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Extrapolation of absolutely convergent Fourier series by identically zero

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Introduction

Let A be the Banach space of all functions f with period 2π and a representation $f(x)=\sum_{-\infty}^{\infty} a_n e^{inx}$, where $\|f\|=\sum_{-\infty}^{\infty}|a_n|<\infty$. We shall denote by $\varphi(a, b)$ the function with period 2π, whose restriction to $(-\pi, \pi)$ is equal to the characteristic function of the interval (a, b) and by φ the function $1-\varphi(-\pi/2, \pi/2)$.

Let f be a function in A with zeros at a and b. We ask for conditions on f such that $f\varphi(a, b)$ is also in A. Theorem 1 gives a sufficient condition, which under certain circumstances is necessary (Theorem 2).

If $f \in \operatorname{Lip} \alpha$, where $\alpha > 1/2$ or even if f has the modulus of continuity $\omega(h)$ where $\sum_{n=1}^{\infty}(1/\sqrt{n})\,\omega(1/n)<\infty$, we know (Bernstein [2]) that $f \in A$. Putting a function equal to zero between two of its zeros does not increase its modulus of continuity and thus we may always modify, in this way, the functions that satisfy the conditions above, without leaving A. On the other hand, we prove in Theorem 3 that under the condition that $\omega(h)/h$ is non-increasing, the convergence of $\sum_1^{\infty}(1/\sqrt{n})\,\omega(1/n)$ is a necessary condition for the above modification. Theorem 3 thus contains a new proof that the divergence of $\sum_1^{\infty}(1/\sqrt{n})\,\omega(1/n)$ is a sufficient condition, provided that $\omega(h)/h$ is non-increasing, for the existence of a function $f \notin A$, with modulus of continuity $O(\omega(h))$. See Stetchkin [6].

Throughout the paper we shall use the letters C_ν, $\nu=1, 2, 3, \ldots$ for constants.

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Preliminaries

Let f be a function in A with zeros at a and b. A translation does not affect the absolute values of the Fourier coefficients, so for our purpose we may assume that $b=-a$ and $|a|\leqslant\pi/2$. We multiply f by the continuous function $\alpha(x)$ in A, which we define as 1 in $|x|\leqslant a$, as 0 in $\frac{3}{2}a\leqslant|x|\leqslant\pi$ and linear on the remaining intervals. The partition of f, $f=f\alpha+f(1-\alpha)$, shows that we need only deal with $f\alpha$. Without loss of generality we may assume that $f(x)\equiv 0$ for $\frac{3}{2}a\leqslant|x|\leqslant\pi$.

We shall make repeated use of the following stronger version of a theorem of Rudin [5 p. 56].

Lemma 1. *Suppose f is a bounded function with period 2π, $0<\delta<\pi$ and $f(x)=0$ for $\pi-\delta\leqslant|x|\leqslant\pi$. Let $g(x)$ be defined for all x by*

$$g(x)=\begin{cases} f(x) & \text{for} \quad |x|\leqslant\pi \\ 0 & \text{for} \quad |x|>\pi. \end{cases}$$

Let $K(t)$ be an even, positive function, non-decreasing for $t>0$ and such that

$$K(2t)\leqslant CK(t) \tag{1}$$

for some constant C. Then

$$f(x)=\sum_{n=-\infty}^{\infty} a_n e^{inx}, \quad \text{where} \quad \sum_{n=-\infty}^{\infty}|a_n|\,K(n)<\infty$$

if and only if

$$g(x)=\int_{-\infty}^{\infty}\hat{g}(t)\,e^{itx}\,dt, \quad \text{where} \quad \int_{-\infty}^{\infty}|\hat{g}(t)|\,K(t)\,dt<\infty.$$

Proof. Condition (1) implies that

$$K(t)<C_1(|t|^p+1) \tag{2}$$

for some constants C_1 and p. Let $h(x)$ be a function with infinitely many derivatives, such that $h(x)\equiv 1$ for $|x|\leqslant\pi-\delta$ and $h(x)=0$ for $|x|\geqslant\pi$. Then

$$|\hat{h}(t)|=\frac{1}{2\pi}\left|\int_{-\infty}^{\infty}h(x)\,e^{-ixt}\,dx\right|<\frac{C_2}{|t|^{p+2}+1}. \tag{3}$$

Since $g(x)=f(x)\,h(x)$, we have

$$\hat{g}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}g(x)\,e^{-itx}\,dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}a_n\,e^{i(n-t)x}\,h(x)\,dx=\sum_{n=-\infty}^{\infty}a_n\,\hat{h}(n-t)$$

and thus

$$\int_{-\infty}^{\infty}|\hat{g}(t)|\,K(t)\,dt=\sum_{n=-\infty}^{\infty}|a_n|\int_{-\infty}^{\infty}|\hat{h}(n-t)|\,K(t)\,dt.$$

For $n>0$ we make the following estimates

$$\int_{-\infty}^{\infty}|\hat{h}(n-t)|\,K(t)\,dt\leqslant\int_{-\infty}^{0}|\hat{h}(n-t)|\,K(t)\,dt$$
$$+K(2n)\int_{0}^{2n}|\hat{h}(n-t)|\,dt+\int_{2n}^{\infty}|\hat{h}(n-t)|\,K(t)\,dt.$$

By (2) and (3) the first and third terms are easily seen to be uniformly bounded. The second term is by (1), $O(K(n))$. For $n<0$ we have analogous estimates and thus

$$\int_{-\infty}^{\infty} |\hat{g}(t)|\, K(t)\, dt \leqslant C_3 \sum_{n=-\infty}^{\infty} |a_n|\, K(n). \tag{4}$$

On the other hand, we have

$$a_n = \int_{-\infty}^{\infty} \hat{g}(t)\, \hat{h}(t-n)\, dt$$

and

$$\sum_{-\infty}^{\infty} |a_n|\, K(n) \leqslant \int_{-\infty}^{\infty} |\hat{g}(t)| \left(\sum_{n=-\infty}^{\infty} |\hat{h}(t-n)|\, K(n) \right) dt.$$

Estimates as above give

$$\sum_{n=-\infty}^{\infty} |a_n|\, K(n) \leqslant C_4 \int_{-\infty}^{\infty} |\hat{g}(t)|\, K(t)\, dt. \tag{5}$$

The inequalities (4) and (5) prove the lemma.

We now return to our problem for a function f which vanishes on $3a/2 \leqslant |x| \leqslant \pi$ and use Lemma 1 with $K(t) \equiv 1$. In this case the lemma states that $f \in A$ if and only if $g(x) = \sum_{-\infty}^{\infty} \hat{g}(t)\, e^{itx}\, dt$, where $\sum_{-\infty}^{\infty} |\hat{g}(t)|\, dt < \infty$. We put

$$g_1(x) = g\left(\frac{\pi}{2a} x\right) = \int_{-\infty}^{\infty} \hat{g}(t)\, e^{it\frac{\pi}{2a}x}\, dt = \int_{-\infty}^{\infty} \hat{g}_1(u)\, e^{iux}\, du.$$

Then we have $g_1(x) = 0$ for $|x| \geqslant 3\pi/4$ and $g(\pm\pi/2) = 0$. Lemma 1 gives a corresponding function $f_1(x) \in A$. Suppose that $f_2 = f_1 \cdot \varphi \in A$. Applying Lemma 1 again, we get a (corresponding) Fourier transform $g_2(x)$. Then $g_2((2a/\pi)x)$ is also a Fourier transform and has a (corresponding) function $f_3(x) \in A$. Now $f_3(x)$ is exactly the function $[1-\varphi(-a, a)]f$. Thus we may assume that $a = \pi/2$. Our problem is then reduced to the following:

Suppose $f \in A$, $f(\pm\pi/2) = 0$. Under what conditions on f do we have $\|f\varphi\| < \infty$?

We first give a sufficient condition:

Theorem 1. *Let* $f \in A$, $f(a) = f(b) = 0$, $f(x) = \sum_{-\infty}^{\infty} a_k e^{ikx}$ *and*

$$\sum_{-\infty}^{\infty} |a_k| \log |k| < \infty.$$

Then $f\varphi(a, b)$ *belongs to* A.

Proof. We begin by proving the theorem for $a = -b = \pi/2$. We have

$$\varphi(x) \sim \sum_{-\infty}^{\infty} b_n e^{inx}, \quad \text{where} \quad b_n = \frac{\sin(n\pi/2)}{n\pi} \quad \text{for} \quad n \neq 0 \quad \text{and} \quad b_0 = \tfrac{1}{2}.$$

Thus

$$\|f\varphi\| = \frac{1}{\pi}\sum_{-\infty}^{\infty}\left|\sum_{k\neq n} a_k \frac{\sin (n-k)\pi/2}{n=k} + \tfrac{1}{2}a_n\right|.$$

Since $\sum_{-\infty}^{\infty}|a_n|<\infty$, the series converges and diverges as $\sum_{-\infty}^{\infty}|c_n|$, where

$$c_n = \sum_{k\neq n} a_k \frac{\sin (n-k)\pi/2}{n-k}. \tag{6}$$

We replace the function f by the sum $f_0(x)$ of four functions with periods $\pi/2$ and zeros at $x=0$.

$$f_0(x) = \sum_{-\infty}^{\infty}{}' a_{4n} e^{i4nx} + \sum_{-\infty}^{\infty}{}' a_{4n+1} e^{i(4n+1)x} + \sum_{-\infty}^{\infty}{}' a_{4n+2} e^{i(4n+2)x} + \sum_{-\infty}^{\infty}{}' a_{4n+3} e^{i(4n+3)x}. \tag{7}$$

The $'$ indicates that the terms a_p, $p=0, 1, 2, 3$, have been replaced by $-\sum_{n\neq 0} a_{4n+p}$ respectively. Instead of c_n we then obtain c_n'. The difference is:

$$c_{4n}' - c_{4n} = \frac{\sum_{k\neq 0} a_{4k+1}}{4n-1} + \frac{a_1}{4n-1} - \frac{\sum_{k\neq 0} a_{4k+3}}{4n-3} - \frac{a_3}{4n-3} = \frac{\sum_{-\infty}^{\infty} a_{4k+1}}{4n-1} - \frac{\sum_{-\infty}^{\infty} a_{4k+3}}{4n-3}.$$

The condition $f(\pm\pi/2)=0$ gives

$$\sum_{-\infty}^{\infty} a_{4k+1} = \sum_{-\infty}^{\infty} a_{4k+3} = \alpha \quad \text{and} \quad \sum_{-\infty}^{\infty} a_{4k} = \sum_{-\infty}^{\infty} a_{4k+2} = \beta.$$

We thus have

$$c_{4n}' - c_{4n} = \frac{\alpha}{(4n-1)(4n-3)} \quad \text{and} \quad \sum_{-\infty}^{\infty} |c_{4n} - c_{4n}'| < \infty.$$

Analogous considerations for c_{4n+1}', c_{4n+2}' and c_{4n+3}' imply that $\sum_{-\infty}^{\infty}|c_n|$ converges and diverges as $\sum_{-\infty}^{\infty}|c_n'|$. f_0 being of the form (7) implies that $\sum_{-\infty}^{\infty}|c_n'|$ is bounded by the sum of four series of similar form. We consider one.

$$\sum_{-\infty}^{\infty}|d_n| = \sum_n \left|\sum_{4k\neq n}{}' a_{4k}\frac{\sin (n\pi/2)}{n-4k}\right| = \sum_{-\infty}^{\infty}|d_{4n+1}| + \sum_{-\infty}^{\infty}|d_{4n+3}|.$$

Here

$$\sum_{-\infty}^{\infty}|d_{4n+1}| = \sum_{n=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty}{}' a_{4k}\frac{1}{4n+1-4k}\right| = \sum_{n=-\infty}^{\infty}\left|\sum_{k\neq 0} a_{4k}\left(\frac{1}{4n+1-4k} - \frac{1}{4n+1}\right)\right|$$

$$\leqslant \sum_{k\neq 0}|a_{4k}| \sum_{n=-\infty}^{\infty}\left|\frac{1}{4n+1-4k} - \frac{1}{4n+1}\right| \leqslant C_5 \sum_{k\neq 0}|a_{4k}|\log|k|.$$

An analogous estimate of $\sum_{-\infty}^{\infty}|d_{4n+3}|$ shows that

$$\sum_{-\infty}^{\infty}|d_n|<\infty \quad \text{if} \quad \sum_{k\neq 0}|a_{4k}|\log|k|<\infty.$$

The other three series of (7) are similarly convergent under the corresponding conditions and thus $\sum_{-\infty}^{\infty}|a_k|\log|k|<\infty$ implies that $f\varphi \in A$. This concludes the proof when $a=-b=\pi/2$.

$K_1(t)=\max\{1, \log|t|\}$ satisfies the conditions of Lemma 1. Thus the convergence of $\sum_{-\infty}^{\infty}|a_k|\log|k|$ implies the convergence of the corresponding integral $\int_{-\infty}^{\infty}|\hat{g}(t)|K_1(t)\,dt$. Using the fact that $K_1((\pi/2a)t)<C_6K_1(t)$ for some positive constant C_6 and arguing as before, we see that the theorem holds for arbitrary a and b.

The condition $\sum_{-\infty}^{\infty}|a_k|\log|k|<\infty$ cannot, in general, be improved. This follows from

Theorem 2. *Let $f(x)=\sum_{-\infty}^{\infty}a_k e^{in_kx}$ be a gap series such that $n_{k+1}/n_k>\lambda>1$ for positive values of n_k and $n_{k-1}/n_k>\lambda>1$ for negative values of n_k. Further, let $f\in A$ and $f(x)=0$ for $x=\pm\pi/2$. Then $f\varphi\in A$ if and only if $\sum_{-\infty}^{\infty}|a_k|\log|n_k|<\infty$.*

Proof. We give the proof for the case $n_k\equiv 0 \pmod 4$. Put $f\varphi\sim\sum_{-\infty}^{\infty}c_n e^{inx}$. Since $a_0=-\sum_{n\neq 0}a_n$ we have by (6)

$$|c_{4n+1}|=\left|\sum_{k=-\infty}^{\infty}\frac{a_k}{4n+1-n_k}\right|=\frac{1}{|4n+1-n_0|}\left|\sum_{k\neq 0}\frac{(n_k-n_0)a_k}{4n+1-n_k}\right|.$$

We consider the terms $|c_{4n+1}|$, where n satisfies the inequality $|4n+1-n_p|<\sqrt{|n_p|}$ for some p, and put $4n+1-n_p=m$. We then obtain

$$|c_{4n+1}|\geqslant\frac{C_7}{|n_p|}\left(\frac{|n_p-n_0||a_p|}{|m|}-\left|\sum_{k\neq p}\frac{(n_k-n_0)a_k}{4n+1-n_k}\right|\right),\quad C_7>0.$$

Using the gap condition, it will easily be seen that the second term in parenthesis is uniformly bounded. Thus

$$\sum_{|4n+1-n_p|<\sqrt{|n_p|}}|c_{4n+1}|\geqslant C_8|a_p|\log|n_p|-C_9\frac{1}{\sqrt{|n_p|}},\quad \text{where}\quad C_8>0.$$

It follows that

$$\sum_{-\infty}^{\infty}|c_{4n+1}|>C_8\sum_{-\infty}^{\infty}|a_p|\log|n_p|-C_9\sum_{-\infty}^{\infty}\frac{1}{\sqrt{|n_p|}}.$$

The second series is convergent by the gap condition and this proves the necessity of our condition in case $n_k\equiv 0 \pmod 4$. It is easily seen that the same estimates hold for an arbitrary sequence $\{n_k\}_{-\infty}^{\infty}$ satisfying the gap condition. Since the sufficiency follows from Theorem 1, the proof is complete.

Corollary. *Let $\omega(h)$ be a positive non-decreasing function, defined for $0 \leqslant h \leqslant \pi$, such that $\omega(h) \to 0$ as $h \to +0$. Then there exist two functions $f \in A$ and $g \notin A$ with modulus of continuity $\omega(h, f)$ and $\omega(h, g)$, such that $\omega(h) \leqslant \omega(h, g) = \omega(h, f)$.*

Proof. The example is furnished by f and $f\varphi$ in the above theorem. In [1 p. 179] Bary has proved that there exists a function $f(x) = \sum_0^\infty a_k e^{in_k x}$ in A that has a modulus of continuity $\omega(h, f) \geqslant \omega(h)$. It is easily seen that in her proof, we may choose $n_k \equiv 0 \pmod 4$ and satisfying $n_{k+1}/n_k > \lambda > 1$. Thus by Theorem 2, $f\varphi = g$ does not belong to A. See also Bary [1 p. 177–178].

Theorem 3. *Let $\omega(h)$ be a positive non-decreasing function in $0 \leqslant h \leqslant \pi$ such that*

$$\sum_1^\infty \frac{1}{\sqrt{n}}\,\omega\left(\frac{1}{n}\right) = \infty \quad \textit{and} \quad \omega(h)/h \textit{ is non-increasing.}$$

Then there exists a function $f \in A$ with zeros at $\pm \pi/2$, whose modulus of continuity is $O(\omega(h))$ and yet $f\varphi \notin A$.

The proof is based on a lemma of Shapiro–Rudin [4], previously used by Kahane–Salem in [3 p. 129–138] in a similar context. The lemma states that there exists a sequence $\{\varepsilon_n\}_{-\infty}^{\infty}$ where $\varepsilon_n = \pm 1$, such that

$$\left| \sum_{n=\mu}^{\nu-1} \varepsilon_n e^{inx} \right| \leqslant 16\sqrt{\nu - \mu}, \text{ for all } x, \mu \text{ and } \nu, \nu > \mu. \tag{8}$$

I. *Construction of f*

We let $f = \sum_{n=0}^\infty a_n e^{i4nx}$, where $a_0 = -\sum_{n=1}^\infty a_n$, and a_n is chosen as follows for $n \neq 0$. In the interval $2^q < n \leqslant 2^{q+1}$ we choose $\alpha(q)$ equidistant integers

$$n_{1,q} = 2^q + \beta(q),\ n_{2,q} = 2^q + 2\beta(q),\ \ldots,\ n_{\alpha(q),q} = 2^q + \alpha(q)\beta(q)$$

such that $\beta(q) \geqslant 1$ and $\alpha(q) = [2^q/\beta(q)]$. We put

$$a_n = \begin{cases} \gamma(q)\dfrac{1}{n}\varepsilon_k & \text{for } n = n_{k,q} \\ 0 & \text{otherwise,} \end{cases}$$

where ε_k are the numbers occurring in the above mentioned lemma and $\gamma(q)$ a positive function of q.

II. *Estimate of the modulus of continuity of f*

Let $h > 0$ be an arbitrary number. Then for some q_0 we have $2^{-(q_0+1)} < h \leqslant 2^{-q_0}$ and

$$|f(x+h) - f(x)| \leqslant 4h \sup_x \left| \sum_{n=1}^{2^{q_0}} n a_n e^{i4nx} \right| + 2 \sup_x \left| \sum_{n=2^{q_0}+1}^{\infty} a_n e^{4inx} \right|. \tag{9}$$

By the triangle inequality and (8) we obtain

$$\left|\sum_{n=1}^{2^{q_0}} na_n e^{i4nx}\right| \leqslant \sum_{q=0}^{q_0-1} \gamma(q) \left|\sum_{k=1}^{\alpha(q)} \varepsilon_k \exp\{4i(2^q + k\beta(q)x\}\right|$$

$$= \sum_{q=0}^{q_0-1} \gamma(q) \left|\sum_{k=1}^{\alpha(q)} \varepsilon_k e^{i4k\beta(q)x}\right| \leqslant 16 \sum_{q=0}^{q_0-1} \gamma(q) \sqrt{\alpha(q)} = o\left(\sum_{q=0}^{q_0-1} \gamma(q)\cdot 2^{q/2}(\beta(q))^{-\frac{1}{2}}\right)$$

if
$$\beta(q) = o(2^q). \tag{10}$$

The second series in (9) is

$$\left|\sum_{2^{q_0}+1}^{\infty} a_n e^{i4nx}\right| \leqslant \sum_{q=q_0}^{\infty} \gamma(q) \left|\sum_{k=1}^{\alpha(q)} \frac{\varepsilon_k e^{i4k\beta(q)x}}{n_{k,q}}\right|.$$

We put $S_{k,q} = \sum_{p=1}^{k} \varepsilon_p e^{i4p\beta(q)x}$. Then by (8) $|S_{k,q}| \leqslant 16\sqrt{k}$ for every q. A summation by parts of the inner series gives, if $\beta(q) = o(2^q)$,

$$\left|\sum_{k=1}^{\alpha(q)} \frac{\varepsilon_k e^{i4k\beta(q)x}}{n_{k,q}}\right| = \left|\sum_{k=1}^{\alpha(q)-1} S_{k,q}\left(\frac{1}{n_{k,q}} - \frac{1}{n_{k+1,q}}\right) + \frac{S_{\alpha(q),q}}{n_{\alpha(q),q}}\right|$$

$$\leqslant \frac{\beta(q)}{2^{2q}} \sum_{k=1}^{\alpha(q)} \sqrt{k} + \frac{16\sqrt{\alpha(q)}}{2^q} = O\{(\beta(q)\cdot 2^q)^{-\frac{1}{2}}\}.$$

Thus
$$\left|\sum_{2^{q_0}+1}^{\infty} a_n e^{i4nx}\right| = O\left\{\sum_{q=q_0}^{\infty} \gamma(q)(\beta(q)\cdot 2^q)^{-\frac{1}{2}}\right\} \tag{11}$$

and we obtain

$$|f(x+h) - f(x)| = O\left\{\frac{1}{2^{q_0}} \sum_{q=0}^{q_0-1} \gamma(q)\cdot 2^{q/2}(\beta(q))^{-\frac{1}{2}} + \sum_{q=q_0}^{\infty} \gamma(q)(\beta(q)\cdot 2^q)^{-\frac{1}{2}}\right\}. \tag{12}$$

In order that our estimates should be valid and $f \in A$ we have to impose conditions on $\beta(q)$ and $\gamma(q)$, namely (10) and $\sum_{q=0}^{\infty}(\gamma(q)/\beta(q)) < \infty$ respectively.

III. *Estimate of* $\|f\varphi\|$

Let $f\varphi \sim \sum_{-\infty}^{\infty} c_n e^{inx}$. We restrict ourselves to studying $|c_n|$ for positive $n \equiv 1 \pmod 4$. We have by (6)

$$|c_{4n+1}| = \left|\sum_{k=1}^{\infty} a_k \left(\frac{1}{4n+1-4k} - \frac{1}{4n+1}\right)\right| = \left|\sum_{k=1}^{\infty} \alpha_{n,k}\right|.$$

Thus

$$|c_{4n+1}| \geqslant \left|\sum_{k=[3n/4]}^{[6n/5]} \alpha_{n,k}\right| - \left|\sum_{k=1}^{[3n/4]-1} \alpha_{n,k}\right| - \left|\sum_{k=[6n/5]+1}^{\infty} \alpha_{n,k}\right| = |d_n| - |e_n| - |f_n|. \quad (13)$$

We first show that $\sum_1^\infty |e_n| < \infty$ and $\sum_1^\infty |f_n| < \infty$ and then that $\sum_1^\infty |d_n|$ is divergent under certain conditions on $\beta(q)$ and $\gamma(q)$. Define s_k as $\sum_{n=1}^k a_k$. Then using the same method that yields (11) we see that $|s_k| = O(1/\sqrt{k})$ if we assume that

$$\gamma(q) \cdot (\beta(q))^{-\frac{1}{2}} = O(1). \quad (14)$$

An estimate of $|e_n|$ gives

$$|e_n| = \left|\sum_{k=1}^{[3n/4]-1} a_k \left(\frac{1}{4n+1-4k} - \frac{1}{4n+1}\right)\right| \leqslant 4 \sum_{k=1}^{[3n/4]} \frac{|s_k|}{n^2} + \frac{|s_{[3n/4]-1}|}{n} = O\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

In a similar way we find that $|f_n| = O(1/n^{\frac{3}{2}})$. It follows that $\sum_1^\infty |e_n|$ and $\sum_1^\infty |f_n| < \infty$. In the series $\sum_1^\infty |d_n|$ we let the summation run only over those n that satisfy:

$$\begin{cases} 2^q < \left[\dfrac{3n}{4}\right] < \left[\dfrac{6n}{5}\right] < 2^{q+1} \\ |n - n_{p,q}| < \dfrac{\beta(q)}{4} \end{cases} \quad (15)$$

for some q and p. Then

$$|d_n| = \frac{4}{4n+1} \left|\sum_{k=[3n/4]}^{[6n/5]} \frac{k a_k}{4n+1-4k}\right| = \frac{4\gamma(q)}{4n+1} \left|\sum_{p=r_1}^{r_2} \frac{\varepsilon_p}{4n+1-4n_{p,q}}\right|,$$

where $1 \leqslant r_1 \leqslant r_2 \leqslant \alpha(q)$. Let the integer $s = s(n)$ be defined by the condition that $n_{s,q}$ is the integer in the sequence $\{n_{p,q}\}$ that is nearest to n and put $n - n_{s,q} = m$. Thus we have, if $\gamma(q) > 0$,

$$\frac{2^{q+1}|d_n|}{\gamma(q)} \geqslant \frac{1}{|4m+1|} - \left|\sum_{p=1}^{r_3} \frac{\varepsilon_{s+p}}{4m+1-4p\beta(q)}\right| - \left|\sum_{p=1}^{r_4} \frac{\varepsilon_{s-p}}{4m+1+4p\beta(q)}\right|,$$

where $|r_3| < \alpha(q)$ and $|r_4| < \alpha(q)$. We estimate the first sum by Abelian transformation and put $\sigma_p = \sum_{k=1}^p \varepsilon_{s+k}$. By (8) $|\sigma_p| \leqslant 16\sqrt{p}$ for every s and using $|m| < \beta(q)/4$ we obtain

$$\left|\sum_{p=1}^{r_3} \frac{\varepsilon_{s+p}}{4m+1-4p\beta(q)}\right| \leqslant \sum_{p=1}^{r_3} \left|\frac{4\sigma_p \beta(q)}{[4m+1-4p\beta(q)][4m+1-4(p+1)\beta(q)]}\right| + \left|\frac{\sigma_{r_3}}{r_3 \cdot \beta(q)}\right|$$

$$\leqslant 6 \sum_{p=1}^{\alpha(q)} \frac{\sqrt{p}\,\beta(q)}{[p\beta(q)]^2} + \frac{16}{\beta(q)} = O\left(\frac{1}{\beta(q)}\right).$$

An anologous estimate holds for the second sum and thus

$$\frac{2^{q+1}}{\gamma(q)}\left|d_{n_{s,q}+m}\right| \geqslant \frac{1}{4m+1} - C_{10}\cdot\frac{1}{\beta(q)}.$$

Hence, even if $\gamma(q)=0$,

$$\sum_{|m|<\beta(q)/4} \left|d_{n_{s,q}+m}\right| \geqslant C_{11}\frac{\gamma(q)\log\beta(q)}{2^{q+1}},$$

where $C_{11}>0$ and does not depend on q or s. Since s assumes at least $\alpha(q)/4$ different values when n lies in the interval defined by (15) we obtain

$$\sum_{n=2^q}^{2^{q+1}} |d_n| \geqslant \frac{C_{11}\cdot 2^q\cdot\gamma(q)\log\beta(q)}{\beta(q)\cdot 2^{q+3}} = C_{12}\frac{\gamma(q)\log\beta(q)}{\beta(q)}, \quad C_{12}>0.$$

It follows that $\sum_1^\infty |d_n|$ is divergent if

$$\sum_1^\infty \frac{\gamma(q)\log\beta(q)}{\beta(q)}$$

is divergent. If (14) is satisfied we then have by (13) that $\sum_{-\infty}^{\infty}|c_n| = \infty$. The function f constructed above thus belongs to A but $f\varphi$ does not, if the following conditions are satisfied:

(i) $1 \leqslant \beta(q) = o(2^q)$,

(ii) $\displaystyle\sum_1^\infty \frac{\gamma(q)}{\beta(q)} < \infty.$

(iii) $0 \leqslant \gamma(q)(\beta(q))^{-\frac{1}{2}} \leqslant 1$,

(iv) $\displaystyle\sum_1^\infty \frac{\gamma(q)}{\beta(q)}\log\beta(q) = \infty.$

IV. *Proof of the theorem*

The divergence of $\sum_1^\infty 1/\sqrt{n}\,\omega(1/n)$ is equivalent to the divergence of $\sum_{q=0}^\infty 2^{q/2}\omega(2^{-q})$. We form a new function $\omega_1(h) = \min(\omega(h), \sqrt{h})$. Then $\omega_1(h)/h$ is non-increasing and $\sum_{q=0}^\infty 2^{q/2}\omega_1(2^{-q}) = \infty$. Thus, without loss of generality, we may assume that $\omega(h) \leqslant \sqrt{h}$. By Lemma 2 below we may choose a sequence $\{q_\nu\}_1^\infty$ such that $\sum_{\nu=1}^\infty 2^{q_\nu/2}\omega(2^{-q_\nu}) = \infty$ and

$$\omega(2^{-q_{\nu+1}})\left(\tfrac{5}{4}\right)^{q_{\nu+1}-q_\nu} \leqslant \omega(2^{-q_\nu}) \leqslant \omega(2^{-q_{\nu+1}})\left(\tfrac{5}{3}\right)^{q_{\nu+1}-q_\nu}. \tag{16}$$

Put $\gamma(q)/\sqrt{\beta(q)}=\lambda(q)$ and define $\lambda(q)$ as $2^{q/2}\omega(2^{-q})$ for $q=q_\nu$ and $\lambda(q)=0$ for $q\neq q_\nu$. Then $0\leqslant\lambda(q)\leqslant 1$ and $\sum_1^\infty\lambda(q)=\infty$. We choose $\beta(q)\geqslant 1$, non-decreasing and such that

$$\sum_1^\infty \frac{\lambda(q)}{\sqrt{\beta(q)}}<\infty \quad\text{and}\quad \sum_1^\infty \frac{\lambda(q)\log\beta(q)}{\sqrt{\beta(q)}}=\infty.$$

Since $\lambda(q)\leqslant 1$ we may choose $\beta(q)=o(2^q)$. Thus the conditions (i)–(iv) are satisfied. The function f constructed by means of the above $\beta(q)$ and $\gamma(q)=\sqrt{\beta(q)}\,\lambda(q)$ has, by (12), a modulus of continuity $\omega(h,f)$, satisfying

$$\omega(h,f)=O\left\{\frac{1}{2^{q_0}}\sum_0^{q_0-1}2^{q/2}\lambda(q)+\sum_{q_0}^\infty 2^{-q/2}\lambda(q)\right\}$$

for $2^{-(q_0+1)}<h\leqslant 2^{-q_0}$. Now we have

$$\sum_0^{q_0-1}2^{q/2}\lambda(q)=\sum_{q_\nu<q_0}2^{q_\nu}\omega(2^{-q_\nu}). \tag{17}$$

By the second inequality in (16) the terms in the series increase at least as the terms of a geometric series with ratio 6/5. The sum is dominated by its last term and since $2^q\omega(2^{-q})$ is non-decreasing the sum is $O\{2^{q_0}\omega(2^{-q_0})\}$. By the first inequality in (16) we obtain, since $\omega(2^{-q})$ is non-increasing,

$$\sum_{q_0}^\infty 2^{-q/2}\lambda(q)=\sum_{q_\nu\geqslant q_0}\omega(2^{-q_\nu})=O\{\omega(2^{-q_0})\}.$$

We have now proved that $\omega(h,f)=O(\omega(h))$, which concludes the proof.

Note. The condition that $\omega(h)/h$ is non-increasing enters only in the estimate of (17). It can thus be replaced by the slightly less restrictive

$$2^q\omega(2^{-q})=O\{2^p\omega(2^{-p})\}\ \text{for}\ p\leqslant q.$$

Corollary. *Let $\omega(h)$ be a positive non-decreasing function on $0\leqslant h\leqslant\pi$ such that $\sum_1^\infty 1/\sqrt{n}\,\omega(1/n)=\infty$ and $\omega(h)/h$ is non-increasing. Then there exists a function $g\notin A$, whose modulus of continuity is $O(\omega(h))$.*

Proof. The function $f\varphi$ constructed in Theorem 3 is such a function. Other examples have previously been constructed by Bernstein [2] and Stetchkin [6]. See Bary [1 p. 165–177].

Lemma 2. *Let $\{a_k\}_1^\infty$ be a sequence of positive numbers, such that $a_k\leqslant 2^{-k/2}$ and $\sum_{k=0}^\infty 2^{k/2}a_k$ diverges. Then there exists an increasing sequence $\{k_\nu\}_1^\infty$ of positive integers such that*

$$(\tfrac{5}{4})^{k_{\nu+1}-k_\nu}\leqslant\frac{a_{k_\nu}}{a_{k_{\nu+1}}}\leqslant(\tfrac{5}{3})^{k_{\nu+1}-k_\nu} \tag{18}$$

and such that $\sum_{\nu=1}^\infty 2^{k_\nu/2}a_{k_\nu}$ diverges.

Proof. We put $2^{k/2}a_k = b_k$, choose $\delta > 0$ and construct a sequence $\{k_\nu\}_1^\infty$, such that

$$(1-\delta)^{k_{\nu+1}-k_\nu} \leqslant \frac{b_{k_\nu}}{b_{k_{\nu+1}}} \leqslant (1+\delta)^{k_{\nu+1}-k_\nu}. \tag{18'}$$

First we define the sequence $\{n_\mu\}_1^\infty$ by

$$\begin{cases} n_1 = 1 \\ n_{\mu+1} = \min\limits_{n < n_\nu} \{n \mid b_n \geqslant (1-\delta)^p b_{n+p} \text{ for all } p \geqslant 0\}. \end{cases}$$

This sequence is infinite because the contrary would imply the existence of an integer n and a sequence $\{p_\nu\}_1^\infty$, $p_\nu \to \infty$, such that $b_{n+p_\nu} \geqslant (1-\delta)^{-p_\nu} b_n$. This however violates the condition: $b_n \leqslant 1$ for every n. Let m be the largest integer in $n_\mu \leqslant n < n_{\mu+1}$ such that $b_m \geqslant (1-\delta)^{n_{\mu+1}-m} b_{n_{\mu+1}}$. Then

$$b_{m+p} \leqslant (1-\delta)^{n_{\mu+1}-m-p} b_{n_{\mu+1}} \leqslant (1-\delta)^{-p} b_m \text{ i.e.}$$

$$b_m \geqslant (1-\delta)^p b_{m+p} \text{ for } 0 \leqslant p \leqslant n_{\mu+1} - m.$$

Since also

$$b_m \geqslant (1-\delta)^{n_{\mu+1}-m} b_{n_{\mu+1}} \geqslant (1-\delta)^p b_{m+p} \text{ for } p \geqslant n_{\mu+1} - m$$

we have by definition $m = n_\mu$ and

$$\sum_{n_\mu+1}^{n_{\mu+1}} b_n \leqslant b_{n_{\mu+1}} \sum_{p=0}^{\infty} (1-\delta)^p = O(b_{n_{\mu+1}}).$$

Thus

$$\sum_{\mu=1}^{\infty} b_{n_\mu} = \infty. \tag{19}$$

We define $\{k_\nu\}_1^\infty$ as a subsequence of $\{n_\mu\}_1^\infty$ by

$$\begin{cases} k_1 = n_1 \\ k_{\nu+1} = \min\limits_{n_\mu > k_\nu} \{n_\mu \mid b_{n_\mu} \geqslant (1+\delta)^{k_\nu - n_\mu} b_{k_\nu}\}. \end{cases}$$

This sequence is infinite. The contrary would imply the existence of an n_μ, such that

$$b_{n_{\mu+p}} \leqslant (1+\delta)^{n_\mu - n_{\mu+p}} b_{n_\mu}$$

for all $p > 0$. This is impossible since $\sum_{\mu=1}^\infty b_{n_\mu} = \infty$.

Now the sequence $\{k_\nu\}_1^\infty$ satisfies (18') as an immediate consequence of the construction. By the definition of $\{k_\nu\}_1^\infty$ we obtain:

$$\sum_{k_\nu \leqslant n_\mu < k_{\nu+1}} b_{n_\mu} \leqslant b_{k_\nu} \sum_{p=0}^{\infty} (1+\delta)^{-p} = O(b_{k_\nu}).$$

Using (19) we realise that $\sum_{\nu=1}^\infty b_{k_\nu}$ diverges. Choose $\delta < 0{,}1$ and the proof is complete.

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