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On an inequality concerning the integrals of moduli of regular analytic functions

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With 1 figure in the text

1. Let $f(z)$ be an analytic function, regular in a convex domain D and on its boundary C. Let L be a rectifiable curve in D. Our problem is to estimate

$$\int_L |f(z)\,dz|$$

by means of the integral

$$\int_C |f(t)\,dt|.$$

Professor F. CARLSON (1) has shown that in the case D being a circle, then

$$\int_L |f(z)\,dz| \leq \frac{1}{\pi}\int_C V(t)\,|f(t)\,dt$$

where $V(t)$ is the upper limit of the sum of the angles at which the elements of L are seen from a point t on C. He has called my attention to the possibility of solving the problem for convex domains by the same method as the one he uses for a circle.

GABRIEL, in a first work (2), has treated the problem for a circle and in a second work (3) for convex regions.

2. Let L be a rectilinear segment in D. We may suppose that L is parallel to the real axis. Let $F(\zeta, z)$ be a function that for each $z \in D$ is an analytic function of ζ, regular in D and continuous on C. Then, by Cauchy's theorem, we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{F(t,z)}{F(z,z)}\frac{f(t)}{t-z}\,dt.$$

Hence

$$|f(z)| \leq \frac{1}{2\pi}\int_C \left|\frac{F(t,z)}{F(z,z)}\,\frac{1}{t-z}\right| |f(t)\,dt|$$

and

$$\int_L |f(z)\,dz| \leq \frac{1}{2\pi}\int_C \lambda(t)\,|f(t)\,dt|$$

where

1) $$\lambda(t) = \int_L \left|\frac{F(t,z)}{F(z,z)}\,\frac{dz}{t-z}\right|.$$

Let $z \in L$ and $t \in C$. If $t - z = r\,e^{i\theta}$, then

$$\left|\frac{dz}{t-z}\right| = \left|\frac{d\theta}{\sin\,\theta}\right|.$$

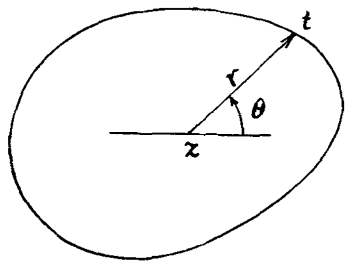

Fig. 1.

Let $u(\zeta, z)$ be an harmonic function of ζ, regular for $\zeta \in D$, $z \in D$. When $\to t \in C$ along a path in D, then $u(\zeta, z) \to \log\frac{1}{|\sin\,\theta|}$. Let $v(\zeta, z)$ be the conjugated harmonic function and put

$$F(t,z) = e^{-(u+iv)}.$$

Then

2) $$\lambda(t) = \int_L e^{u(z,z)}\,|d\,\theta|.$$

If D is the circle $|\zeta| \leq R$, then we obtain by this construction

$$F(\zeta, z) = \tfrac{1}{2}\left[1 - \frac{\zeta(\zeta - z)}{R^2 - \zeta\,\bar{z}}\right];\ e^{u(z,z)} = 2$$

and hence

$$\lambda(t) = 2\int_L |d\,\theta| = 2\,V(t)$$

where $V(t)$ is the angle at which the segment L is seen from a point t on C. This is exactly F. CARLSON's result which is easily extended to the case, where L is a polygon and finally a rectifiable curve. Now we shall prove the following theorem

Theorem. *Let L be a rectificable curve in a convex domain D. Then*

$$\int_L |f(z)\,dz| \leq \frac{A}{2\pi}\int_C V(t)\,|f(t)\,dt|$$

where $A = 4$ *and* $V(t)$ *is the upper limit of the sum of the angles, at which the elements of* L *are seen from the point* t *on* C.

It is sufficient to prove that $u(z, z) \leq \log 4$. Further, we can assume that the curve C is smooth. The potential of a double layer $\mu_z(t) = \frac{1}{\pi} \log \frac{1}{|\sin \theta|}$ on C is

$$U_z(\zeta) = \mathcal{J}\left\{\int_C \mu_z(t) \frac{dt}{t - \zeta}\right\} = \int_C \mu_z(t)\, d\, [\arg(t - \zeta)].$$

When $\zeta \to t_0 \in C$ along a path in D, then

$$U_z(\zeta) \to \log \frac{1}{|\sin \theta|_{t_0}} + \int_C \mu_z(t)\, d\, [\arg(t - t_0)] \geq \log \frac{1}{|\sin \theta|_{t_0}} = u(t_0, z)$$

since C is convex and the double layer is non-negative.

Hence follows

$$u(\zeta, z) \leq U_z(\zeta), \qquad \zeta \in D.$$

Putting $\zeta = z$, we get

$$u(z, z) \leq U_z(z) = \frac{1}{\pi} \int_0^{2\pi} \log \frac{1}{|\sin \theta|}\, d\theta = \log 4.$$

This proves the theorem.

If the inner curve L *is convex, then* $V(t) \leq 2\pi$ *and*[1]

3) $$\int_L |f(z)\, dz| \leq A \int_C |f(t)\, dt|; \qquad A = 4.$$

3. The obtained value $A = 4$ is not the best possible. Let a_θ be that arc of C for which $0 \leq \arg t - z \leq \theta$ and let $\omega(\zeta, a_\theta, D) = \omega(\zeta, a_\theta)$ be the harmonic measure of a_θ with respect to the region D. Then

4) $$u(z, z) = \int_C \log \frac{1}{|\sin \theta|}\, d\omega(z, a_\theta).$$

For convex regions the following lemma holds:

Lemma 1. $\omega(z, a_0)$ *is an absolutely continuous function of* θ *and*

$$0 \leq \frac{d\omega(z, a_\theta)}{d\theta} \leq \frac{1}{\pi}.$$

Let $\Delta\theta > 0$ and consider the function

$$\Delta\omega = \omega(z, a_{\theta+\Delta\theta}) - \omega(z, a_\theta)$$

[1] GABRIEL (3) has shown this inequality with $A = \pi(1 + e) + e$.

which is an harmonic function of z, regular in D. Since $\Delta\omega$ is the harmonic measure of the arc $\Delta\alpha = \alpha_{\theta+\Delta\theta} - \alpha_\theta$, it is non-negative and the lower bound of $\frac{d\omega}{d\theta}$ is immediately obtained.

Let H be the half-plane that is bounded by the straight line through the end-points of $\Delta\alpha$, and that does not contain $\Delta\alpha$ in its interior. If $\Delta\theta$ is sufficiently small, then $z \in H$. Let α' be the rectilinear segment between the end-points of $\Delta\alpha$. Then it is well known that

$$\omega(z, \alpha'; H) - \Delta\omega \geq 0.$$

For the left member is a regular harmonic function of z in the region $H \cdot D$, where it has non-negative boundary-values. But

$$\omega(z, \alpha'; H) = \frac{\Delta\theta}{\pi}.$$

Hence

$$\frac{\Delta\omega}{\Delta\theta} \leq \frac{1}{\pi}$$

and the lemma follows immediately.

Now we may write 4) as

$$5) \qquad u(z, z) = \int_0^{2\pi} \log\frac{1}{|\sin\theta|}\,\frac{d\omega(z, \alpha_\theta)}{d\theta}\,d\theta.$$

From lemma 1 follows that

$$0 \leq u(z, z) \leq \frac{1}{\pi}\int_0^{2\pi} \log\frac{1}{|\sin\theta|}\,d\theta = \log 4.$$

This is the result that we have already obtained. But $\frac{d\omega}{d\theta}$ cannot take its maximum-value $\frac{1}{\pi}$ in the whole interval of integration since

$$\int_0^{2\pi} \frac{d\omega(z, \alpha_\theta)}{d\theta}\,d\theta = \omega(z, C) = 1.$$

We use the following lemma:

Lemma 2. *If* $g(\theta)$ *and* $h(\theta)$ *are integrable,* $g(\theta)$ *non-increasing, and*

$$0 \leq h(\theta) \leq k, \quad \int_0^a h(\theta)\,d\theta = M$$

then

$$\int_0^a g\,h\,d\theta \leq k\int_0^{M/k} g\,d\theta.$$

Put

$$H(\theta) = \int_0^\theta h\,d\theta.$$

Then $H(a) = M$ and

$$0 \le H(\theta) \le \begin{cases} k\theta & \text{in } 0 \le \theta \le M/k \\ M & \text{in } M/k \le \theta \le a. \end{cases}$$

Since $g(\theta)$ is non-increasing, it follows that

$$\int_0^a g h\,d\theta = \Big|_0^a g H - \int_0^a H\,dg \le \Big|_0^a g H - k\int_0^{M/k} \theta\,dg - M\int_{M/k}^a dg =$$

$$= \Big|_0^a g H - k\Big|_0^{M/k} \theta g - H(a)\Big|_{M/k}^a g + k\int_0^{M/k} g\,d\theta = k\int_0^{M/k} g\,d\theta.$$

This proves the lemma.

We can write the integral 5)

$$u(z, z) = \int_0^{\pi/2} \log\frac{1}{\sin\theta}\,h(\theta)\,d\theta \tag{6}$$

where

$$h(\theta) = \left(\frac{d\omega}{d\theta}\right)_\theta + \left(\frac{d\omega}{d\theta}\right)_{\pi-\theta} + \left(\frac{d\omega}{d\theta}\right)_{\pi+\theta} + \left(\frac{d\omega}{d\theta}\right)_{2\pi-\theta}.$$

From lemma 1 follows that $0 \le h(\theta) \le \frac{4}{\pi}\cdot$ Further

$$\int_0^{\pi/2} h(\theta)\,d\theta = \omega(z, C) = 1.$$

Now, applying lemma 2 to the integral 6), we obtain

$$0 \le u(z, z) \le \frac{4}{\pi}\int_0^{\pi/4} \log\frac{1}{\sin\theta}\,d\theta.$$

By integrating the well-known development

$$\log\frac{1}{\sin\theta} = \log\frac{1}{\theta} + \sum_{n=1}^{\infty} \frac{2^{2n-1} B_n}{n\,\underline{|2n}}\,\theta^{2n}, \qquad |\theta| < \pi$$

where the B_n are Bernoulli's numbers, we obtain

$$\frac{4}{\pi}\int_0^{\pi/4} \log\frac{1}{\sin\theta}\,d\theta = \log\frac{4}{\pi} + 1 + \sum_{n=1}^{\infty} \frac{2^{2n} B_n}{n\,\underline{|2n+1}}\left(\frac{\pi}{4}\right)^{2n} = \log 4 - K.$$

A numerical calculation gives $K = 0.1100$. Thus

$$e^{u(z,z)} \leq 4\, e^{-K} = 3.5833 < 3.6.$$

This upper limit gives the improved value of A.

By the formal calculation of $u(z, z)$ we have supposed for the sake of simplicity that the element of L at the point z is parallel to the real axis. If the angle between this element and the real axis is β, then

$$u_\beta(z, z) = \int_C \log \frac{1}{|\sin(\theta - \beta)|}\, d\omega(z, \alpha_\theta).$$

Hence

$$\frac{1}{2\pi} \int_0^{2\pi} u_\beta(z, z)\, d\beta = \log 2 \int_C d\omega(z, \alpha_\theta) = \log 2.$$

Thus, *the mean-value of* $u_\beta(z, z)$ *for all elements at* z *is* $\log 2$.

If D is a circle, then $u_\beta(z, z) = \log 2$ for all β and $z \in D$. But in the general case $u(z, z)$ is not constant. The function

$$z = \frac{w}{(1 - \delta w)^2}$$

is schlicht for $|\delta| \leq 1$ and represents the unit circle $|w| \leq 1$ on a convex domain D for $|\delta| \leq 2 - \sqrt{3}$. If $\delta = 2 - \sqrt{3}$, then we can show by elementary calculus, that the value of $u(z, z)$ at $z = 0$ for a segment L, coinciding with the real axis, is $\log\left(1 + \frac{2}{\sqrt{3}}\right)$. Hence the best possible value of A in the theorem, obtained by this method, is $\geq 1 + \frac{2}{\sqrt{3}}$.

Lemma 3. *Let* $z = w + a_2 w^2 + \cdots$ *be schlicht and map the unit circle* $|w| \leq 1$ *on a convex region* D. *Then for* $0 < \varrho \leq 1$:

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{1}{\mathcal{I} z(\varrho e^{i\varphi})} \right| d\varphi \leq A + \log \frac{1}{\varrho}$$

where A *is the constant calculated in the preceding.*

Putting $\theta_\varphi = \arg z(\varrho e^{i\varphi})$ we have $\mathcal{I} z = |z| \sin \theta_\varphi$. In the z-plane the circle $|w| \leq \varrho$ is represented on a convex region $D_\varrho < D$. We denote by α_θ the arc

$$0 \leq \arg z \leq \theta,\ z \in \text{the boundary of } D_\varrho.$$

Now, it is easily seen that

$$\frac{1}{2\pi}\int_0^{2\pi} \log \left| \frac{\varrho}{z(\varrho e^{i\varphi}) \sin \theta_\varphi} \right| d\varphi = \int_0^{2\pi} \log \frac{1}{|\sin \theta|} d_\theta \omega (z = 0, \alpha_\theta, D_\varrho) \leq A.$$

Hence the lemma follows. It is evident that in the lemma we can substitute $\mathcal{J}z$ by the more general $\cos \beta \cdot \mathcal{J}z + \sin \beta \mathcal{R} z$ where β is real.

REFERENCES. (1) **Carlson**, Quelques inégalités concernant les fonctions analytiques. Ark. Mat. Astr. Fys. 29 B, N:o 11 (1943). — (2) **Gabriel**, Some results concerning the integrals of moduli of regular functions along curves of certain types. Proc. London Math. Soc. 28 (1928). — (3) ——, Concerning integrals of moduli of regular functions along convex curves. Proc. London Math. Soc. 39 (1937).

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