslant t_i, i>1$.

By Harnack's inequality $\omega(z)\leqslant 2\omega(a_i)$ in the disc, $\sigma_i'$, with centre $a_i$ and radius $r_i/3$, and it follows that

$$\int_{D-\sigma_0}k(C\omega(z))\,dA\geqslant\sum_2^\infty\int_{\sigma_i'}k(C\omega(z))\,dA\geqslant\frac{\pi}{9}\sum_2^\infty k(t_i)\,r_i^2.$$

But the sequence $\{t_i\}_2^\infty$ can be chosen so that $\sum_2^\infty k(t_i)\,r_i^2=\infty$, and this proves the theorem.

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