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Asymptotic estimates for spectral functions connected with hypoelliptic differential operators

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1. Introduction

Let $x=(x_1,\ldots,x_n)$ be coordinates in R^n and put $D_k=(2\pi i)^{-1}\partial/\partial x_k$ and $D^\alpha=D_1^{\alpha_1}\ldots D_n^{\alpha_n}$. We shall consider a hypoelliptic differential operator $M(D)=\sum M_\alpha D^\alpha$ with constant coefficients. Let us assume that the coefficients M_α are real, so that $M(D)$ is formally self-adjoint. Moreover, we suppose that $M(\xi)\to+\infty$ when $|\xi|\to\infty$, $\xi\in R^n$.

If S is an open subset of R^n and we define $M(D)$ on $C_0^\infty(S)$, we get a symmetric linear operator a_0 in the Hilbert space $L^2(S)$. We let A be a self-adjoint extension of a_0. Then by the spectral theorem A has a spectral resolution $E(\lambda)$ of commuting projection operators increasing with λ (Nagy [7]). The operator $E(\lambda)$ is given by a kernel $e_\lambda(x,y)$, the spectral function of A: $E(\lambda)u(x)=\int_S e_\lambda(x,y)u(y)\,dy$, where e_λ is infinitely differentiable in $S\times S$. This is proved in Hörmander [5] in the case where A is semi-bounded and in this paper in the general case.

If in particular $S=R^n$, there is a unique self-adjoint extension A_0 of a_0 with a corresponding spectral function $e_{0,\lambda}$ which is easily computed by a Fourier transformation:

$$e_{0,\lambda}(x,y)=\int_{M(\xi)\leqslant\lambda}\exp\left(2\pi i\langle x-y,\xi\rangle\right)d\xi.$$

We are going to give a result on the behaviour of $e(\lambda)=e_{0,\lambda}(x,x)$ when $\lambda\to+\infty$. We shall show (Theorem 1) that there are real numbers a and t, $a>0$ and t an integer $\geqslant 0$ such that for some number $k>0$

$$k^{-1}\lambda^a(\log\lambda)^t\leqslant e(\lambda)\leqslant k\lambda^a(\log\lambda)^t \quad (\lambda \text{ large})$$

and

$$e'(\lambda)=0(1)\,\lambda^{a-1}(\log\lambda)^t \quad (\lambda\to+\infty).$$

An analogous result holds for the derivatives of $e_{0,\lambda}$ with respect to x,y:

$$e_{0,\lambda}^{(\alpha,\alpha)}(x,x)=\int_{M(\xi)\leqslant\lambda}\xi^{2\alpha}d\xi.$$

If $n=2$, we shall prove a sharper result, namely $e(\lambda)=k(1+o(1))\,\lambda^a(\log\lambda)^t$ for a positive number k, and where $t=0$ or $t=1$.

The proof of Theorem 1 uses analytic continuation properties of the function

$e(\lambda)$, which follow from results in the author's paper [8]. In particular cases the asymptotic behaviour of $e(\lambda)$ has been investigated by Gortjakov [3], who then also computed the numbers a and t. Further we prove an asymptotic result for $e_\lambda(x,y)$ when S is arbitrary. For this we show an estimate for a fundamental solution of $(M(D)-\lambda)$ when $\lambda\to-\infty$, and apply a Tauberian theorem of Ganelius for the Stieltjes transformation. We get the following result (Theorem 2), valid also in the not semi-bounded case,

$$|e_\lambda(x,x)-e_{0,\lambda}(x,x)|=O(1)\,\lambda^{a-b}(\log\lambda)^t \quad (\lambda\to+\infty),$$

where a, t correspond to the polynomial $M(\xi)$ as above and $b>0$ is the largest number such that

$$|\text{grad}\, M(\xi)|\leqslant C(|M(\xi)|+1)^{1-b} \quad (\forall\xi\in R^n)$$

for some number C. When $\lambda\to-\infty$, we have with some $c>0$

$$e_\lambda(x,y)=O(1)\exp(-c|\lambda|^b).$$

2. Notations. The spectral function

We introduce the following customary notations. If α is a multi-index $(\alpha_1,\ldots,\alpha_n)$, where the α_i are non-negative integers, we put $|\alpha|=\alpha_1+\ldots+\alpha_n$ and $\xi^\alpha=\xi_1^{\alpha_1}\cdot\ldots\cdot\xi_n^{\alpha_n}$ with $\xi=(\xi_1,\ldots,\xi_n)$. We write $D_j=(2\pi i)^{-1}\partial/\partial x_j\,(j=1,\ldots,n)$ and $D=(D_1,\ldots,D_n)$. Let $M(\xi)=\Sigma M_\alpha\xi^\alpha$ be a complex polynomial in $\xi_1,\ldots,\xi_n,n\geqslant 2$. Then it corresponds to a differential operator $M(D)=\Sigma M_\alpha D^\alpha$ with constant coefficients. We shall assume that it is hypoelliptic, i.e. (Hörmander [5]) that

$$M^{(\alpha)}(\xi)/M(\xi)\to 0 \quad (|\xi|\to\infty, \xi \text{ real}), \tag{1}$$

where $M^{(\alpha)}(\xi)=D^\alpha M(\xi)$, and the relation (1) holds for all α with $|\alpha|>0$. Moreover, we suppose that M is real. Then it follows from (1) that either $M(\xi)\to+\infty$ or $M(\xi)\to-\infty$ when $|\xi|\to\infty$ (ξ real). Let us choose the sign of M so that $M(\xi)\to+\infty$. Let S be an open subset of R^n. We shall then work in the Hilbert space $L^2(S)$ with inner product $(u,v)=\int_S u(x)\overline{v(x)}dx$ and norm $\|u\|=(u,u)^{\frac{1}{2}}$. If we define $M(D)$ on the set $C_0^\infty(S)$ of all infinitely differentiable functions, which vanish outside compact subsets of S, we get a linear operator a_0 in $L^2(S)$, which is also symmetric, since M is real so that $M(D)$ is formally self-adjoint. Let us assume that A is a self-adjoint extension of a_0 and that A is bounded from below, $A\geqslant\lambda_0 I$, say, where I is the identity operator in $L^2(S)$. In the sense of Nagy [7], to A there corresponds a spectral resolution $E(\lambda)$, which is a projection-valued, non-decreasing function on the real line. We have $E(\lambda)=0$ for $\lambda<\lambda_0$. Since M is hypoelliptic, the following statement holds (Hörmander [5]).

To every multi-index α there is a positive integer r such that $D^\alpha u$ is continuous (i.e. there is a continuous function v such that $D^\alpha u=v$ in the distributional sense) for every distribution u such that $M(D)^r u$ is locally square integrable, and we have an inequality

$$\sup_{x\in K}|D^\alpha u(x)|\leqslant C(\|M(D)^r u\|+\|u\|), \tag{2}$$

where K is any compact subset of S, and C is independent of u but may depend on

α, r, K, and S. Of course, $\sup_{x\in K}|D^\alpha u(x)|$ means $\sup_{x\in K}|v(x)|$, where v is the continuous function equivalent to $D^\alpha u$. By (2) it may be shown (Hörmander [5]) that $E(\lambda)$ is given by a kernel $e_\lambda(x,y)$, the spectral function of A, such that

$$E(\lambda)\,u(x)=\int_S e_\lambda(x,y)u(y)dy \quad (u\in L^2(S)),$$

where e_λ is defined and infinitely differentiable in $S\times S$. Further $e_\lambda(x,y)=\overline{e_\lambda(y,x)}$ for all $x,y\in S$. We also have an estimate

$$e_\lambda^{(\alpha,\beta)}(x,y)\equiv i^{|\alpha+\beta|}D_x^\alpha D_y^\beta\, e_\lambda(x,y)=0(1)\,\lambda^{p+k(|\alpha+\beta|)} \tag{3}$$

when $\lambda\to+\infty$, uniformly on compact subsets of $S\times S$, with some positive numbers p and k and all α,β. For e_λ we also have the following lemma.

Lemma 1. For any α and any $x\in S$, $e_\lambda^{(\alpha,\alpha)}(x,x)$ is an increasing function of λ, and the variation with respect to λ on any real interval Λ satisfies the inequality

$$\operatorname*{var}_\Lambda e_\lambda^{(\alpha,\beta)}(x,y)\leqslant(\operatorname*{var}_\Lambda e_\lambda^{(\alpha,\alpha)}(x,x)\cdot\operatorname*{var}_\Lambda e_\lambda^{(\beta,\beta)}(y,y))^{\frac{1}{2}}$$

for all $x,y\in S$ and all α,β.

Proof. For the proof we refer to Bergendal [1], the Lemmas 1.2.2 and 1.2.1. There the lemma is proved for the spectral function of an elliptic operator, but the proof only uses that $(e_\lambda-e_\mu)$ is the kernel of an orthogonal projection if $\lambda>\mu$, and so it works as well in our case.

In particular it follows from the lemma that for any x,y,α,β the function $e_\lambda^{(\alpha,\beta)}(x,y)$ is locally of bounded variation. If $\lambda<\lambda_0$, then $G(\lambda)=(A-\lambda I)^{-1}$ exists as a bounded operator in $L^2(S)$, and $\|(A-\lambda I)^{-1}\|\leqslant(\lambda_0-\lambda)^{-1}$. If the integral of a real function with respect to a spectral measure is defined as in Nagy [7], then $G(\lambda)=\int_{\lambda_0}^{+\infty}(\mu-\lambda)^{-1}dE(\mu)$. If in (3) the number p is smaller than 1, then

$$G_\lambda(x,y)=\int_{\lambda_0}^{+\infty}(\mu-\lambda)^{-1}de_\mu(x,y) \tag{4}$$

is defined as a continuous function in $S\times S$ (this is seen e.g. by an integration by parts). From the definition of the integral with respect to a spectral measure (Nagy [7]) it follows that on $C_0^\infty(S)$ (and also on $L^2(S)$), G_λ is the kernel of $(A-\lambda)^{-1}$. We shall call G_λ Green's function corresponding to A. We see that for $\varphi\varepsilon C_0^\infty(S)$ the function $\psi(x)=(G_\lambda(x,\cdot),\varphi)$ is continuous (no correction is needed). If in (3) also $(p+k|\alpha+\beta|)<1$, we get from (3) that $G_\lambda^{(\alpha,\beta)}$ is continuous in $S\times S$, and

$$G_\lambda^{(\alpha,\beta)}(x,y)=\int_{\lambda_0}^{+\infty}(\mu-\lambda)^{-1}de_\mu^{(\alpha,\beta)}(x,y). \tag{5}$$

If instead of A we consider the operator $B=A^r$ with a positive integer r, then B is self-adjoint and bounded from below, and B is further an extension of $M(D)^r$, defined on $C_0^\infty(S)$. Since $M(D)^r$ is hypoelliptic, B has a spectral function $e_{r,\lambda}(x,y)$, and for large λ we have $e_{r,\lambda}=e_{\lambda^{1/r}}$. Hence, taking r large enough, we may make the exponent in (3) smaller than 1, if we have $e_{r,\lambda}$ instead of e_λ. Hence, for any M,α and β, (5) holds for the Green's function and the spectral function of A^r if we take r large enough.

We have a particularly simple case when the set S is the whole of R^n. Then the Fourier transform

$$\mathfrak{F}f(\xi)=\int \exp(-2\pi i\langle x,\xi\rangle)f(x)\,dx$$

taken in the sense of Schwartz [11] is a unitary mapping of $L^2(S)$ into $L^2(R^n)$, and $\mathfrak{F}a_0\mathfrak{F}^{-1}$ is multiplication by $M(\xi)$. Hence a_0 has a unique self-adjoint extension A_0, and since the spectral resolution $\hat{E}_r(\lambda)$ of $\hat{A}_0^r=\mathfrak{F}A_0^r\mathfrak{F}^{-1}$ is multiplication by the characteristic function of the set $\{\xi \mid M(\xi)^r\leqslant\lambda\}$ and the operator $(\hat{A}^r-\lambda)^{-1}$ is multiplication by $(M(\xi)^r-\lambda)^{-1}$, we have

$$e_{0,r,\lambda}(x,y)=\int_{M(\xi)^r\leqslant\lambda}\exp(2\pi i\langle x-y,\xi\rangle)\,d\xi$$

and

$$G_{0,r,\lambda}(x,y)=\int(M(\xi)^r-\lambda)^{-1}\exp(2\pi i\langle x-y,\xi\rangle)\,d\xi. \tag{6}$$

The integral is absolutely convergent for large negative λ if r is large enough, since for a hypoelliptic polynomial $M(\xi)$ we have $|M(\xi)|\geqslant C|\xi|^c$ for all large real ξ with some positive constants c and C (Hörmander [5]).

We now give a result on the asymptotic behaviour of $e_{0,\lambda}(x,x)$, when λ tends to $+\infty$.

Theorem 1. Let $P(\xi_1,\dots,\xi_n)$ be a real polynomial such that $P(\xi_1,\dots,\xi_n)\to+\infty$ when $|\xi|\to\infty$ (ξ real) and let α be a multi-index. If

$$e(\lambda)=\int_{P(\xi)\leqslant\lambda}\xi^{2\alpha}d\xi,$$

then there are positive numbers c, C, and a, and a non-negative integer t such that

$$C^{-1}\lambda^a(\log\lambda)^t\leqslant e(\lambda)\leqslant C\lambda^a(\log\lambda)^t \quad (\lambda>c)$$

and

$$e'(\lambda)=0(1)\lambda^{a-1}(\log\lambda)^t \quad (\lambda\to+\infty).$$

If $n=2$, then $t=0$ or $t=1$ and

$$e(\lambda)=(k+o(1))\lambda^a(\log\lambda)^t \quad (\lambda\to+\infty)$$

with some positive constant k.

Remark. It is clear that the numbers a and t are uniquely determined by P and α. We shall call $a=a(P,\alpha)$ and $t=t(P,\alpha)$ the E-numbers of the pair (P,α).

The proof of Theorem 1 depends on the following lemma which is a particular case of results in the author's paper [8] (Theorems 1 and 2 and Lemma 2).

Lemma 2. Consider a real algebraic manifold $V(\lambda)$: $p(\lambda,\xi)=0$ in R^n depending on $\lambda\in R$. Here $p(\lambda,\xi)$ is a real polynomial in $\lambda\in R$ and $\xi\in R^n$. Suppose that, for some λ_0, $V(\lambda_0)$ is not empty and that $\operatorname{grad}_\xi p(\lambda_0,\xi)\neq 0$ for all $\xi\in V(\lambda_0)$. Further assume that there is a bounded subset Ω of R^n such that $V(\lambda)\subset\Omega$ for all λ in a neighbourhood of λ_0. For λ in a neighbourhood of λ_0, let $\omega_\lambda(\xi)$ be a differential $(n-1)$-form on $V(\lambda)$ such that in any local coordinate system on $V(\lambda_0)$ with coordinates ξ' picked among the ξ_i (also defining a local coordinate system on $V(\lambda)$ for λ in a neighbourhood of λ_0) the coefficients of $\omega_\lambda(\xi)$ are regular analytic algebraic functions of (λ,ξ'). Define the function

$$g(\lambda)=\int_{V(\lambda)}\omega_\lambda(\xi)$$

in a (sufficiently small) neighbourhood of λ_0. Let $G(\lambda)$ be a primitive function of $g(\lambda)$. Then there is a finite set W of points $\xi_1,\ldots,\xi_r \in C$ such that $G(\lambda)$ may be continued analytically along any path in C not passing through any point of W. Moreover, all the determinations of $G(\lambda)$ in the neighbourhood of any $\lambda \in (C-W)$ span a finite dimensional linear space over C. Put $\varrho=2\cdot\max_j|\xi_j|$. Then, if $\lambda_1>\varrho$ and if $G_1(\lambda)$ is a function element of $G(\lambda)$ at λ_1, there is a real number c and to every positive integer N a number K such that

$$|G(\lambda)|\leqslant K|\lambda|^c$$

for all λ with $|\lambda|>\varrho$ and for all determinations of $G(\lambda)$ that may be obtained from $G_1(\lambda)$ by analytic continuation at most N rounds in the region $|\lambda|>\varrho$. These properties hold also for the function $g(\lambda)$ itself. If $n=2$, then there is a positive integer N such that for $|\lambda|>\varrho$

$$T^N g(\lambda)=g(\lambda)+h(\lambda),\quad T^N h(\lambda)=h(\lambda),$$

where T is analytic continuation one round in the positive sense along circles $|\lambda|=$ constant.

For the proof of Theorem 1 we shall also need the following lemma.

Lemma 3. Let $q(\xi_1,\ldots,\xi_n)$ be a complex polynomial. Then there is a number σ such that when $|\lambda|>\sigma$ we have grad $q(\xi)\neq 0$ for all $\xi\in V(\lambda)$: $q(\xi)=\lambda$.

Proof. Consider the algebraic manifold $\text{grad}(q(\xi))=0$. It consists of a finite number of connected components $F_1,\ldots,F_s$, and $q(\xi)$ is constant $=\lambda_j$ on every F_j. Then for $|\lambda|>\max(|\lambda_j|)$ we have that grad $q(\xi)$ is different from zero for all $\xi\in V(\lambda)$. The lemma is proved.

Now let us turn to the proof of Theorem 1. It follows from lemma 3 that $e(\lambda)$ is real analytic for λ greater than some σ. Consider the derivative $f(\lambda)=e'(\lambda)$. We may write $f(\lambda)$ as an integral over $V(\lambda)$: $P(\xi_1,\ldots,\xi_n)=\lambda$,

$$f(\lambda)=\int_{V(\lambda)}\xi^{2\alpha}(d\xi/dP(\xi)).$$

It is clear that the differential $(n-1)$-form $\omega_\lambda(\xi)=(d\xi/dP(\xi))_{V(\lambda)}$ on $V(\lambda)$ has regular algebraic coefficients in any local coordinate system with coordinates among the ξ_i. Hence $e(\lambda)$ has the properties stated in Lemma 2.

From the fact that all the determinations of $e(\lambda)$ span a finite dimensional linear space over the complex numbers it follows (see e.g. Goursat [4], p. 447–460) that in a neighbourhood of infinity $e(\lambda)$ is a finite sum of terms of the type $\lambda^\beta(\log\lambda)^\nu H(\lambda)$, where β is a complex number, ν a non-negative integer, and $H(\lambda)$ is analytic and single-valued in a neighbourhood of ∞. Hence every such function $H(\lambda)$ may be developed into a Laurent series $\sum_{k=-\infty}^{+\infty}a_k\lambda^k$, convergent in a neighbourhood of ∞. Further all the functions $H(\lambda)$ are linear combinations of functions of the form $\lambda^{\gamma}(\log\lambda)^{\mu}h(\lambda)$, where γ is a complex number, μ an integer, and $h(\lambda)$ some branch of $e(\lambda)$. Because of the estimate of $e(\lambda)$ obtained in lemma 2 the Laurent series of any $H(\lambda)$ contains only a finite number of non-zero terms with a positive exponent.

Let us write every term $\lambda^{\beta}(\log\lambda)^{\nu}H(\lambda)$ so that $H(\lambda)=\sum_{k=-\infty}^{0}a_k\lambda^k$ with $a_0\neq 0$, which we can always do, choosing β conveniently. Then among the terms of e we select the 'largest' ones, first taking those having $\operatorname{Re}(\beta)$ maximal, $=a$, say, and among these keep those who have ν maximal, $=t$, say. Then, in every such 'maximal' term we replace $H(\lambda)$ by the constant term in the Laurent expansion. The sum of the selected terms is then a function

$$\varphi(\lambda)=\lambda^{a}(\log\lambda)^{t}(c_1\lambda^{ik_1}+\ldots+c_1\lambda^{ik_1})=\lambda^{a}(\log\lambda)^{t}\cdot\Phi(\log\lambda),$$

where the c_i and k_i are constants and the k_i real, and we may suppose that Φ is not identically zero. By our method of picking the terms in φ we have

$$e(\lambda)-\varphi(\lambda)=o(1)\,\lambda^{a}(\log\lambda)^{t}\quad(\lambda\to+\infty). \tag{7}$$

We have $\Phi(\mu)=\Phi_1(\mu)+i\Phi_2(\mu)$, where Φ_1 and Φ_2 are functions of the type

$$g(\mu)=A_1\sin(d_1\mu+e_1)+\ldots+A_q\sin(d_q\mu+e_q), \tag{8}$$

where the A_j, d_j and e_j are real constants and q some positive integer. It is well known (see e.g. Besicovitch [2], p. 5, Th. 12) that a function g of the type (8) has the following property. If ω_0 is in the range of g, then there is to every $\varepsilon>0$ an increasing sequence $\mu_1,\mu_2,\ldots$ of real numbers and a positive number K, such that $\mu_j\to+\infty$ when $j\to+\infty$, $\mu_{j+1}-\mu_j<K$ for all j and

$$|g(\mu_j)-\omega_0|<\varepsilon, j=1,2,\ldots.$$

By this property we get that $\Phi_2\equiv 0$. For, if there were a number $y_0\neq 0$ in the range of Φ_2, then there would be a sequence (λ_j), tending to $+\infty$, such that

$$|\operatorname{Im}(\varphi(\lambda_j))|>|y_0|\lambda_j^{a}(\log\lambda_j)^{t}/2,$$

and from (7) it would then follow that $e(\lambda_j)$ is non-real, if j is sufficiently large, which is a contradiction, since e is real. Hence $\Phi_2\equiv 0$, and $\Phi=\Phi_1$. An analogous argument shows that $\Phi\geqslant 0$, as a consequence of the inequality $e(\lambda)\geqslant 0$. Now let us consider $e'(\lambda)$.

From the way of picking the terms in φ we find

$$e'(\lambda)=a\lambda^{a-1}(\log\lambda)^{t}\Phi(\log\lambda)+\lambda^{a-1}(\log\lambda)^{t}\Phi'(\log\lambda)$$
$$+o(1)\lambda^{a-1}(\log\lambda)^{t}\quad(\lambda\to+\infty). \tag{9}$$

By the same type of arguments as above for Φ and Φ_2 we get from (9) and $e'(\lambda)\geqslant 0$ ($e(\lambda)$ is evidently increasing)

$$\Phi'(\mu)\geqslant -a\Phi(\mu). \tag{10}$$

Since Φ is not identically zero, and $\Phi\geqslant 0$, there is an increasing sequence $\mu_1,\mu_2,\ldots$ such that $\mu_j\to+\infty$ when $j\to+\infty$, and with two positive numbers C and K

$$(\mu_{j+1}-\mu_j)<K,\quad \Phi(\mu_j)>C,\quad \text{for all } j.$$

Let μ be an arbitrary number $>\mu_1$, and let μ_l be the largest μ_j which is $\leqslant \mu$. The solution of the differential equation $u' = -au$ which passes through the point $(\mu_l, \Phi(\mu_l))$ is

$$u(x) = \Phi(\mu_l)\exp(-a(x-\mu_l)).$$

From (10) it then follows $\Phi(x) \geqslant u(x)$ for $x \geqslant \mu_l$, and so a must be non-negative, since otherwise Φ would grow exponentially, but we know that it is bounded. Further $x = \mu$ gives

$$\Phi(\mu) \geqslant C\exp(-aK) > 0.$$

By (7) we now get the statement about $e(\lambda)$ in the theorem. That about $e'(\lambda)$ follows from (9).

It remains to show the stronger assertions in the case $n=2$. By Lemma 2 there is an integer $N>0$ such that in a neighbourhood of ∞ we have $T^N e' = e' + h$, $T^N h = h$. Hence $h(\lambda)$ is in a neighbourhood of ∞ a single-valued function of $\lambda^{1/N}$, and so is $F(\lambda) = e'(\lambda) - (2N\pi i)^{-1} h(\lambda) \log \lambda$. Thus h and F may be developed into Puiseux series in a neighbourhood of ∞. From the estimate by Lemma 2 holding for $e'(\lambda)$ it follows that h and F are of polynomial growth, and so their Puiseux expansions contain only a finite number of non-zero terms with a positive exponent. Hence in a neighbourhood of ∞ we have

$$e'(\lambda) = \sum_{k=-\infty}^{k_0} a_k \lambda^{k/N} + (\log \lambda) \sum_{k=-\infty}^{k_0'} b_k \lambda^{k/N}$$

By integration we find

$$e(\lambda) = F_1(\lambda) + (\log \lambda) F_2(\lambda) + b_{-N} (\log \lambda)^2/2,$$

where F_1 and F_2 are Puiseux series, convergent in a neighbourhood of ∞ and containing only a finite number of terms with a positive exponent. Since $e(\lambda)$ grows faster than some positive power of λ, the term $b_{-N}(\log\lambda)^2/2$ is not the leading one, and the particular statement for $n=2$ follows. (It may be shown that actually $b_{-N}=0$.) The theorem is proved.

Now we return to our hypoelliptic polynomial $M(\xi)$ and the unique self-adjoint extension A_0 in $L^2(R^n)$ of $M(D)$, defined on $C_0^\infty(R^n)$, and the spectral function $e_{0,\lambda}(x,y) = \int_{M(\xi)\leqslant\lambda} \exp(2\pi i\langle x-y, \xi\rangle)\, d\xi$. For an arbitrary multi-index α we have

$$e_{0,1}^{(\alpha,\alpha)}(x,x) = \int_{M(\xi)\leqslant\lambda} \xi^{2\alpha} d\varkappa.$$

Hence Theorem 1 gives a result on the behaviour of $e_\lambda^{(\alpha,\alpha)}(x,x)$ when $x \to +\infty$, and to the pair (M,α) we have a pair of E-numbers $a(M,\alpha)$ and $t(M,\alpha)$.

3. An estimate for a certain fundamental solution

We consider $(M(\xi)^r - \lambda)$ with a positive integer r and λ large and negative. The operator $(M(D)^r - \lambda)$ has a temperate fundamental solution with pole zero which is the inverse Fourier transform of $(M(\xi)^r - \lambda)^{-1}$. Hence the fundamental solution with pole x is

$$h_{r,\lambda}(x,y) = \int (M(\xi)^r - \lambda)^{-1} \exp(2\pi i\langle y-x, \xi\rangle)\, d\xi,$$

where the integral is absolutely convergent if r is large enough. It is clear that $h_{r,\lambda}$ is the complex conjugate of the Green's function of A_0^r given by (6). We are going to show that outside the pole the fundamental solution tends exponentially to zero, when $\lambda \to -\infty$. For that we shall need the following lemma.

Lemma 4. If $M(\xi)$ is a hypoelliptic polynomial of degree m, then there is a largest number $b = b(M)$ such that $0 < b \leqslant 1/m$ and

$$|M^{(\alpha)}(\xi)| \leqslant C(|M(\xi)|+1)^{1-b|\alpha|} \tag{11}$$

for some number C and all real ξ and all α. If r is a positive integer, then $b(M^r) = b(M)/r$.

Proof. For a proof we refer to Hörmander [6], Theorem 3.2, except for the last statement, but this is easily checked using that it is proved in Hörmander [6] that if b is the largest number such that (11) holds for all α with $|\alpha| = 1$, then (11) holds for all α with the same b.

We also have

Lemma 5. Let N be a hypoelliptic polynomial and put $b = b(\mathrm{N})$. Then

$$|N^{(\alpha)}(\xi + \tau z \xi_0) - N^{(\alpha)}(\xi)| \leqslant C|z|(|\xi|+1)^{-c|\alpha|}(|N(\xi)| + \tau^{1/b})$$

for some constant C, all α, all real ξ and $\tau \geqslant 1$ and all complex z with $|z| \leqslant 1$. Here ξ_0 is arbitrary in R^n and C and c are positive and independent of ξ, z, and τ.

Proof. By Taylor's formula

$$N^{(\alpha)}(\xi + \tau z \xi_0) - N^{(\alpha)}(\xi) = \sum_{j=1}^{m} (\tau z)^j N_j(\xi),$$

where m is the degree of N and N_j is a linear combination of derivatives of N of order $(|\alpha| + j)$. By Lemma 4 we have

$$|\tau^j z^j N_j(\xi)| \leqslant C\tau^j |z|^j (|N(\xi)|+1)^{1-b(|\alpha|+j)} \tag{12}$$

with some constant C. From the well-known inequality $x^a y^{1-a} \leqslant x + y$ for $x, y > 0$ and $0 \leqslant a \leqslant 1$, then, from (12), putting $x = \tau^{1/b}$, $y = (|N(\xi)| + 1)$ and $a = jb$, we get

$$|\tau^j z^j N_j(\xi)| \leqslant C|z|^j (|N(\xi)|+1)^{-b|\alpha|}(|N(\xi)| + 1 + \tau^{1/b}).$$

Since $|N(\xi)| \geqslant |\xi|^k$ with some positive k for large $|\xi|$ the proof is complete.

From Lemma 4 we also get

Lemma 6. If $N(\xi)$ is hypoelliptic, then

$$|N^{(\alpha)}(\xi)| \leqslant C(|\xi|+1)^{-c|\alpha|}(|N(\xi)|+1)$$

for all α and all real ξ, where c and C are positive constants. We may now give an estimate for the fundamental solution considered.

Lemma 7. Let $N(\xi)$ be real and hypoelliptic and let $N(\xi) \to +\infty$ when $|\xi| \to \infty$. Put $b = b(N)$. Then

$$h_\lambda(x,y) = \int \exp\,(2\pi i\langle y-x,\xi\rangle)\,(N(\xi)-\lambda)^{-1}d\xi$$

is (with respect to y) a temperate fundamental solution with pole x of $(N(D)-\lambda)$ when λ is large and negative, and

$$D_y^\alpha\, h_\lambda(x,y) = O(1)\exp\,(-c|\lambda|^b) \quad (\lambda\to-\infty)$$

for $x\neq y$, all α and some $c>0$. The estimate is uniform on compact subsets of the region $x\neq y$.

Proof. Take an arbitrary $\xi_0\in R^n$, let z be a complex number and put $H_\lambda(\xi,z) = N(\xi+|\lambda|^b z\xi_0)-\lambda$. By Lemma 5 it follows, taking $\tau=|\lambda|^b$, that there are positive numbers C', C and $(-\lambda_1)$ such that

$$C^{-1}(|N(\xi)|+|\lambda|)\leqslant|H_\lambda(\xi,z)|\leqslant C(|N(\xi)|+|\lambda|) \quad (\lambda<\lambda_1) \tag{13}$$

for all real ξ and all z with $|z|\leqslant C'$.

Now, for $|\text{Im}\,(z)|\leqslant c'$ the inverse Fourier transform of $1/H_\lambda(\xi,z)$ (with respect to ξ) is equal to $\exp\,(2\pi iz|\lambda|^b\langle y,\xi_0\rangle)h_\lambda(0,y)$. In fact, a translation by $z|\lambda|^b\xi_0$ corresponds by the Fourier transformation to multiplication by $\exp\,(2\pi iz|\lambda|^b\langle y,\xi_0\rangle)$, since $H_\lambda(\xi,z)$ keeps away from zero when $|z|\leqslant c'$ (see Nilsson [8], p. 114).

Let $B(y)$ be a positive definite homogeneous polynomial of degree f. Then $B(y)\exp\,(2\pi i|\lambda|^b\langle y,\xi_0\rangle)h_\lambda(0,y)$ is (as a function of y) the inverse Fourier transform of $B(D_\xi)\,(1/H_\lambda(\xi,z))$. From the rules of differentiation we see that $B(D_\xi)\,(1/H_\lambda(\xi,z))$ is a linear combination of terms $(H_\lambda^{(\alpha_1)}(\xi,z)\cdot\ldots\cdot H_\lambda^{(\alpha_f)}(\xi,z))/H_\lambda(\xi,z)^{f+1}$, where $\sum|\alpha_i|=f$.

Now it follows from (13) and Lemma 5 that

$$|H^{(\alpha_i)}(\xi,z)/H_\lambda(\xi,z)|\leqslant C(|\xi|+1)^{-c|\alpha_i|}$$

for all real ξ, all $\lambda<\lambda_1$ and all z with $|z|\leqslant c'$, and where C is a constant. So we may conclude that if f is sufficiently large we have

$$|B(D_\xi)\,(1/\mathrm{H}_\lambda(\xi,z))|\leqslant C(|\xi|+1)^{-n-1}$$

with some number C, independent of ξ, λ and z for $|z|\leqslant c'$ and $\lambda<\lambda_1$. But then we get, putting $z=ic'$:

$$|B(y)\exp\,(2\pi c'|\lambda|^b\langle y,\xi_0\rangle)h_\lambda(0,y)|\leqslant C \quad (\lambda<\lambda_1)$$

for all y, where C is some number, independent of λ and y.

Since ξ_0 is arbitrary, the lemma follows in the case $\alpha=0$. To get it for arbitrary α we need only notice that for $y\neq0$ we have $N(D_y)^s h_\lambda(0,y)=\lambda^s h_\lambda(0,y)$ and then use (2).

4. Asymptotic estimates for the spectral function when the domain S is arbitrary

First we are going to establish a relation between the Green's functions of A^r and A_0^r, $G_{r,\lambda}(x,y)$ and $G_{0\,r,\lambda}(x,y)=\overline{h_{r,\lambda}(x,y)}$, respectively.

Lemma 8 (see Odhnoff [10]). In $L^2(S)$ one has the following identity.

$$\overline{G_{r,\lambda}(x,\cdot)} = \psi h_{r,\lambda}(x,\cdot) + (B-\lambda)^{-1} k_{r,\lambda}(x,\cdot), \tag{14}$$

where x is arbitrary in S, $\psi \in C_0^\infty(S)$, ψ real and $\psi(y)=1$ in a neighbourhood of x. Further $B = A^r$, and

$$k_{r,\lambda}(x,y) = (\psi(y)\, B_y - B_y \psi(y)) h_{r,\lambda}(x,y).$$

(In particular $k_{r,\lambda}(x,\cdot) \in C_0^\infty(S)$.)

Proof. Let us denote the right side of (14) by $f_{r,\lambda}(x,\cdot)$ and prove that

$$((B-\lambda)\,u, f_{r,\lambda}(x,\cdot)) = u(x) \tag{15}$$

when $u \in \mathcal{D}(B^\infty) = \bigcap_j^\infty \mathcal{D}(B^j) \subset C^\infty(S)$ (the last relation by (2)). In fact, we have seen that $((B-\lambda)u, \overline{G_{r,\lambda}(x,\cdot)}) = u(x)$ for all u such that $(B-\lambda)u \in C_0^\infty(S)$, and, $(B-\lambda)^{-1}$ being bounded, we should then have $(v, f_{r,\lambda}(x,\cdot) - \overline{G_{r,\lambda}(x,\cdot)}) = 0$ for all $v \in C_0^\infty(S)$, and the lemma would follow. To verify (15) we first consider (with $u \in \mathcal{D}(B^\infty)$)

$$\begin{aligned}((B-\lambda)\,u, \psi h_{r,\lambda}(x,\cdot)) &= (\psi(B-\lambda)\,u, h_{r,\lambda}(x,\cdot))\\ &= ((B-\lambda)\,\psi u, h_{r,\lambda}(x,\cdot)) + ((\psi B - B\psi)\,u, h_{r,\lambda}(x,\cdot))\\ &= u(x) + ((\psi B - B\psi)\,u, h_{r,\lambda}(x,\cdot)),\end{aligned} \tag{16}$$

where in the last step we have used that $h_{r,\lambda}(x,\cdot)$ is a fundamental solution of $(M(D)^r - \lambda)$ with pole x. Now we consider

$$\begin{aligned}((B-\lambda)\,u, (B-\lambda)^{-1} k_{r,\lambda}(x,\cdot)) &= (u, k_{r,\lambda}(x,\cdot))\\ &= (u, (\psi B - B\psi)\, h_{r,\lambda}(x,\cdot)) = ((B\psi - \psi B)\,u, h_{r,\lambda}(x,\cdot)),\end{aligned} \tag{17}$$

where the last step is permitted since the differential operator $(B\psi - \psi B)$ vanishes outside a compact subset of $S - \{x\}$. The lemma now follows from (16) and (17).

Next we are going to estimate the term $(B-\lambda)^{-1} k_{r,\lambda}(x,\cdot)$ in (14). By Lemma 7 we have

$$\|k_{r,\lambda}(x\cdot)\| = O(1)\exp(-c\,|\lambda|^{b/r}) \quad (\lambda \to -\infty),$$

where c is a positive constant and b corresponds to M by Lemma 2 and the estimate is uniform in the neighbourhood of any point in S. It follows that

$$\|(B-\lambda)^{-1} k_{r,\lambda}(x,\cdot)\| = O(1)\exp(-c\,|\lambda|^{b/r}) \quad (\lambda \to -\infty).$$

Let α be an arbitrary multi-index, and let us consider $D^\alpha (B-\lambda)^{-1} k_{r,\lambda}(x,\cdot)$. By (2) we then get, if r is large enough,

$$D_y^\alpha (B_y - \lambda)^{-1} k_{r,\lambda}(x,y) = O(1)\exp(-c'\,|\lambda|^{b/r}) \quad (\lambda \to -\infty)$$

with a positive constant c', and the estimate is uniform on compact subsets of $\omega \times S$, where ω is a neighbourhood of an arbitrary point in S. By Lemma 8 it is then easy to see that

$$D_y^\alpha (G_{r,\lambda}(x,y) - G_{0,r,\lambda}(x,y)) = O(1)\exp(-k\,|\lambda|^{b/r}) \quad (\lambda \to -\infty)$$

uniformly on compact subsets of $S \times S$, where k is a positive constant.

Since $(G_{r,\lambda}(y,x)-G_{0,r,\lambda}(y,x))=\overline{(G_{r,\lambda}(x,y)-G_{0,r,\lambda}(x,y))}$, it also follows that

$$D_x^\alpha(G_{r,\lambda}(x,y)-G_{0,r,\lambda}(x,y))=O(1)\exp(-k|\lambda|^{b/r})\quad(\lambda\to-\infty)$$

uniformly on compact subsets of $S\times S$. Hence, with an arbitrary positive integer s, if r is large enough

$$(\Delta_x^s+\Delta_y^s)(G_{r,\lambda}(x,y)-G_{0,r,\lambda}(x,y))=O(1)\exp(-k|\lambda|^{b/r})\quad(\lambda\to-\infty)$$

uniformly on compact subsets of $S\times S$. By well-known estimates for elliptic operators (of the type (2)) it then follows that for any pair (α,β) of multi-indices

$$(G_{r,\lambda}^{(\alpha,\beta)}(x,y)-G_{0,r,\lambda}^{(\alpha,\beta)}(x,y))=O(1)\exp(-k|\lambda|^{b/r})\quad(\lambda\to-\infty),\tag{18}$$

if r is large enough.

If we assume $A>0$, $A_0>0$, we have by (6) and (18)

$$\int_0^{+\infty}(\mu-\lambda)^{-1}d(e_{r,\mu}^{(\alpha,\beta)}(x,y)-e_{0,r,\mu}^{(\alpha,\beta)}(x,y))=O(1)\exp(-k|\lambda|^{b/r})$$

when $\lambda\to-\infty$. To get information for $(e_{r,\lambda}-e_{0,r,\lambda})$ from this estimate we shall use a Tauberian theorem by Ganelius. The theorem to be quoted is unpublished but will appear in the Mathematica Scandinavica; the corresponding theorem for the Laplace transformation has been announced in [12]. (If we are content with the result $e_\lambda(x,x)=(1+o(1))e_{0,\lambda}(x,x)$ we can use a Tauberian theorem by Keldish [13], where the Tauberian condition is

$$O\leqslant\left(\frac{\partial}{\partial\lambda}e_{r,\lambda}(x,x)\right)/e_{r,\lambda}(x,x)\leqslant 1.$$

It follows from Theorem 1 that this condition is satisfied, if r is large enough.)

First we define a slowly oscillating function as a positive, continuous function L on the positive real line such that $L(c\omega)/L(\omega)\to1$ when $\omega\to+\infty$ for every $c>0$. Then we have

Lemma 9. Let the function $\sigma(\mu)$ be locally of bounded variation for $\mu>0$. Suppose that $\int_0^{+\infty}(\mu+\omega)^{-1}d\sigma(\mu)$ is convergent for $\omega=$ some $x_0>0$. (and hence for every ω not on the negative real axis). Let c, $\varkappa$, and ν be real numbers, $c>0$, $0<\varkappa\leqslant\frac{1}{2}$ and $\nu<1$. Let $L(\omega)$ be a slowly oscillating function. Then, if

$$\int_0^{-\infty}(\mu+\omega)^{-1}d\sigma(\mu)=O(1)\exp(-c|\omega|^\varkappa)\quad(\omega\to+\infty)\tag{19}$$

and

$$\sup_{\omega\leqslant\Omega\leqslant\omega+\omega^{1-\varkappa}}\left(\int_\omega^\Omega d\sigma(\mu)\right)\leqslant O(1)\,\omega^{\nu-\varkappa}L(\omega)\quad(\omega\to+\infty),\tag{20}$$

then

$$\sigma(\omega)=O(1)\,\omega^{\nu-\varkappa}L(\omega)\quad(\omega\to+\infty).$$

Now we are going to apply this Tauberian theorem to the function

$$\sigma(\mu)=(e_{0,r,\mu}^{(\alpha,\alpha)}(x,x)-e_{r,\mu}^{(\alpha,\alpha)}(x,x))$$

with α arbitrary and $x \in S$. We let (a,t) be the E-numbers of (M,α). With c = the number k of (18), $\varkappa = b/r$, $\nu = a/r$ and $L(\omega) = (\log \omega)^t$ we have that c, $\varkappa$, ν, and L satisfy the conditions of the lemma, if r is large enough, and also the other conditions are satisfied, (19) because of (18) and (20) because of the estimate for $(\partial/\partial\lambda)\, e_{0,r,\lambda}^{(\alpha,\alpha)}(x,x)$ in Theorem 1 and the fact that $e_{r,\lambda}^{(\alpha,\alpha)}(x,x)$ is a non-decreasing function of λ, which was stated in Lemma 1. Hence we get the conclusion of Lemma 9:

$$(e_{0,r,\lambda}^{(\alpha,\alpha)}(x,x) - e_{r,\lambda}^{(\alpha,\alpha)}(x,x)) = O(1)\,\lambda^{(a-b)/r}(\log\lambda)^t \quad (\lambda \to +\infty). \tag{21}$$

By (21) and Lemma 1 we now get

$$\operatorname*{var}_{(\lambda,\lambda+\lambda^{1-b/r})} e_{r,\lambda}^{(\alpha,\beta)}(x,y) = O(1)\,\lambda^{(a_\alpha + a_\beta - 2b)/2r}(\log\lambda)^{(t_\alpha+t_\beta)/2}$$

when $\lambda \to +\infty$. Here (a_2,t_1) and (a_2,t_2) are the E-numbers of (M,α) and (M,β), respectively, and α and β are arbitrary multi-indices, x and y belong to S, and r is sufficiently large. The same estimate holds for $e_{0\,r,\lambda}^{(\alpha,\alpha)}(x,y)$, and so, taking

$$\sigma(\mu) = (e_{0,r,\mu}^{(\alpha,\beta)}(x,y) - e_{r,\mu}^{(\alpha,\beta)}(x,y)),$$

we get by the Tauberian theorem

$$(e_{0,r,\lambda}^{(\alpha,\beta)}(x,y) - e_{r,\lambda}^{(\alpha,\beta)}(x,y)) = O(1)\,\lambda^{(a_1+a_2-2b)/2r}(\log\lambda)^{(t_1+t_2)/2} \tag{22}$$

when $\lambda \to +\infty$. However, we want the results for $e_\lambda = e_{1,\lambda}$ and not for $e_{r,\lambda}$. From the relation $e_{r,\lambda} = e_{\lambda^{1/r}}$ we immediately find that (22) is valid not only for r sufficiently large but also for $r=1$. Our restriction that $A>0$, $A_0>0$, may also be removed, since by a translation in the eigenvalue parameter λ we may make these two inequalities satisfied, and the translation does not change the asymptotic formulas.

We can also take care of the case where A is not bounded from below. We have the following lemma.

Lemma 10. Let A be an arbitrary self-adjoint extension in $L^2(S)$ of a_0 and $E(\lambda)$ the corresponding spectral resolution. Then for any λ, $E(\lambda)$ is given by a kernel e_λ:

$$E(\lambda)\,u(x) = \int_S e_\lambda(x,y)\,u(y)\,dy \quad (u \in L^2(S)),$$

where e_λ is infinitely differentiable in $S\times S$ and where

$$e_\lambda^{(\alpha,\beta)}(x,y) = O(1)\exp\left(-c|\lambda|^{b(M)}\right) \quad (\lambda \to -\infty)$$

uniformly on compact subsets of $S\times S$. Here c is a positive constant and α,β are arbitrary multi-indices.

Proof. For a proof we refer to Nilsson [8], the Theorems 3 and 4, where the corresponding theorem is proved for an elliptic differential operator $P(D)$. The proof, however, works as well in our case. For it uses essentially three facts:

(*a*) To every $x \in S$ we have a fundamental solution $g_\lambda(x,y)$ with pole x of $(P(D)-\lambda)$, defined when λ is large and negative and decreasing exponentially outside the pole when $\lambda \to -\infty$, (*b*) the fundamental solution $g_\lambda(x,y)$ above satisfies an inequality

$$|g_\lambda(x,y)| \leq C \cdot |x-y|^{-n+\delta} \quad (x \neq y)$$

uniformly on compact subsets of $S \times S$ and for all λ. Here C and δ are positive constants,

(*c*) we have for $P(D)$ an interior a priori L^2-estimate of the type (2) (this paper).

Now (*a*) holds also for $M(D)$ (Lemma 7) and so does (*c*). Further in [8] (*b*) is only used to make certain that the mapping

$$u \to \int_K g_\lambda(\cdot, y)\, u(y)\, dy \quad (K = \text{compact} \subset S)$$

is continuous from local L^2 to local L^2 and that the continuity is uniform with respect to λ. In our case we have that $M(\xi) \geq C|\xi|^k$ when ξ is large, with some positive c, C, and it follows that the temperate fundamental solution of $(M(D)^r - \lambda)$ is uniformly bounded with respect to λ, if r is large enough and λ large and negative.

This result may then replace (*b*) in question of $M(D)^r$, but via the elementary connection between spectral functions of $M(D)$ and $M(D)^r$, with r odd, we get the desired result also for $M(D)$.

By Lemma 10 we may now see that (22) is valid also if A is not bounded from below. For let us consider A^r with r even; then A^r is bounded from below so that (22) holds for $e_{r,\lambda}$. But $e_\lambda = e_{-\lambda} + e_{r,\lambda^r}$ for $\lambda > 0$, and so by lemma 10 we get (22) also for e_λ.

We collect our results in the following theorem.

Theorem 2. Let $M(\xi)$ be a real hypoelliptic polynomial in R^n, $n \geq 2$, such that $M(\xi) \to +\infty$ when $|\xi| \to \infty$. Let S be an open subset of R^n and let a_0 be the operator in $L^2(S)$ defined by the differential operator $M(D)$, acting on $C_0^\infty(S)$. Suppose that A is a self-adjoint extension in $L^2(S)$ of a_0, not necessarily bounded from below. Then the spectral resolution $E(\lambda)$ of A is given by a kernel $e_\lambda(x,y)$:

$$E(\lambda)\, u(x) = \int_S e_\lambda(x,y)\, u(y)\, dy \quad (u \in L^2(S)),$$

where e_λ is infinitely differentiable in $S \times S$, and

$$e_\lambda^{(\alpha,\beta)}(x,y) = O(1) \exp\left(-c|\lambda|^{b(M)}\right) \quad (\lambda \to -\infty)$$

for any multi-indices α, β. Here $c > 0$, and $b(M)$ is the largest positive number b such that with some constant C

$$|M^{(\alpha)}(\xi)| \leq C(|M(\xi)| + 1)^{1-b|\alpha|}$$

for all α and all real ξ. (If M is elliptic and of degree m, we have $b(M) = 1/m$). Further, if A_0 is the unique self-adjoint extension in $L^2(R^n)$ of $M(D)$, defined on $C_0^\infty(R^n)$, and $e_{0,\lambda}$ its spectral function, then

$$(e_\lambda^{(\alpha,\beta)}(x,y) - e_{0,\lambda}^{(\alpha,\beta)}(x,y)) = O(1)\lambda^{(t_\alpha + t_\beta - 2b(M))}(\log \lambda)^{(t_\alpha + t_\beta)/2}$$

when $\lambda \to +\infty$. Here α,β are arbitrary. The pair (a_γ, t_γ) is characterized by the property that

$$K^{-1}\lambda^{a_\gamma}(\log \lambda)^{t_\alpha} \leqslant e_{0,\lambda}^{(\gamma,\gamma)}(x,x) \leqslant K\lambda^{a_\gamma}(\log \lambda)^{t_\gamma}$$

for some $K>0$, all large positive λ and $\gamma=\alpha,\beta$. That such numbers exist was proved in Theorem 1.

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