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On a closure problem

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Let $f(x)$ be a measurable function defined on the real line and such that $|f|^p$ is summable for any $p \geq 1$. Under this condition we shall consider the closure properties of the set

$$f(x+t) \qquad (-\infty < t < \infty) \tag{1}$$

in the different spaces L^p for $p \geq 1$. By C_f^p we shall denote the linear closed subset of L^p spanned by (1) in the strong topology of this space.

According to a theorem of F. Riesz and Banach C_f^p is a proper subset of L^p if and only if there is a non-trivial solution $g \in L^q$ $(1/p + 1/q = 1)$ of the integral equation

$$0 = \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi = \int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi. \tag{2}$$

If this is the case for a certain $p > 1$ we get another non-trivial solution by setting

$$h(x) = \int_{x}^{x+1} g(x)\, dx,$$

which is bounded and therefore belongs to any space $L^{q'}$, $q' > q$. On applying the cited theorem once again we find that $C_f^p \neq L^p$ implies $C_f^{p'} \neq L^{p'}$ for $1 \leq p' < p$.

From this we conclude that there will exist in the general case a number $\gamma > 1$ such that the system (1) is closed on L^p for all $p > \gamma$ but not for any $p < \gamma$. If (1) is always, respectively never, closed on L^p $(p > 1)$, we obviously have to define γ as 1 or $+\infty$. This number γ shall be called the "closure exponent" of f, and our object is to study the relation between γ and the Hausdorff dimension α of the set E where the Fourier transform of f vanishes. According to two theorems of Wiener [5] we know that $\gamma = 1$ if E is empty, while $\gamma \leq 2$ if E is of vanishing linear measure. It is also known (Segal [4]) that this latter condition does not imply $\gamma < 2$ in the general case. We shall now prove the following

Theorem. *If $0 \leq a < 1$ the closure exponent satisfies the inequality*

$$\gamma \leq \frac{2}{2-a}. \tag{3}$$

In the proof we shall avail ourselves of the following concept of spectral sets (cf. [1]). If $g(x)e^{-\varepsilon|x|}$ is summable for any $\varepsilon > 0$, the spectral set Λ_g of g is formed by those numbers λ for which it is true that

$$\int_{-\infty}^{\infty} g(x)\, e^{-\sigma|x| - itx}\, dx$$

does not converge uniformly to 0 in any interval $|t-\lambda| < \delta$ as $\sigma \to +0$. The above theorem is essentially a corollary of the following result in harmonic analysis in certain Hilbert spaces (cf. [1], pp. 20–27). Let $\Phi(x)$ be an even function which is convex and decreasing to 0 for $x > 0$, and furthermore summable around $x = 0$, and let $w(x)$ be the Fourier transform of Φ defined by the formula:

$$w(x) = \frac{1}{2x^2} \int_0^{\infty} (1 - \cos x\xi)\, d\,\Phi'(\xi).$$

The capacity $C_\Phi(A)$ of a compact set A is defined as $1/V$, where V is the least upper bound of the energy integral

$$\iint \Phi(x-y)\, d\mu(x)\, d\mu(y)$$

for all positive distributions μ of the unit mass on A. It now holds for any compact A that the class of functions g with the properties $\Lambda_g < A$ and

$$0 < \int_0^{\infty} |g(x)|^2\, w(x)\, dx < \infty$$

is empty, if and only if $C_\Phi(A) = 0$.

Let us now assume that our theorem is wrong. In that case there will exist numbers p and β such that $C_f^p \neq L^p$ and

$$2 > p > \frac{2}{2-\beta} > \frac{2}{2-a}. \tag{4}$$

This implies that (2) has a non-trivial solution g in the conjugate space L^q. From this solution we can derive others by taking convolutions $h = g * k$, where $k \in L^1$ and has a bounded spectral set. Any h of this form is an entire function bounded on the real axis and has itself a compact spectral set. Since we also can chose k such that $h \not\equiv 0$, we may without loss of generality assume that g itself has these properties. In case $f \in L^1$ it is known for any solution $g \in L^\infty$ of the integral equation (2) that $\Lambda_g < E$ (cf. [2], [3]). Since also $g \in L^q$ we derive from (4) on applying Hölders inequality over $|x| \geq 1$, that

$$0 < \int_{-\infty}^{\infty} |g(x)|^2 \frac{dx}{|x|^{1-\beta}} < \infty. \tag{5}$$

Except for a numerical factor the functions $|x|^{\beta-1}$ and $|x|^{-\beta}$ are Fourier transforms of each other and thus (5) implies that the capacity of Λ_g measured with respect to $\Phi(x) = |x|^{-\beta}$ must be positive. As is well known from potential theory, this implies that the Hausdorff dimension of Λ_g is $\geq \beta$. Thus

$$\alpha = \dim E \geq \dim \Lambda_g \geq \beta$$

which is contradictary to (4) and so theorem is proved.

It should finally be noted that the theorem just established does not remain true for all f if we replace the right hand member in (3) by any smaller number. This is a consequence of Theorem I of the paper by SALEM in the same issue of this journal.

REFERENCES. [1] **A. Beurling:** Sur les spectres des fonctions. Colloques internationaux du Centre national de la recherche scientifique. Analyse harmonique, Nancy, 1947. — [2] ——: Sur la composition d'une function sommable et d'une fonction bornée. C. R. Acad. Sci., Paris 1947. — [3] ——: Sur une classe de fonctions presque-périodiques. Ibid. — [4] **I. E. Segal:** The span of the translations of a function in a Lebesgue space. Proc. Nat. Acad. Sci. U.S.A. 30, 1944. — [5] **N. Wiener:** Tauberian theorems, Ann. of Math. 33, 1932.

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