ARKIV FÖR MATEMATIK Band 6 nr 1

1.65 04 2 Communicated 24 February 1965 by TRYGVE NAGELL and LENNART CARLESON

Estimates of harmonic measures

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Introduction

Estimates of harmonic measures in terms of Euclidean quantities are useful in many situations. In the two-dimensional case one can apply methods of conformal mapping and extremal lengths, and many sharp results are well known. Different means of harmonic measures can be studied in the n-dimensional case. This paper is intended to provide a survey of methods available to estimate harmonic measures.

The two-dimensional case is treated in Chapter I. The second paragraph contains well-known distortion inequalities from the theory of conformal mapping, and § 3 contains well-known results from the theory of extremal lengths. In § 4 we prove two symmetrization theorems with the aid of a result from § 3. In § 5 we apply results from § 3 to comb domains.

Chapter II gives n-dimensional methods. In § 6 a method of Carleman [6.1] is applied to harmonic measures. The derivation of Carleman's method in Theorem 6.1 follows that of Dinghas [6.3]. The estimates of harmonic measures in Theorems 6.2 and 6.3 are new in the case $n>2$. In § 7 we treat Nevanlinna's mean value in a special case. In § 8 we prove some symmetrization results with probabilistic methods.

The main problem is to provide upper bounds for harmonic measures. Lower bounds are discussed in § 2 and § 7.

Bearing in mind the possibility of exhausting a given domain with more regular domains we have not aimed at generality in assumptions about the domains considered.

The subject of this paper was suggested by Professor L. Carleson, to whom I am deeply grateful for all his advice.

1. *Definitions*

R^n is the n-dimensional Euclidean space, $n \geqslant 2$, with points $z=(x_1, y_1, \ldots, y_{n-1})=(x_1, y)$. In Chapter I we treat the case $n=2$ and prefer to write $z=x+iy$. The following definitions are then to be understood with $\operatorname{Re} z$ instead of x_1.

D denotes a domain (open connected set) and ∂D the boundary of D.

$\Theta_x=\{z \mid x_1=x,\ z\in D\}$.

D_x is the subdomain of $\{z \mid x_1<x,\ z \in D\}$ that contains a given point z_0.

$\theta_x=\{z \mid z\in\Theta_x,\ z\in\partial D_x\}$.

Without D being specified D_ξ denotes a domain in $\{z \mid x_1<\xi\}$ with part of its boundary on $\{z \mid x_1=\xi\}$ and θ_ξ then denotes the interior of $\{z \mid x_1=\xi,\ z\in\partial D_\xi\}$.

Given D_ξ, $\vartheta_x = \{z \mid x_1 = x,\ z \in D_\xi\}$, $x < \xi$.

θ_x^i, $i = 1, 2, \ldots, n(x)$, are the components or unions of components of $\Theta_x(\vartheta_x)$ that separate two given points or surfaces. (Cf. § 2.)

$\Theta(x)$, $\theta(x)$, $\vartheta(x)$, $\theta_i(x)$ are the measures of the respective sets.

Θ_x will be used in preference to ϑ_x and θ_x, if D is such that all the Θ_x are connected. Θ_x can also be used to denote a set in the $(n-1)$-dimensional y-space.

$\omega(z;\ \alpha;\ D)$ denotes the harmonic measure at the point z of $\alpha \subset \partial D$ with respect to D.

c may denote various constants.

Symmetrization of an n-dimensional open set A with respect to an $(n-1)$-dimensional hyperplane p (Steiner symmetrization) [1.1, p. 5, pp. 151–152] means the following: A is transformed into A^* so that any straight line perpendicular to p that intersects A also intersects A^*. Both intersections have the same measure (length) and the intersection with A^* is a single line-segment symmetric with respect to p. When $n = 2$ this reduces to the definition of symmetrization with respect to a straight line.

A continuous function f is symmetrized with respect to a hyperplane p by symmetrizing the sets $\{z \mid f(z) > a\}$, $\inf f(z) \leqslant a < \sup f(z)$, in the manner described above.

Symmetrization of an n-dimensional open set, $n > 2$, with respect to a straight line l (Schwarz symmetrization) [1.1, pp. 151–152] means the following: A is transformed into A^* so that any $(n-1)$-dimensional hyperplane perpendicular to l that intersects A also intersects A^*. Both intersections have the same measure and the intersection with A^* is a sphere with its centre on l.

Chapter I. The two-dimensional case

2. *Distortion theorems in the theory of conformal mapping*

Problems of distortion in the theory of conformal mapping have been widely studied. We shall refer to the survey given by Lelong–Ferrand [2.2, Ch. VI, in particular pp. 185–202, pp. 216–217].

Let D be a simply connected domain in the z-plane not containing the point at infinity. Let A and B be two accessible boundary points. We limit the discussion to the following situation: A and B are the only boundary points of D at infinity and $\operatorname{Re} z \to -\infty$, when $z \to A$, $z \in D$, and $\operatorname{Re} z \to +\infty$, when $z \to B$, $z \in D$. Let L be a Jordan arc in D joining A and B. θ_x^i, $i = 1, 2, \ldots, n(x)$, are the segments of Θ_x that separate A and B. θ_x^1 is the first of the θ_x^i that is met when moving along L from A to B. θ_x^1 can also be defined as that segment among the θ_x^i that separates the largest subdomain of D from A. Cf. Fig. 2.1. $\theta_1(x)$, the length of θ_x^1, is lower semicontinuous. A detailed discussion of the definition of θ_x^1 is given by Lelong–Ferrand [2.2, pp. 185–186].

D is mapped conformally onto G in the w-plane, $w = u + iv$. $G = \{w \mid |v| < \tfrac{1}{2}\pi\}$ and A corresponds to $u = -\infty$ and B to $u = +\infty$. γ_x is the image of θ_x^1. u_1 and u_2 are defined by

$$u_1(x) = \inf_{w \in \gamma_x} u,\quad u_2(x) = \sup_{w \in \gamma_x} u.$$

Let D_ξ^1 be the subdomain of D separated from B by θ_ξ^1. $G_a = \{w \mid u < a,\ |v| < \tfrac{1}{2}\pi\}$ and $l_a = \{w \mid u = a,\ |v| < \tfrac{1}{2}\pi\}$. We use conformal invariance of harmonic measures, the

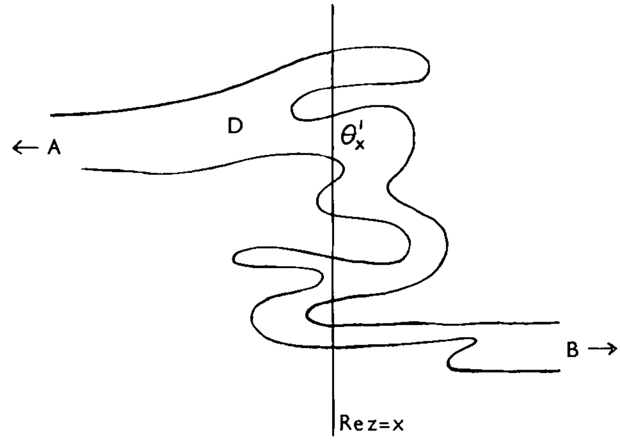

Fig. 2.1

extension principle, and explicit harmonic measures in G_a to establish the following relations:

$$\max_{z\in\theta^1_x} \omega(z;\ \theta^1_\xi;\ D^1_\xi) \leqslant \omega(u_2(x);\ l_{u_1(\xi)};\ G_{u_1(\xi)})$$
$$= \frac{4}{\pi} \operatorname{arctg} \exp\,(-u_1(\xi)+u_2(x)),\ u_2(x)<u_1(\xi); \tag{2.1}$$

$$\max_{z\in\theta^1_x} \omega(z;\ \theta^1_\xi;\ D^1_\xi) \geqslant \omega(u_1(x);\ l_{u_2(\xi)};\ G_{u_2(\xi)}) = \frac{4}{\pi} \operatorname{arctg} \exp\,(-u_2(\xi)+u_1(x)). \tag{2.2}$$

Ahlfors' first distortion inequality [2.1, pp. 7–12], [2.2, pp. 187–190], [2.3, pp. 93–100], states that

$$u_1(\xi)-u_2(x) > \pi \int_x^\xi \frac{dt}{\theta_1(t)} - 4\pi, \quad \text{when} \quad \int_x^\xi \frac{dt}{\theta_1(t)} > 2. \tag{2.3}$$

By (2.1) this yields an upper bound for $\omega(z;\ \theta^1_\xi;\ D^1_\xi)$. (Also cf. [2.3, pp. 76–78].) However, a more general result is proved in Theorem 3.2 with the method of extremal lengths. In connection with estimates of harmonic measures, distortion inequalities in the other direction are more useful, since there are few other methods for finding lower bounds of harmonic measures.

Distortion inequalities in the other direction require various restrictive assumptions about D. Ahlfors' original second inequality [2.1, pp. 12–17] is contained (with a different constant term) in an inequality by Lelong-Ferrand [2.2, pp. 194–198]. Another variant was proved by Warschawski [2.4, pp. 291–296], [2.2, p. 202]. With (2.2) this yields the following theorem.

Theorem 2.1. *Let D be bounded by the curves $y=\varphi_2(x)$ and $y=\varphi_1(x)$, $\varphi_2(x)>\varphi_1(x)$, $-\infty<x<\infty$. Let φ_1 and φ_2 have bounded derivatives; $|\varphi_1'(x)|<m$, $|\varphi_2'(x)|<m$, $-\infty<x<\infty$. $\psi(x)=\frac{1}{2}(\varphi_1(x)+\varphi_2(x))$, $\Theta(x)=\varphi_2(x)-\varphi_1(x)$. Then*

$$\max_{z\in\theta_x} \omega(z;\, \Theta_\xi;\, D_\xi) \geqslant c \exp\left(-\pi\int_x^\xi \frac{dt}{\Theta(t)} - \pi\int_x^\xi \left(\frac{\psi'^2(t)}{\Theta(t)} + \frac{\Theta'^2(t)}{12\Theta(t)}\right) dt\right),$$

where $c = \exp\left(-8\pi(1+\frac{4}{3}m^2)\right)$.

Another method for determining lower bounds of harmonic measures will be discussed in § 7. Lower bounds of the form $c \exp(-\pi \int_x^\xi dt/\Theta(t))$ can not be established if Θ oscillates too much. An example illustrating this is given in § 5.

3. *Relations between extremal lengths and harmonic measures*

This subject dates back to Beurling's thesis [3.2], and appears to have been well known to many mathematicians before an account of it was published by Hersch [3.3].

Let Γ be a family of locally rectifiable curves (denoted γ) in a domain D (i.e. each compact subcurve of a γ is rectifiable).

Consider non-negative functions ϱ in D for which

$$L_\varrho = L_\varrho(\Gamma) = \inf_\gamma \int_\gamma \varrho\,|dz|$$

$$A_\varrho = A_\varrho(D) = \iint_D \varrho^2 dxdy$$

are defined and are not both 0 or both ∞. (For a locally rectifiable γ $\int \varrho\,|dz|$ is defined as the supremum of the integrals over subcurves of γ.) The extremal length $\lambda(\Gamma)$ is defined by

$$\lambda(\Gamma) = \sup_\varrho \frac{L_\varrho^2}{A_\varrho}.$$

The functions ϱ define a conformal metric by $d\sigma = \varrho(z)\,|dz|$.

The definition of extremal length is due to Ahlfors and Beurling [3.1, p. 114]. The extremal length is a conformal invariant. Thus, if relations between some extremal lengths and harmonic measures are known in, for instance, the circle, these relations can be used to find estimates of harmonic measures.

In the following, when discussing a family of curves we shall assume that they are locally rectifiable. Let D be simply connected and let all points of ∂D be accessible. Let four points be picked on ∂D, so as to divide ∂D into four parts, $\alpha_1, \beta_1, \alpha_2, \beta_2$, in this order. D is then called a quadrangle. Let Γ be the family of curves joining α_1 to α_2 within D. Then

$$\lambda(\Gamma) = \lambda_D(\alpha_1, \alpha_2) = \lambda(\alpha_1, \alpha_2)$$

is called the extremal distance between α_1 and α_2 in D.

We now list a few well-known properties of extremal lengths and distances [3.1, p. 115], [3.3, pp. 305–308]. We only discuss extremal lengths different from zero and infinity. Considering those ϱ for which $\int_\gamma \varrho\,|dz| \geqslant 1$, for all $\gamma \in \Gamma$, $\lambda(\Gamma)$ is defined by

$$\lambda(\Gamma) = \sup_\varrho A_\varrho^{-1}.$$

Such a ϱ will be called admissible with respect to Γ.

Lemma 3.1. *$\lambda(\Gamma)$ is a conformal invariant.*

Lemma 3.2. *If $\Gamma_1 \subset \Gamma_2$, then $\lambda(\Gamma_1) \geqslant \lambda(\Gamma_2)$.*

The first two lemmata follow immediately from the definition.

Lemma 3.3. *Let the Γ_k be in disjoint domains D_k, $k=1, 2, \ldots, n$. Let $\Gamma = \{\gamma\}$ be such that each γ contains at least one $\gamma_k \in \Gamma_k$ for each k, $k=1, 2, \ldots, n$. Then*

$$\lambda(\Gamma) \geqslant \sum_{k=1}^{n} \lambda(\Gamma_k).$$

Proof. Let ϱ_k be admissible with respect to Γ_k, $k=1, 2, \ldots, n$. Let t_k, $k=1, 2, \ldots, n$, be positive numbers with $\sum_{k=1}^{n} t_k = 1$. Then $\varrho = \sum_{k=1}^{n} t_k \varrho_k$ is admissible with respect to Γ. The lemma is proved by choosing $t_k = \lambda(\Gamma_k) \cdot (\sum_{k=1}^{n} \lambda(\Gamma_k))^{-1}$, $k=1, 2, \ldots, n$.

Lemma 3.4. *Let D be a rectangle with sides α_1 and α_2 of length a, and sides β_1 and β_2 of length b. Then*

$$\lambda(\alpha_1, \alpha_2) = b a^{-1}.$$

Proof. Let D be $\{z \mid 0 < \operatorname{Re} z < a,\ 0 < \operatorname{Im} z < b\}$. Then, by the Schwarz inequality,

$$L_\varrho^2 \leqslant \left(\int_0^b \varrho\, dy \right)^2 \leqslant b \int_0^b \varrho^2\, dy.$$

Integrating over x we obtain

$$a L_\varrho^2 \leqslant b A_\varrho \tag{3.1}$$

and hence
$$\lambda(\alpha_1, \alpha_2) \leqslant b a^{-1}.$$

Equality in (3.1) holds for $\varrho = b^{-1}$. This proves the lemma.

Lemma 3.5. *Let D be a quadrangle with ∂D divided into $\alpha_1, \beta_1, \alpha_2, \beta_2$ in this order. β_1 and β_2 are assumed to be analytic arcs and α_1 and α_2 simple arcs. Let u be harmonic in D with boundary values 1 on α_2, 0 on α_1, and $\partial u / \partial n = 0$ on β_1 and β_2. Then*

$$\lambda^{-1}(\alpha_1, \alpha_2) = \iint_D |\operatorname{grad} u|^2\, dx dy.$$

Proof. By Lemma 3.1 it is sufficient to prove this in the case of a rectangle. Let α_1 in Lemma 3.4 be on the real axis. Then $u = y b^{-1}$ and the lemma is correct.

Lemma 3.6. *Let $^-$ denote reflection in the real axis. Let D be symmetric with respect to the real axis and Γ such that $\gamma \in \Gamma \Rightarrow \bar{\gamma} \in \Gamma$. Then it is sufficient to consider ϱ's symmetric with respect to the real axis to determine $\lambda(\Gamma)$.*

Proof. Let ϱ be admissible with respect to Γ. Then $\bar{\varrho}$ and $\frac{1}{2}(\varrho + \bar{\varrho})$ are also admissible. Furthermore

$$A_{(\varrho+\bar{\varrho})/2}=\frac{1}{4}\iint_D(\varrho+\bar{\varrho})^2dxdy\leqslant\frac{1}{2}\iint_D(\varrho^2+\bar{\varrho}^2)dxdy=A_\varrho.$$

This proves the lemma.

Next we collect the information that we shall need about elliptic integrals. We assume that $0<k<1$. Define

$$K(k)=\int_0^1(1-x^2)^{-\frac{1}{2}}(1-k^2x^2)^{-\frac{1}{2}}dx,\ K'(k)=K((1-k^2)^{\frac{1}{2}}),\tag{3.2}$$

and

$$t(k)=\frac{K'(k)}{K(k)}.\tag{3.3}$$

Then

$$t(k)\leqslant\frac{2}{\pi}\log\frac{4}{k},\tag{3.4}$$

$$t(k)-\frac{2}{\pi}\log\frac{4}{k}=A(k)k^2;\ |A(k)|\leqslant C_0,\quad\text{when}\quad 0\leqslant k\leqslant k_0<1.\tag{3.5}$$

(3.4) and (3.5) follow, for instance, from [3.4, p. 54] and are used by Hersch [3.3, pp. 316–319].

The following theorem giving an explicit relation between a harmonic measure and an extremal length was proved by Hersch [3.3, pp. 319–320].

Theorem 3.1. *Let D be simply connected and let all points of ∂D be accessible. ∂D is divided into two connected parts α and β. z_0 is a fixed point in D and $\omega=\omega(z_0;\alpha;D)$. Let Γ be the family of curves in D joining points on α and separating z_0 from β. t is defined by* (3.2) *and* (3.3). *Then*

$$\lambda(\Gamma)=2t\left(\sin\frac{\pi\omega}{2}\right).\tag{3.6}$$

Proof. D is mapped conformally onto G, the interior of the unit circle in the w-plane, so that $z=z_0$ corresponds to $w=0$ and α to $\alpha_1=\{w|-\pi\omega<\arg w<\pi\omega,\ |w|=1\}$. $\beta_1=\partial G-\alpha_1$ and $\eta_1=\{w|-1<\operatorname{Re} w\leqslant 0,\ \operatorname{Im} w=0\}$. Γ_1 is the family of curves γ_1 in G joining points on α_1 and separating $w=0$ from β_1. C is the family of curves c joining η_1 and α_1 in $G_1=G-\eta_1$.

By Lemma 3.6 it is sufficient to consider ϱ symmetric with respect to the real axis to determine $\lambda(\Gamma_1)$ and $\lambda(C)$. A curve $\gamma_1\in\Gamma_1$ contains two curves c' and c'' in C. Let $^-$ denote reflection in the real axis. Then, for a symmetric ϱ,

$$\int_{\gamma_1}\varrho|dw|\geqslant\int_{c'}\varrho|dw|+\int_{c''}\varrho|dw|=\int_{\bar{c}'}\varrho|dw|+\int_{\bar{c}''}\varrho|dw|$$
$$\geqslant\min\left(\int_{c'\cup\bar{c}'}\varrho|dw|,\ \int_{c''\cup\bar{c}''}\varrho|dw|\right).$$

Hence

$$L_\varrho(\Gamma_1)=2L_\varrho(C)$$

and

$$\lambda(\Gamma_1)=4\lambda(C).\tag{3.7}$$

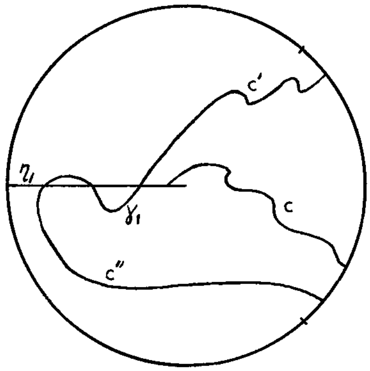

Fig. 3.1

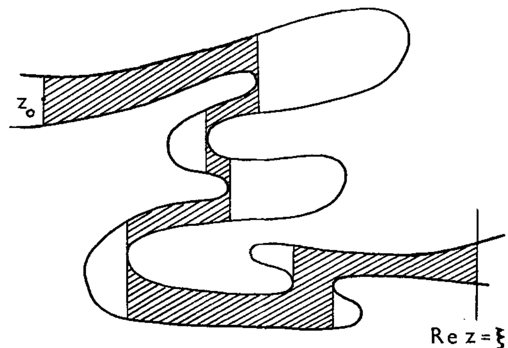

Fig. 3.2

$\lambda(C)=\lambda_{G_1}(\eta_1, \alpha_1)$ is now determined by a conformal mapping of the quadrangle G_1 onto a rectangle. By virtue of conformal invariance the theorem follows.

The following corollary will be useful in the next paragraph.

Corollary 3.1. Let D_ξ (containing z_0) satisfy the assumptions of Theorem 3.1 with $\theta_\xi=\alpha$ and let ∂D_ξ be piecewise smooth. D_ξ is mapped conformally onto $G=\{w \mid |w|<1\}$ so that $z=z_0$ corresponds to $w=0$ and θ_ξ to $\{w \mid -\pi\omega<\arg w<\pi\omega,\ |w|=1\}$, where $\omega=\omega(z_0; \theta_\xi; D_\xi)$. $\varkappa_1$ denotes $\{w \mid 0\leqslant \operatorname{Re} w<1,\ \operatorname{Im} w=0\}$ and $\varkappa$ is the image of $\varkappa_1$ in D_ξ. Let u be harmonic in $D_\xi-\varkappa$ with boundary values 1 on $\varkappa$, 0 on $\partial D_\xi-\theta_\xi$, and $\partial u/\partial n=0$ on θ_ξ (except at the endpoint of $\varkappa$). Then

$$2t\left(\sin\frac{\pi\omega}{2}\right)=\iint_{D_\xi-\varkappa}|\operatorname{grad} u|^2\,dxdy. \tag{3.8}$$

Proof. We consider G. Then by (3.6) and (3.7)

$$\lambda_{G-\varkappa_1}(\varkappa_1, \beta_1)=\frac{1}{2}t\left(\sin\frac{\pi(1-\omega)}{2}\right)=\frac{1}{2}t^{-1}\left(\sin\frac{\pi\omega}{2}\right).$$

The corollary now follows by Lemma 3.1 and Lemma 3.5.

Corollary 3.2. With the notation of Theorem 3.1

$$\omega(z_0; \alpha; D)\leqslant 4\exp\left(-\frac{\pi}{4}\lambda(\Gamma)\right).$$

Proof. This follows from (3.6) and (3.4).

Theorem 3.2. *Let D be simply connected and let all points of ∂D be accessible. Assume that D has no boundary point at infinity with finite* Re z. *Let D_ξ (containing $z_0=x_0+iy_0$) be such that θ_ξ consists of one segment. Let θ_x^i, $i=1, 2, \ldots, n(x)$, separate z_0 and θ_ξ. Assume that $n(x)=0$, $x<x_0$. Then*

$$\omega(z_0;\, \theta_\xi;\, D_\xi) \leqslant 4 \exp\left(-\pi \int_{x_0}^{\xi} \left(\sum_{i=1}^{n(x)} \frac{1}{\theta_i(x)}\right) dx\right).$$

Proof. The θ_x^i cover the shaded area in Fig. 3.2. Let Γ be the family of curves in D_ξ joining points on θ_ξ and separating z_0 from $\partial D_\xi - \theta_\xi$. We use Corollary 3.2 with $\alpha = \theta_\xi$ and $D = D_\xi$. Thus

$$\omega(z_0;\, \theta_\xi;\, D_\xi) \leqslant 4 \exp\left(-\frac{\pi}{4}\lambda(\Gamma)\right).$$

A simple estimate of $\lambda(\Gamma)$ is obtained by the following choice of ϱ:

$$\varrho(z) = \begin{cases} \dfrac{1}{\theta_i(x)}, & z \in \theta_x^i,\ i = 1, 2, \ldots, n(x),\ x_0 < x < \xi \\ 0 \text{ otherwise.} \end{cases}$$

We note that each $\theta_i(x)$ is lower semicontinuous. Now

$$\lambda(\Gamma) \geqslant L_\varrho^2 A_\varrho^{-1} \geqslant 4 \int_{x_0}^{\xi} \left(\sum_{i=1}^{n(x)} \frac{1}{\theta_i(x)}\right) dx,$$

and the theorem is thus proved.

Remark 1. The possibility of such a choice of metric (in the case of $n(x) \equiv 1$) was noted by Hersch [3.3, pp. 325–326].

Remark 2. Let the D in Theorem 3.2 have a boundary point B such that $\operatorname{Re} z \to +\infty$, $z \to B$, $z \in D$. Let θ_x^i, $i = 1, 2, \ldots, n(x)$, separate z_0 and B. D_x^i is the subdomain of D separated from B by θ_x^i. We drop the assumption that $n(x) = 0$, $x < x_0$. Let I be the interval generated by the θ_x^i separating z_0 and a fixed θ_ξ^j. Then

$$\omega(z_0;\, \theta_\xi^j;\, D_\xi^j) \leqslant 4 \exp\left(-\pi \int_I \left(\sum_i{}' \frac{1}{\theta_i(x)}\right) dx\right),$$

where $'$ means that the sum is to be taken over the θ_x^i separating z_0 and θ_ξ^j.

Remark 3. The estimate of $\omega(z_0; \theta_\xi; D_\xi)$ in Theorem 3.2 in terms of the lengths of the θ_x^i separating z_0 and θ_ξ can be generalized to higher dimensions, cf. Theorem 6.2.

Remark 4. It does not appear possible to extend the method of § 3 to higher dimensions. In Theorem 3.1 $\lambda(\Gamma)$ differs little from $4\lambda_D(\alpha_1, \alpha)$, when D is a quadrangle with two opposite boundary arcs α_1 and α, such that the distance between α_1 and α is large in comparison with the length of α_1 and z_0 is near α_1. Now let G be a vertical right cylinder of height b and let the area of the two horizontal sides α_1 and α be A. Then, in analogy with Lemma 3.4, $\lambda_G(\alpha_1, \alpha) = bA^{-1}$. However, the harmonic measure of α with respect to G depends not only on the size but also on the shape of a cross-section of the cylinder.

4. *Symmetrization results for harmonic measures*

Symmetrization with respect to a straight line is defined in § 1. The following two theorems are proved with the aid of Corollary 3.1. Another method to prove symmetrization results is dicussed in § 8.

Theorem 4.1. *Let D_ξ (containing z_0) be bounded by a piecewise smooth simple closed curve. Let θ_ξ consist of one segment. * denotes symmetrization with respect to the real axis. Then*

$$\max_{\operatorname{Re} z_0 = x_0} \omega(z_0;\, \theta_\xi;\, D_\xi) \leqslant \omega(x_0;\, \theta_\xi^*;\, D_\xi^*).$$

Equality holds if and only if D_ξ is a translate of D_ξ^*.

Proof. We use the same notation as in Corollary 3.1. Let $\varkappa$ be the image of $\varkappa_1$ in D_ξ. The slit $\varkappa' = \{z \,|\, x_0 \leqslant \operatorname{Re} z < \xi,\ \operatorname{Im} z = 0\}$ is the image of $\varkappa_1$ in D_ξ^*. Let u be harmonic in $D_\xi - \varkappa$ with boundary values 1 on $\varkappa$, 0 on $\partial D_\xi - \theta_\xi$, and $\partial u/\partial n = 0$ on θ_ξ (except at the endpoint of $\varkappa$). Let v be harmonic in $D_\xi^* - \varkappa'$ with boundary values 1 on $\varkappa'$, 0 on $\partial D_\xi - \theta_\xi^*$, and $\partial v/\partial n = 0$ on θ_ξ^* (except at the endpoint of $\varkappa'$). Then by (3.8),

$$2t\left(\sin \frac{\pi\omega}{2}\right) = \iint_{D_\xi - \varkappa} |\operatorname{grad} u|^2\, dxdy,$$

$$2t\left(\sin \frac{\pi\omega^*}{2}\right) = \iint_{D_\xi^* - \varkappa'} |\operatorname{grad} v|^2\, dxdy.$$

where $\omega = \omega(z_0;\, \theta_\xi;\, D_\xi)$ and $\omega^* = \omega(x_0;\, \theta_\xi^*;\, D_\xi^*)$.

Let u be symmetrized with respect to the real axis. The symmetrized function is u^*. $\varkappa$ corresponds to $\varkappa^*$ on the real axis so that $u^* = 1$ on $\varkappa^*$. It is possible that $\varkappa^*$ extends to the left of x_0. We now reflect D_ξ and D_ξ^* in $l = \{z \,|\, \operatorname{Re} z = \xi\}$ and denote the reflected domains by $\hat{D}_\xi$ and $\hat{D}_\xi^*$. Set $G = D_\xi \cup \theta_\xi \cup \hat{D}_\xi$ and $G^* = D_\xi^* \cup \theta_\xi^* \cup \hat{D}_\xi^*$. $\varkappa$, $\varkappa'$, and $\varkappa^*$ (including endpoints) are also reflected in l, and s, s', s^* denote the unions of the given slits and their reflections. By reflection in l u is defined to be harmonic in $G - s$ and v is defined to be harmonic in $G^* - s'$. The domain of u^* is also extended in this way. According to a result of Pólya and Szegö [4.4, p. 186–187]

$$\iint_{G-s} |\operatorname{grad} u|^2\, dxdy \geqslant \iint_{G^*-s^*} |\operatorname{grad} u^*|^2\, dxdy. \tag{4.1}$$

When $s^* - s' \neq \phi$ $\partial v/\partial n = 0$ on $s^* - s'$. By Dirichlet's principle (with free boundary values)

$$\iint_{G^*-s^*} |\operatorname{grad} u^*|^2\, dxdy \geqslant \iint_{G^*-s'} |\operatorname{grad} v|^2\, dxdy$$

and hence
$$\iint_{G-s} |\operatorname{grad} u|^2\, dxdy \geqslant \iint_{G^*-s'} |\operatorname{grad} v|^2\, dxdy. \tag{4.2}$$

From this we obtain

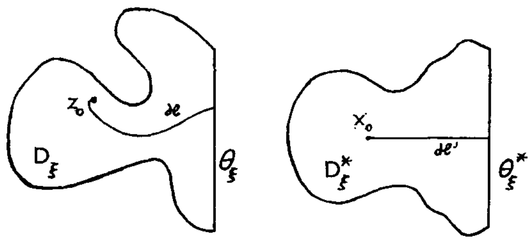

Fig. 4.1

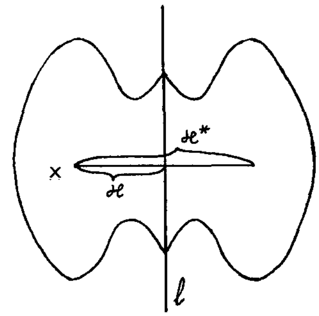

Fig. 4.2

$$t\left(\sin\frac{\pi\omega}{2}\right) \geqslant t\left(\sin\frac{\pi\omega^*}{2}\right).$$

Since $t(k)$ is strictly increasing, $0<k<1$, we obtain the result $\omega \leqslant \omega^*$.

The Dirichlet integrals in (4.2) are the inverted values of the modules of the doubly connected domains $G-s$ and G^*-s'. These modules are (without restrictions in the boundary assumptions) equal if and only if $G-s$ is a translate of G^*-s'. This follows from a result by Jenkins [4.2, p. 106, p. 115]. Theorem 4.1 is now proved.

Remark 1. We mention another possibility of discussing equality in Theorem 4.1. This is to make a detailed examination of a proof of (4.1) along the lines of a proof given in [4.1, pp. 416–419]. Such an investigation was made by Ohtsuka [4.3, pp. 202–205] in the case of circular symmetrization (cf. the following remark). By the result of Jenkins above the case of equality can be settled for more general domains.

Remark 2. Theorem 4.1 can also be formulated for circular symmetrization with respect to the positive real axis. To define circular symmetrization, in the definition of symmetrization in § 1 straight lines are replaced by circles with their centres at the origin [4.4, pp. 193–195].

Theorem 4.2. *Let D_ξ (containing $z_0=x_0$) be bounded by a piecewise smooth closed curve. Let θ_ξ consist of one segment. D_ξ is assumed to be symmetric with respect to the real axis. D_ξ is reflected in l:* $\operatorname{Re} z=\xi$. *The reflected domain is $\hat{D}_\xi$ and $G=D_\xi \cup \theta_\xi \cup \hat{D}_\xi$. G is symmetrized with respect to l, the symmetrized domain being G^*; $\theta_\xi^*=\{z \mid z\in G^*,$* $\operatorname{Re} z=\xi\}$ and $D_\xi^*=\{z \mid z \in G^*, \operatorname{Re} z<\xi\}$. *Then*

$$\omega(x_0;\, \theta_\xi;\, D_\xi) \leqslant \omega(x_0;\, \theta_\xi^*;\, D_\xi^*)$$

with equality if and only if $D_\xi = D_\xi^*$.

Proof. We use Corollary 3.1 and its notation again. Let $\varkappa$ be the image of $\varkappa_1$ in D_ξ. Let u be harmonic in $D_\xi-\varkappa$ with boundary values 1 on $\varkappa$, 0 on $\partial D_\xi-\theta_\xi$, and $\partial u/\partial n=0$ on θ_ξ (except at $z=\xi$). Set $\varkappa^*=\{z \mid |\operatorname{Re} z-\xi| \leqslant \xi-x_0,\ \operatorname{Im} z=0\}$. By reflection in l, u is defined to be harmonic in $G-\varkappa^*$. Let v be harmonic in $G^*-\varkappa^*$ with boundary values 1 on $\varkappa^*$ and 0 on ∂G^*. Then by (3.8)

$$4t\left(\sin\frac{\pi\omega}{2}\right)=\iint_{G-\varkappa^*}|\operatorname{grad} u|^2\,dxdy,$$

$$4t\left(\sin\frac{\pi\omega^*}{2}\right)=\iint_{G^*-\varkappa^*}|\operatorname{grad} v|^2\,dxdy,$$

where $\omega=\omega(z_0;\ \theta_\xi;\ D_\xi)$ and $\omega^*=\omega(x_0;\ \theta_\xi^*;\ D_\xi^*)$.

Let u be symmetrized with respect to l. The symmetrized function is u^*. We then obtain in the same way as in the proof of Theorem 4.1

$$\iint_{G-\varkappa^*}|\operatorname{grad} u|^2\,dxdy\geqslant\iint_{G^*-\varkappa^*}|\operatorname{grad} v|^2\,dxdy.$$

The proof can now be completed in the same way as in the proof of Theorem 4.1.

5. *An application to comb domains*

Let a simply connected domain D satisfy the following conditions. $\Theta_x=\phi$ outside $A<x<B$ $(-\infty\leqslant A<B\leqslant\infty)$. $\Theta_x=\{z\,|\,\operatorname{Re} z=x,\ |\operatorname{Im} z|<\infty\}$ for all x in $A<x<B$ except $x=x_m$. The number of points x_m in a finite interval is finite. Each Θ_{x_m} consists of one bounded line-segment. We then call D a *comb domain.*

First we mention an explicit example illustrating § 2. Let D be bounded by the straight lines $\{z\,|\operatorname{Re} z=x_m=-2ma,\ |\operatorname{Im} z|\geqslant b\}$, $m=1, 2, \ldots$, $(a>0)$ and the imaginary axis, where $\alpha=\{z\,|\operatorname{Re} z=0,\ |\operatorname{Im} z|<b\}$. $\omega(z;\ \alpha;\ D)$ can be determined explicitly when $z=x_m$. We write $\exp(-\pi ba^{-1})=k$ and use the notation in (3.2) and (3.3).

By a conformal mapping of $\{z\,|-2a<\operatorname{Re} z<0\}$ onto a rectangle $\{w\,|\,|\operatorname{Re} w|<kK,\ 0<\operatorname{Im} w<kK'\}$ in the w-plane and by analytic continuation, we obtain a conformal mapping of D onto a strip $\{w\,|\operatorname{Re} w<kK,\ 0<\operatorname{Im} w<kK'\}$. Hence

$$\omega(x_n;\ \alpha;\ D)=\frac{4}{\pi}\operatorname{arctg}\exp\frac{\pi x_n}{at(k)}.$$

When $a\to 0$, the term on the right tends to $4\pi^{-1}\operatorname{arctg}\exp(\pi x_n/2b)$, by (3.5).

Now let G be a domain such that $\Theta_x=\{z\,|\operatorname{Re} z=x,\ |\operatorname{Im} z|<\Theta(x)/2\}$, $-\infty<x<0$, and $\Theta_x=\phi$, $x\geqslant 0$ Let $\inf\Theta(x)=2b$ be attained at the points x_m, $m=1, 2, \ldots$. Then $\omega(x_n;\ \alpha;\ G)\leqslant\omega(x_n;\ \alpha;\ D)$. For small values of a, the above estimate for $\omega(x_n;\ \alpha;\ G)$ can, for suitably chosen $\Theta(x)$, be considerably smaller than $c\exp(-\pi\int_{x_n}^0 dx/\Theta(x))$.

Theorem 5.1. *Let the comb domain D be bounded by the lines $\{z\,|\operatorname{Re} z=x_m,\ |\operatorname{Im} z|\geqslant b_m\}$, $m=1, 2, \ldots$, $0>x_1>x_2>\ldots$, and the imaginary axis. Set $\alpha=\{z\,|\operatorname{Re} z=0,\ |\operatorname{Im} z|\leqslant b_0\}$, $\Theta_m=\{z\,|\operatorname{Re} z=x_m,\ |\operatorname{Im} z|<b_m\}$, and $b_m'=\max(b_m, b_{m-1})$, $m=1, 2, \ldots$. Given x, let n be such that $x_{n+1}<x\leqslant x_n$. Assume that*

$$\sum_{m=1}^{n}\left(\frac{x_{m-1}-x_m}{b_m'}\right)^2<M.$$

Then there is a constant c such that

$$\omega(x;\ \alpha;\ D)\leqslant c\exp\left(-\pi\sum_{m=1}^{n}\frac{x_{m-1}-x_m}{2b_m'}\right).$$

Proof. $\omega = \omega(x;\,\alpha;\,D)$. Let Γ be the family of curves in D joining points on α and separating the point $z = x$ from $\partial D - \alpha$. By Corollary 3.2

$$\omega \leqslant 4\ \exp\left(-\frac{\pi}{4}\lambda(\Gamma)\right).$$

Let C be the family of curves in $\{z \mid z \in D,\ \operatorname{Re} z > x_n\}$ joining Θ_n and α. By the same reasoning as in the proof of (3.7),

$$\lambda(\Gamma) \geqslant 4\lambda(C).$$

We now estimate $\lambda(C)$. Set $x_0 = 0$ and $\Theta_0 = \alpha$. Let D_m denote $\{z \mid x_m < \operatorname{Re} z < x_{m-1}\}$, $m = 1, 2, \dots$. By Lemma 3.3

$$\lambda(C) \geqslant \sum_{m=1}^{n} \lambda_{D_m}(\Theta_m,\ \Theta_{m-1}).$$

Set $\Theta'_m = \{z \mid \operatorname{Re} z = x_m,\ |\operatorname{Im} z| < b'_m\}$ and $\Theta'_{m-1} = \{z \mid \operatorname{Re} z = x_{m-1},\ |\operatorname{Im} z| < b'_m\}$, where $b'_m = \max\,(b_m,\ b_{m-1})$, $m = 1, 2, \dots, n$. By Lemma 3.2

$$\lambda_{D_m}(\Theta_m,\ \Theta_{m-1}) \geqslant \lambda_{D_m}(\Theta'_m,\ \Theta'_{m-1}).$$

$\lambda(\Theta'_m,\ \Theta'_{m-1})$ is determined explicitly by a conformal mapping of D_m onto a rectangle. We write

$$k_m = \exp\left(-\frac{2\pi b'_m}{x_{m-1} - x_m}\right)$$

and define t_m by (3.2) and (3.3). Then

$$\lambda_{D_m}(\Theta'_m,\ \Theta'_{m-1}) = 2t_m^{-1}$$

and hence
$$\omega \leqslant 4\ \exp\left(-2\pi \sum_{m=1}^{n} t_m^{-1}\right).$$

By (3.5)
$$2t_m^{-1} = \frac{x_{m-1} - x_m}{2b'_m} + \left(\frac{x_{m-1} - x_m}{2b'_m}\right)^2 B_m,$$

where $|B_m|$ is less than a constant depending only on M.

Remark. If, in Theorem 5.1, $2b_m = \Theta(x_m)$ for a suitable continuous $\Theta(x)$, we can, by a limiting process, obtain a special case of Theorem 3.2.

Chapter II. The general case

6. *Carleman's method*

This method was first used by Carleman in 1933 in a proof of Denjoy's conjecture concerning the number of finite asymptotic values of an integral function of finite order [6.1]. Denjoy's conjecture had been proved earlier by Ahlfors who used his distortion inequality (2.3). An account of Carleman's method in two dimensions is given in two text-books [6.4, pp. 219–224] and [6.6, pp. 121–126]. The growth of harmonic and subharmonic functions of n variables has been investigated by several authors with Carleman's method. Besides the references mentioned here in the text see [6.9]–[6.32] in the bibliography. In these investigations the main problem has been to study the growth of a given function; constants appearing in the estimates are allowed to depend on the function under consideration. However, a direct application to harmonic measures was given by Tsuji [6.8, pp. 112–117] in polar coordinates in the plane, cf. Lemma 6.7.

Carleman's method consists in establishing a differential inequality for the Carleman mean (6.1) of a harmonic function. This can also be interpreted as a differential inequality for a certain Dirichlet integral. In Theorem 6.1 Carleman's method is applied to harmonic measures. The proof of Theorem 6.1 follows—with some alterations—a proof given by Dinghas [6.3, pp. 3–9]. We use Theorem 6.1 and some lemmata to establish the estimates of harmonic measures in Theorem 6.2 and Theorem 6.3. In the case $n > 2$ these estimates are new.

We assume that D satisfies the condition **A** below.

A. D is such that $\Theta_x = \phi$ for $x \leqslant 0$ and $\Theta_x \neq \phi$ for $x > 0$. D is bounded by a finite number of piecewise smooth surfaces. D has no boundary point at infinity for which x_1 is finite. $\Theta(x)$ is bounded and $\leqslant M$ for all $x > 0$. Set $\theta_0 = \{z \mid x_1 = 0, z \in \partial D\}$. The measure of θ_0 is positive.

To begin with, we collect information about the principal eigenvalues in **B**.

B. Let the domain G in R^n be bounded by a finite number of piecewise smooth surfaces. Let V be the class of functions f such that

(1) f is continuous in $G \cup \partial G$ and piecewise continuously differentiable in G,
(2) $f(z) = 0,\ z \in \partial G$,
(3) $f(z) \neq 0$.

Consider the variational problem of minimizing in V the Rayleigh quotient

$$R(f) = \frac{\int_G |\operatorname{grad} f|^2 dz}{\int_G f^2 dz}$$

Let v be the first (normed) eigenfunction of $\Delta v + \lambda v = 0$ in G, $v = 0$ on ∂G, and λ the principal eigenvalue. It is well known that the Rayleigh quotient is minimized in V by v and that the minimum is λ [6.2, p. 399]. λ decreases when G increases [6.2, p. 409]. λ varies continuously with G [6.2, p. 423].

Furthermore λ decreases when G is symmetrized with respect to an $(n-1)$-dimensional hyperplane (cf. § 1) [6.5, p. 419]. We also mention the Faber–Krahn inequality. Let A be the volume of G. Let V_n be the volume and Λ_n the principal eigenvalue of the n-dimensional unit sphere. Then [6.5, p. 413]

$$A^{2/n} \lambda \geqslant \Lambda_n V_n^{2/n}.$$

For a domain G with a less regular boundary, we define λ as $\inf_\Omega \lambda$, where Ω is of the type considered above and $\Omega \subset G$.

Let $z_0 = (x_0, y_0)$ be a fixed point in D. We write

$$u(z) = \omega(z;\ \theta_\xi;\ D_\xi).$$

We shall establish upper bounds for $u(z)$ at $z = z_0$ by studying the Carleman mean φ, defined by

$$\varphi(x) = \int_{\vartheta_x} u^2(x, y)\, dy,\ 0 < x < \xi. \tag{6.1}$$

The definition of φ is completed by setting $\varphi(0) = 0$ and $\varphi(\xi) = \theta(\xi)$.

C. We define $\lambda(x)$ in the following way. Consider a decreasing sequence, $\{\varepsilon\}$, of positive numbers with limit zero. We choose the ε so that grad $u \neq 0$ on the surfaces $u = \varepsilon$ in D_ξ. (The number of values a for which grad u vanishes at points of the equipotential set $u = a$ in D_ξ is enumerable [6.7, p. 276].) We write $u_\varepsilon = \max(u - \varepsilon, 0)$ and set $\vartheta_{x,\varepsilon} = \{z \mid x_1 = x,\ u_\varepsilon(z) > 0\}$. $\vartheta_{x,\varepsilon}$ consists of a finite number of components $\vartheta^i_{x,\varepsilon}$. To each $\vartheta^i_{x,\varepsilon}$ we define $\lambda_{i,\varepsilon}(x)$ according to **B**. Finally we define

$$\lambda_\varepsilon(x) = \min_i \lambda_{i,\varepsilon}(x),\ \lambda(x) = \lim_{\varepsilon \to 0} \lambda_\varepsilon(x).$$

The existence of $\lambda(x)$ will be established in the proof of Theorem 6.1.

Let λ be defined by **C**. Then we define ψ by

$$\psi(x) = \int_0^x \exp\left(2\int_0^t \lambda^{\frac{1}{2}}(u)\, du\right) dt. \tag{6.2}$$

We can now state

Theorem 6.1. *Let D satisfy the assumptions* **A**. *Let φ and ψ be defined by* (6.1) *and* (6.2). *Then φ is a convex function of ψ for $0 \leqslant x \leqslant \xi$ and*

$$\varphi(x) \leqslant \psi(x)\theta(\xi)\psi^{-1}(\xi). \tag{6.3}$$

To prove Theorem 6.1 we consider

$$\varphi_\varepsilon(x) = \int_{\vartheta_{x,\varepsilon}} u_\varepsilon^2(x, y)\, dy.$$

In the following lemmata 6.1–6.4 we drop the index ε. This means that we work under the assumptions that $u = 0$ on $\partial D_\xi - \theta_\xi$ and that u is harmonic in a neighbourhood of each point of ∂D_ξ for which $x_1 < \xi$. Actually $\vartheta_{x,\varepsilon} = \phi$ for $x \leqslant x_\varepsilon$, $\vartheta_{x,\varepsilon} \neq \phi$ for $x > x_\varepsilon > 0$, but when dropping the index ε we also write 0 instead of x_ε.

Lemma 6.1. *$\vartheta(x)$ is continuous and $\lambda(x)$ is upper semicontinuous*, $0 < x < \xi$.

Proof. For simplicity we treat only the case $n = 3$. Suppose that $\vartheta(x)$ is discontinuous at $x = \sigma$. Then $S = \{z \mid u(z) = 0\}$ contains a surface element of positive measure in the plane $x_1 = \sigma$. This is impossible, since an analytic surface has at most a finite number of points in common with any straight line (not contained in the surface). Thus $\vartheta(x)$ is continuous.

When the plane $x_1=\sigma$ is not tangent to S, each component ϑ_σ^i of ϑ_σ is bounded by a finite number of analytic curves, and $\lambda_i(\sigma)$ is the principal eigenvalue of ϑ_σ^i. If the plane $x_1=\sigma$ is tangent to S, $\partial\vartheta_\sigma^i$ may for instance contain isolated points. In such a case $\lambda_i(\sigma)$ is taken to be $\inf_\Omega \lambda$, where Ω is contained in ϑ_σ^i and $\partial\Omega$ consists of a finite number of analytic curves. Then $\lambda_i(\sigma)=\inf R(f)$ for $f\in V$ in ϑ_σ^i.

Given a number $A>\lambda(\sigma)=\lambda_i(\sigma)$, we can choose Ω so that A is the principal eigenvalue of Ω and Ω is strictly contained in ϑ_σ^i. Then Ω is contained in some component of ϑ_x for all x sufficiently near σ. Thus $A>\lambda(x)$ for all x sufficiently near σ and λ is upper semicontinuous.

Lemma 6.2. $\varphi'(x)=2\int_{\vartheta_x} uu_x\,dy,\ 0<x<\xi.$

Proof. Let $\vartheta_a\backslash\vartheta_b$ be the set of points y belonging to ϑ_a but not to ϑ_b. Then

$$\varphi(x+h)-\varphi(x)=\int_{\vartheta_{x+h}} u^2(x+h,y)\,dy-\int_{\vartheta_x} u^2(x,y)\,dy=\int_{\vartheta_x}(u^2(x+h,y)-u^2(x,y))\,dy$$

$$+\int_{\vartheta_{x+h}\backslash\vartheta_x} u^2(x+h,y)\,dy-\int_{\vartheta_x\backslash\vartheta_{x+h}} u^2(x+h,y)\,dy=I_1+I_2+I_3.$$

Dividing I_1 by h and letting h tend to zero we obtain $2\int_{\vartheta_x} uu_x\,dy$. Now consider I_2 for small h. To each point $(x+h,y)$, $y\in\vartheta_{x+h}\backslash\vartheta_x$, belongs some point (s,y), $x\leqslant s\leqslant x+h$, such that $u(s,y)=0$. As the measure of $\vartheta_{x+h}\backslash\vartheta_x$ is $O(1)$, I_2 is $O(h^2)$. In the same way I_3 is $O(h^2)$. This proves the lemma.

Lemma 6.3. $\varphi''(x)=2\int_{\vartheta_x}|\operatorname{grad} u|^2\,dy,\ 0<x<\xi.$

Proof. By Lemma 6.2 and Green's formula

$$\varphi'(x)=2\int_0^x\left(\int_{\vartheta_x}|\operatorname{grad} u|^2\,dy\right)dx.$$

The integrand $\int_{\vartheta_x}|\operatorname{grad} u|^2\,dy$ being continuous by Lemma 6.1, we obtain our lemma.

Lemma 6.4. (*Carleman's differential inequality.*)

$$\varphi''(x)\geqslant 2\varphi'(x)\,\lambda^{\frac12}(x),\ 0<x<\xi.$$

Proof. By Lemma 6.3

$$\varphi''(x)=2\int_{\vartheta_x} u_x^2\,dy+2\int_{\vartheta_x}|\operatorname{grad}_y u|^2\,dy,\ 0<x<\xi.$$

The first integral is estimated by applying the Schwarz inequality to $\varphi'(x)$ in Lemma 6.2. Thus

$$\int_{\vartheta_x} u_x^2\,dy\geqslant\frac{\varphi'^2}{4\varphi}. \tag{6.4}$$

The second integral is estimated by **B**.

$$\int_{\vartheta_x^i} |\operatorname{grad}_y u|^2 dy \geqslant \lambda_i(x) \int_{\vartheta_x^i} u^2 dy \geqslant \lambda(x) \int_{\vartheta_x^i} u^2 dy.$$

Summing over i we obtain

$$\int_{\vartheta_x} |\operatorname{grad}_y u|^2 dy \geqslant \lambda(x)\,\varphi(x). \tag{6.5}$$

Hence
$$\frac{2\varphi''}{\varphi} \geqslant \frac{\varphi'^2}{\varphi^2} + 4\lambda,\ 0<x<\xi. \tag{6.6}$$

By the proof of Lemma 6.3 φ' is a Dirichlet integral and thus positive for $0<x<\xi$. By the inequality $(\varphi''/\varphi' - \varphi'/\varphi)^2 \geqslant 0$ and taking the square root the lemma now follows from (6.6).

Proof of Theorem 6.1. We now use the index ε again. By **B** and Lemma 6.1 $\{\lambda_\varepsilon^{\frac{1}{2}}\}$ is a decreasing sequence of integrable functions. $\lambda^{\frac{1}{2}} = \lim \lambda_\varepsilon^{\frac{1}{2}}$ is integrable over $0<x<\xi$. (Integrability at $x=0$ is guaranteed by the assumption that the measure of θ_0 is positive.) By Lemma 6.4

$$\varphi_\varepsilon''(x) \geqslant 2\varphi_\varepsilon'(x)\,\lambda_\varepsilon^{\frac{1}{2}}(x) \geqslant 2\varphi_\varepsilon'(x)\,\lambda^{\frac{1}{2}}(x),\ x_\varepsilon < x < \xi.$$

For ψ defined by (6.2)

$$\psi''(x) = 2\lambda^{\frac{1}{2}}(x)\,\psi'(x) \text{ a.e.}$$

Hence
$$\frac{d}{dx}(\log \varphi_\varepsilon' - \log \psi') \geqslant 0 \text{ a.e.}$$

Since $\log \varphi_\varepsilon'$ and $\log \psi'$ are absolutely continuous on an interval $\alpha \leqslant x \leqslant \beta$, $x_\varepsilon < \alpha < \beta < \xi$, φ_ε is thus a convex function of ψ on any such interval. By continuity at $x=0$ and $x=\xi$, $\varphi = \lim \varphi_\varepsilon$ is a convex function of ψ for $0 \leqslant x \leqslant \xi$. Thus (6.3) is true and Theorem 6.1 is proved.

Remark. (Asymptotic equality in Carleman's differential inequality.)

Equality in (6.4) holds if and only if $u = f u_x$, where f depends only on x. When ϑ_x is connected and $\partial\vartheta_x$ smooth, equality holds in (6.5) if and only if $u = gv$, where g depends only on x and v is the first eigenfunction of $\Delta v + \lambda v = 0$ in ϑ_x, $v=0$ on $\partial\vartheta_x$.

Now let D_ξ be a right cylinder, $D_\xi = \{z \mid x_1 < \xi,\ y \in \Theta\}$, where Θ is simply connected and has a smooth boundary. Let $\{\lambda_n\}_1^\infty$ be the eigenvalues $(\lambda_1 = \lambda)$ and $\{v_n\}_1^\infty$ the corresponding eigenfunctions of $\Delta v + \lambda v = 0$ in Θ, $v=0$ on $\partial\Theta$. Then, by the method of separation of variables (the c_n denoting constants)

$$u(x, y) = \sum_1^\infty c_n \exp\,(\lambda_n^{\frac{1}{2}}(x-\xi))\,v_n(y).$$

Thus, for large negative values of $x-\xi$,

$$u(x, y) \sim c_1 \exp\,(\lambda^{\frac{1}{2}}(x-\xi))\,v_1(y),$$

$$u_x(x, y) \sim c_1 \lambda^{\frac{1}{2}} \exp (\lambda^{\frac{1}{2}}(x-\xi))\, v_1(y).$$

φ, as well as φ_ε, is now twice differentiable, and asymptotic equality in (6.4) and (6.5) holds for φ.

Instead of defining φ as an integral over ϑ_x as in (6.1) we now define φ as an integral over θ_x. In order to estimate φ in this case we introduce the following additional assumptions about D.

D. Θ_x is connected, $x \leqslant x_0$. ∂D is smooth. In any finite interval ∂D has a finite number of tangent hyperplanes $x_1 = c$.

E. $\lambda(x)$ is defined as in **C**, but with respect to θ_x instead of ϑ_x. (For $x \leqslant x_0$ θ_x is understood to be equal to Θ_x).

Lemma 6.5. *Let D satisfy* **A** *and* **D**. *Then* (6.3) *in Theorem 6.1 is true for $x = x_0$ with φ, λ, and ψ defined with respect to θ_x instead of ϑ_x.*

Proof. The choice of θ_x is due to Tsuji [6.8, p. 112]. In the case of simply connected θ_ξ and D_ξ he takes into account only the first of the θ_x^i separating θ_0 from θ_ξ, cf. Ahlfors' distortion inequality (2.3). However, some irrelevant components of ϑ_x not separating θ_0 from θ_ξ also enter the discussion. In Fig. 6.1 the θ_x cover the shaded area.

Let $x_1 = \sigma$ be those hyperplanes tangent to ∂D for which D_σ is a proper subdomain of $D_{\sigma+0}$.

Let the σ belonging to the interval $0 < x < \xi$ be $\sigma_1 < \sigma_2 < \dots < \sigma_m$. Then

$$\varphi_\varepsilon(\sigma_\mu) \leqslant \varphi_\varepsilon(\sigma_\mu + 0),\ \varphi_\varepsilon'(\sigma_\mu - 0) \leqslant \varphi_\varepsilon'(\sigma_\mu + 0),\ \mu = 1, 2, \dots, m.$$

Thus φ_ε and $d\varphi_\varepsilon/d\psi$ have positive jumps at $x = \sigma_\mu$, $\mu = 1, 2, \dots, m$. According to the proof of Theorem 6.1 φ_ε is a convex function of ψ on intervals not containing any of the σ. Thus, for $x_\varepsilon < x \leqslant x_0 \leqslant \sigma_1$,

$$\varphi_\varepsilon(x) \leqslant \theta(\xi) \frac{\psi(x) - \psi(x_\varepsilon)}{\psi(\xi) - \psi(x_\varepsilon)}.$$

The lemma now follows by letting ε tend to zero.

Lemma 6.6. $u(z_0) \leqslant c\varphi^{\frac{1}{2}}(x_0)$, *where $z_0 = (x_0, y_0)$ and c depends only on the geometry of D near z_0.*

Proof. This lemma is an immediate consequence of Harnack's inequality for positive harmonic functions. If a closed sphere $S(r; z_0)$ with radius r and centre at z_0 is contained in D, then for $z \in S(r/2; z_0)$

$$u(z) \geqslant (\tfrac{2}{3})^{n-2} \tfrac{1}{3}\, u(z_0),$$

and hence
$$\varphi(x) \geqslant c_n r^{n-1} u^2(z_0),$$

where c_n only depends on n. This proves the lemma.

We now discuss estimating φ from

$$\varphi(x) \leqslant \psi(x) \theta(\xi) \psi^{-1}(\xi). \tag{6.3}$$

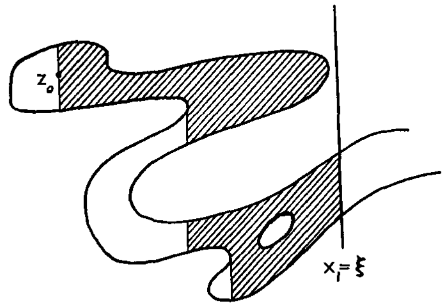

Fig. 6.1

Writing μ instead of $2\lambda^{\frac{1}{2}}$, we have instead of (6.2)

$$\psi(x)=\int_0^x \exp\left(\int_0^t \mu(u)\,du\right)dt.$$

Since $\vartheta(x)\leqslant M$, it follows by the Faber–Krahn inequality that $\mu(x)\geqslant m>0$. Hence

$$\psi(x)=\exp\left(\int_0^x \mu(u)\,du\right)\int_0^x \exp\left(-\int_t^x \mu(u)\,du\right)dt\leqslant m^{-1}\exp\left(\int_0^x \mu(u)\,du\right). \tag{6.7}$$

In the same way, assuming that $\mu(x)\leqslant m_1$, we obtain for $\xi>m_1^{-1}\log 2$

$$\psi(\xi)\geqslant(2m_1)^{-1}\exp\left(\int_0^\xi \mu(u)\,du\right)$$

and hence

$$\varphi(x)\leqslant c\exp\left(\int_x^\xi \mu(u)\,du\right). \tag{6.8}$$

It is, of course, unsatisfactory to require that μ be bounded. In general the estimate (6.8) does not follow from (6.3). For instance take $\theta(x)\equiv\vartheta(x)\equiv 1$ and $\mu(x)\equiv 2x$, $x>0$. Then, for large ξ $\psi(\xi)\sim(2\xi)^{-1}\exp\xi^2$. A general result is given in the following lemma.

Lemma 6.7. *Let D satisfy* **A** *and λ be defined by* **C**. *Then for $0<l<\xi-x_0$, $z_0=(x_0, y_0)$,*

$$u(z_0)\leqslant cl^{-\frac{1}{2}}\exp\left(-\int_{x_0}^{\xi-l}\lambda^{\frac{1}{2}}(x)\,dx\right),$$

where the constant c depends only on **A** *and the geometry of D near z_0. If D also satisfies* **D** *this is true with λ defined by* **E**.

Proof. We use a trivial estimate of $\psi(\xi)$. Writing $\mu=2\lambda^{\frac{1}{2}}$, we have for $l>0$

$$\psi(\xi)\geqslant\exp\left(\int_0^{\xi-l}\mu(u)\,du\right)\int_{\xi-l}^{\xi}\exp\left(\int_{\xi-l}^{t}\mu(u)\,du\right)dt\geqslant l\exp\left(\int_0^{\xi-l}\mu(u)\,du\right).$$

Hence, by (6.3) and (6.7)

$$\varphi(x_0) \leqslant c\, l^{-1} \exp\left(-\int_{x_0}^{\xi-l} \mu(u)\, du\right).$$

Our lemma now follows from Lemma 6.5 and Lemma 6.6.

Remark. Tsuji proved this result in polar coordinates in the plane [6.8, pp. 112–117], but with a more complicated derivation of the estimate of φ.

In Theorem 6.3 we shall estimate φ from (6.3) in some special cases. First, however, we shall make the same choice among the components of ϑ_x as in Theorem 3.2 with Carleman's method. By Lemma 6.2 and Green's formula $\varphi_\varepsilon'/2$ is a Dirichlet integral. By working in terms of convexity in the proof of Theorem 6.1 difficulties due to infinite Dirichlet integrals are avoided. Now we have to introduce auxiliary functions possessing finite Dirichlet integrals. We shall assume that D satisfies **D** and D_ξ **F**.

F. θ_ξ is connected, and the diameter of $D_\xi - D_{\xi-1}$ is less than a fixed constant.

G. Given ξ, let θ_x^i, $i=1, 2, \ldots, n(x)$, denote the components or unions of components of ϑ_x separating θ_0 from θ_ξ. $\lambda_i(x)$ is now defined with respect to θ_x^i in the same way as was $\lambda(x)$ with respect to ϑ_x in **C**. We write

$$\lambda^{\frac{1}{2}}(x) = \sum_{i=1}^{n(x)} \lambda_i^{\frac{1}{2}}(x).$$

Lemma 6.8. *Let D satisfy* **A** *and* **D** *and let D_ξ satisfy* **F**. *Let λ be defined by* **G**. *Then for $z_0=(x_0, y_0)$, $\xi > x_0+1$,*

$$u(z_0) \leqslant c \exp\left(-\int_{x_0}^{\xi-1} \lambda^{\frac{1}{2}}(x)\, dx\right), \tag{6.9}$$

where the constant c depends only on **A**, **F**, *x_0, and the geometry of D near z_0.*

Proof. We first modify D_ξ. Since, by **F**, the diameter of $D_\xi - D_{\xi-1}$ is bounded, each ϑ_x with $\xi-1 \leqslant x \leqslant \xi$ is contained in an $(n-1)$-dimensional sphere of fixed radius R and centre Y. Set $G_\xi = \{z \mid \xi-1 < x_1 < \xi,\ |y-Y| < R\}$. We consider $D_\xi \cup G_\xi$ instead of D_ξ, but do not change our notation. Thus $\theta_\xi = \{z \mid x_1 = \xi,\ |y-Y| < R\}$. Set $\theta_\xi' = \{z \mid x_1=\xi,\ |y-Y| < R/3\}$ and $\theta_\xi'' = \{z \mid x_1 = \xi,\ |y-Y| < 2R/3\}$. We choose a function F, twice continuously differentiable and monotonic for $\frac{1}{3} \leqslant t \leqslant \frac{2}{3}$, with $F(\frac{1}{3})=1$ and $F(\frac{2}{3})=0$.

Now let f be harmonic in D_ξ with boundary values 0 on $\partial D_\xi - \theta_\xi''$, $F(R^{-1}|y|)$ on $\theta_\xi'' - \theta_\xi'$, and 1 on θ_ξ'. We shall need an inequality of the following type:

$$u(z) \leqslant c f(z). \tag{6.10}$$

Applying the method of separation of variables in the cylinder G_ξ, we obtain that $u(z)$ and $f(z)$ tend to zero in the same way when $z \in \vartheta_{\xi-\frac{1}{2}}$ tends to $\partial\vartheta_{\xi-\frac{1}{2}}$. Therefore we can, with the aid of Harnack's inequality, establish the inequality (6.10) on $\vartheta_{\xi-\frac{1}{2}}$ and hence in $D_{\xi-\frac{1}{2}}$.

Let D_x^i denote the subdomain of D separated from θ_ξ by θ_x^i. We assume that the θ_x^i, $i=1, 2, \ldots, n(x)$, are taken in such order that $D_x^i \subset D_x^{i+1}$, $i=1, 2, \ldots, n(x)-1$.

We now define

$$D_{\xi,\varepsilon}=\{z\,|\,z\in D_\xi,\ f(z)>\varepsilon\},\ \theta^i_{x,\varepsilon}=\{z\,|\,z\in\theta^i_x,\ f(z)>\varepsilon\},$$

$$D^i_{x,\varepsilon}=\{z\,|\,z\in D^i_x,\ f(z)>\varepsilon\},\ f_\varepsilon=\max\ (f-\varepsilon,\ 0),$$

and

$$\varphi_{i,\varepsilon}(x)=\int_{\theta^i_{x,\varepsilon}}|f_\varepsilon|^2\,dy,\ d_{i,\varepsilon}(x)=\int_{D^i_{x,\varepsilon}}|\operatorname{grad} f_\varepsilon|^2\,dz.$$

$\{\varepsilon\}$ is taken to be a sequence of the type considered in **C** (with respect to f).

Let $x_1=\sigma$ be those hyperplanes tangent to ∂D for which one of the following situations occurs for some i. I. D^i_σ is a proper subdomain of $D^i_{\sigma+0}$, and $d_{i,\varepsilon}(\sigma)\leqslant d_{i,\varepsilon}(\sigma+0)$. II. D^i_σ is a proper subdomain of $D^i_{\sigma-0}$, and $d_{i,\varepsilon}(\sigma)\leqslant d_{i,\varepsilon}(\sigma-0)$. III. The interior of $D^\sigma_{i+1}-D^i_\sigma$ contains no θ^j_x ($j=1,2,\ldots,n(x)$), and $d_{i,\varepsilon}(\sigma)\leqslant d_{i+1,\varepsilon}(\sigma)$. Cf. Fig. 3.2.

On an interval I (not containing any points σ) where D^i_x increases with x

$$2d_{i,\varepsilon}(x)=\varphi'_\varepsilon(x)$$

by Lemma 6.2. Let λ_i be defined by **G**. By virtue of Lemma 6.4 we obtain

$$d'_{i,\varepsilon}(x)\geqslant 2\lambda_i^{\frac{1}{2}}(x)\,d_{i,\varepsilon}(x). \tag{6.11}$$

On an interval J (not containing any points σ) where D^i_x increases with $-x$ we obtain in the same way

$$-d'_{i,\varepsilon}(x)\geqslant 2\lambda_i^{\frac{1}{2}}(x)\,d_{i,\varepsilon}(x). \tag{6.12}$$

Now we run through $D_{\xi-1}$ from z_0 to $\theta_{\xi-1}$ so that each θ^i_x is passed once. Then the Dirichlet integral of f_ε over $D^i_{x,\varepsilon}$ increases with $D^i_{x,\varepsilon}$ on intervals I and J according to (6.11) and (6.12). At the points σ the Dirichlet integral of f_ε has non-negative increments as described in I–III. When running through $D_{\xi-1}$ in this manner we integrate in (6.11) and (6.12). Writing d_ε instead of $d_{1,\varepsilon}$ we obtain

$$d_\varepsilon(x_0)\leqslant d_\varepsilon(\xi-1)\ \exp\left(-2\int_{x_0}^{\xi-1}\left(\sum_{i=1}^{n(x)}\lambda_i^{\frac{1}{2}}(x)\right)dx\right)\leqslant d_\varepsilon(\xi)\ \exp\left(-2\int_{x_0}^{\xi-1}\lambda^{\frac{1}{2}}(x)\,dx\right).$$

By Green's formula (with inner normal derivatives) we obtain

$$d_\varepsilon(\xi)=-\int_{\theta_\xi}f_\varepsilon\frac{\partial f_\varepsilon}{\partial n}dy=-\int_{\theta_\xi}f_\varepsilon\frac{\partial f}{\partial n}dy\to-\int_{\theta_\xi}f\frac{\partial f}{\partial n}dy,$$

when $\varepsilon\to 0$. By the maximum principle

$$-\int_{\theta_\xi}f\frac{\partial f}{\partial n}dy\leqslant-\int_{\theta_\xi}g\frac{\partial g}{\partial n}dy,$$

where g is harmonic in G_ξ with boundary values f on θ_ξ and boundary values 0 on $\partial G_\xi-\theta_\xi$. Hence for sufficiently small ε, $d_\varepsilon(\xi)\leqslant c_0$.

Finally we estimate $f(z_0)$. Since $2d_\varepsilon=\varphi'_\varepsilon$ is increasing,

$$\varphi_\varepsilon(x_0)\leqslant x_0\,\varphi'_\varepsilon(x_0)\leqslant c\,d_\varepsilon(x_0).$$

We let ε tend to zero and then estimate $f(z_0)$ by Lemma 6.6. Thus we obtain (6.9) for $f(z_0)$. By (6.10), (6.9) is also valid for $u(z_0)$ in the modified domain and hence for the original $u(z_0)=\omega(z_0;\,\theta_\xi;\,D_\xi)$.

Remark. The original domain D_ξ was modified so as to 1° facilitate defining auxiliary functions f, 2° assure the boundedness of $d_\varepsilon(\xi)$, and 3° allow estimating u in terms of f. In the two-dimensional case we can instead begin by considering domains D bounded by a finite number of analytic curves. f can now be defined in the original domain D_ξ as above. Discussion of f_ε is unnecessary. We can use the technique of Lemma 6.9 below. We define N in the following way: $\xi \in N$ if and only if an isosceles triangle Δ_ξ with base along θ_ξ and a fixed opposite angle 2α is contained in D_ξ. When $\xi\in N$, the Dirichlet integral of f is bounded by a fixed constant and furthermore the inequality (6.10) is correct. If $\xi \notin N$ a simple expedient is to consider $D_\xi \cup \Delta_\xi$ instead of D_ξ.

From Lemma 6.7 and Lemma 6.8 we now obtain the following

Theorem 6.2. *Let D satisfy* **A** *p. 13 and let λ be defined by* **C** *p. 14. $z_0=(x_0, y_0)$ is a fixed point in D. Then for $\xi > x_0+1$*

$$\omega(z_0;\,\theta_\xi;\,D_\xi)\leqslant c\,\exp\left(-\int_{x_0}^{\xi-1}\lambda^{\frac{1}{2}}(x)\,dx\right). \tag{6.13}$$

If D satisfies **A** *p. 13 and* **D** *p. 17 and λ is defined by* **E** *p. 17,* (6.13) *is also correct.*

Finally, (6.13) *is correct if D satisfies* **A** *p. 13,* **D** *p. 17, and D_ξ satisfies* **F** *p. 19, and λ is defined by* **G** *p. 19.*

The constant c depends only on the geometry of D near z_0 and constants appearing in the conditions satisfied by D and D_ξ; when **F** *is used c also depends on x_0.*

Remark. The conditions involving smoothness in **A** and **D** were introduced for simplicity and are not essential. If **A**, **D**, **F** (or some of them) – apart from smoothness conditions—are satisfied, we can exhaust D_ξ with a monotone sequence of subdomains $D_\xi^{(\nu)}$, $\nu=1, 2, \ldots$, in which the theorem above can be applied. Now $\omega(z;\,\theta_\xi;\,D_\xi)$ and $\lambda(x)$ in D_ξ are defined from the corresponding quantities $\omega^{(\nu)}$ and $\lambda^{(\nu)}(x)$ in $D_\xi^{(\nu)}$ by a limiting process. $\{\lambda^{(\nu)}(x)\}$ is a non-increasing sequence. It follows that (6.13) is valid even though the smoothness conditions are not satisfied.

If λ is bounded we can integrate up to ξ in (6.13) and let c also depend on the least upper bound of λ. We shall now show that integration up to ξ in (6.13) is possible in some special cases when λ is not bounded.

Theorem 6.3. *Let D satisfy* **A** *p. 13 and Θ_x be connected, $x>0$. Let $\lambda(x)$ be defined by* **C** *p. 14. Let $\Theta(x)$ and $\lambda(x)$ be continuous, $x>0$. $z_0=(x_0, y_0)$ is a fixed point in D. Then*

$$\omega(z_0;\,\Theta_\xi;\,D_\xi)\leqslant c\,\exp\left(-\int_{x_0}^{\xi}\lambda^{\frac{1}{2}}(x)\,dx\right) \tag{6.14}$$

is implied by any one of the following three conditions.

(*a*) $n=2$.

(*b*) $n>2$, $\lambda^{\frac{1}{2}}(x)\,\Theta^{r}(x)\leqslant M_0$, $x>0$, *for some* $r<1$.

(*c*) $n > 2$, $\lambda^{\frac{1}{2}}(x)\,\Theta(x) \leqslant M_0$, $\lambda(x)$ *non-decreasing*, $x > 0$.

The condition (*d*) *implies* (6.15).
(*d*) $n > 2$, $\lambda^{\frac{1}{2}}(x) \equiv \lambda^{\frac{1}{2}}(x_0)\,k(x) \equiv \lambda^{\frac{1}{2}}(x_0)\,(\Theta(x)/\Theta(x_0))^{-1/)n-1)}$, $k(x)$ *non-decreasing*, $x > 0$.

$$\omega(z_0;\, \Theta_\xi;\, D_\xi) \leqslant c\, k^{2-n}(\xi)\, \exp\left(-\int_{x_0}^{\xi} \lambda^{\frac{1}{2}}(x_0)\, k(x)\, dx\right). \tag{6.15}$$

The constants c depend only on the geometry of D near z_0, x_0, *and constants occurring in* **A**, (*b*), *and* (*c*).

We first prove a lemma.

Lemma 6.9. *Let f be continuous and bounded*, $0 < f(x) \leqslant K$, $x \geqslant 0$, *and* $f(x) = K$, $0 \leqslant x \leqslant a$. *Then for* $p > 1$ *and* $\xi \geqslant a$

$$\max_{x \leqslant \xi} f^{-p}(x) \int_0^x \exp\left(-\int_t^{\xi} f^{-1}(u)\, du\right) dt \geqslant c > 0, \tag{6.16}$$

where c only depends on p, a, and *K*.

Proof. Let Γ denote the curve $y = f(x)$, $x \geqslant 0$, in the xy-plane. Δ_x denotes a triangle with vertices $(x, 0)$, $(x, f(x))$, $(x - f(x) \cot \alpha, 0)$, where $\operatorname{tg} \alpha = q$ is a large constant. Set $N = \{x \,|\, \text{the interior of } \Delta_x \text{ lies below } \Gamma\}$. If we choose $q > K/a$, N will certainly be non-empty.

Now let ξ belong to N. Given ξ, we take $t_0 = \xi - (2q)^{-1} f(\xi)$.

Then
$$\int_{t_0}^{\xi} f^{-1}(x)\, dx \leqslant \int_{t_0}^{\xi} (f(\xi) - q(\xi - x))^{-1} dx \leqslant \log 2.$$

Hence
$$f^{-1}(\xi) \int_0^{\xi} \exp\left(-\int_t^{\xi} f^{-1}(u)\, du\right) dt \geqslant (4q)^{-1}. \tag{6.17}$$

By taking $x = \xi$ in (6.16), the truth of (6.16) follows for $\xi \in N$.

Now consider a $\xi_0 \notin N$, $\xi_0 > a$. By continuity there exists a largest $b < \xi_0$ such that $b \in N$. Then (6.17) is valid for $\xi = b$ and furthermore

$$\exp\left(-\int_b^{\xi_0} f^{-1}(u)\, du\right) \geqslant (1 + q(\xi_0 - b)\, f^{-1}(b))^{-1/q}, \tag{6.18}$$

and $\xi_0 - b \leqslant K \cot \alpha$. Now take $x = b$ and $\xi = \xi_0$ in (6.16) and use (6.17) with $\xi = b$. Then it remains to determine a fixed lower bound for

$$f^{1-p}(b)\, \exp\left(-\int_b^{\xi_0} f^{-1}(u)\, du\right).$$

By virtue of (6.18) this is done by choosing $q \geqslant (p-1)^{-1}$. Thus the lemma is proved.

Proof of Theorem 6.3. We use the technique of Lemma 6.9. By (6.3), (6.7), and Lemma 6.6 we can reduce the proof of our theorem to determining a fixed positive lower bound for

$$\Theta^{-1}(\xi)\int_0^\xi \exp\left(-2\int_t^\xi \lambda^{\frac{1}{2}}(u)\,du\right)dt.$$

If this cannot be done for all large ξ, we can under the condition (a) obtain the desired estimate for $u(z_0)$ by a simple estimate of an auxiliary harmonic measure. Under the condition (b) we instead reduce the proof to determining a fixed positive lower bound for

$$\max_{x\leqslant\xi}\Theta^{-1}(x)\int_0^x \exp\left(-2\int_t^\xi \lambda^{\frac{1}{2}}(u)\,du\right)dt. \tag{6.19}$$

$(a)\Rightarrow(6.14)$. We use the technique of Lemma 6.9 with $f=(2\pi)^{-1}\Theta=\frac{1}{2}\lambda^{-\frac{1}{2}}$. (The assumption that f be constant for small x is not essential. It can be satisfied by a modification of D.) If $\xi\in N$, (6.17) is true, and (6.14) follows. Now consider a $\xi\notin N$, ξ sufficiently large. By continuity there exists a largest $b<\xi$, such that $b\in N$. Then (6.17) is true for b instead of ξ. By the maximum principle

$$\omega(z_0;\,\Theta_\xi;\,D_\xi)\leqslant\omega(z_0;\,\Theta_b;\,D_b)\max_{z\in\Theta_b}\omega(z;\,\Theta_\xi;\,D_\xi).$$

Hence it suffices to prove that

$$\exp\left(\pi\int_b^\xi \Theta^{-1}(u)\,du\right)\max_{z\in\Theta_b}\omega(z;\,\Theta_\xi;\,D_\xi)\leqslant c. \tag{6.20}$$

By the definition of b

$$2\pi\int_b^\xi \Theta^{-1}(u)\,du\leqslant q^{-1}\log\,(1+2\pi q(\xi-b)\,\Theta^{-1}(b)).$$

If $(\xi-b)\Theta^{-1}(b)\leqslant c_1$ (a fixed number), then (6.20) is correct. If $(\xi-b)\Theta^{-1}(b)>c_1$, we use a simple estimate of $\omega(z;\,\Theta_\xi;\,D_\xi)$. Set $l=\{z\,|\,x_1=\xi\}$ and let G be the domain bounded by l and $\{z\,|\,x_1=b,\ y\geqslant\Theta(b)/2\}$. By the extension principle and an explicit conformal mapping of G onto a half-plane we obtain

$$\max_{z\in\Theta_b}\omega(z;\,\Theta_\xi;\,D_\xi)\leqslant\omega(b;\,l;\,G)\leqslant c(\xi-b)^{-\frac{1}{2}}\Theta^{\frac{1}{2}}(b).$$

Hence (6.20) is correct, and this part of the lemma is proved.

$(b)\Rightarrow(6.14)$. We use Lemma 6.9 with $f=\frac{1}{2}\lambda^{-\frac{1}{2}}$. Since $\Theta(x)\leqslant M$, $x>0$, $\lambda^{\frac{1}{2}}(x)$ is bounded from below according to the Faber–Krahn inequality. The condition (b) now implies (6.16). We thus obtain a fixed lower bound for (6.19) and (6.14) follows.

$(c)\Rightarrow(6.14)$, $(d)\Rightarrow(6.15)$. We use the technique of Lemma 6.9 with $f=\frac{1}{2}\lambda^{-\frac{1}{2}}$. For a non-increasing f all sufficiently large ξ are in N and (6.17) gives the desired results. (Instead of f being non-increasing we can require that f' exists and is bounded by a fixed constant.)

7. *The Nevanlinna mean*

In a special case lower bounds of harmonic measures can be established by studying the Nevanlinna mean (7.2). Heins used this name for (7.2) in the case of a rectangle or a half-plane (in polar coordinates) [7.1, p. 4].

We consider domains D such that $\Theta_x \neq \phi$, $-\infty < x < \infty$. Θ_0 is assumed to be Steiner symmetric with respect to the coordinate hyperplanes $y_i = 0$, $i = 1, 2, \ldots, n-1$. By this we mean that the intersection of Θ_0 with a straight line perpendicular to $y_i = 0$ is either a single line-segment symmetrical with respect to $y_i = 0$ or empty, $i = 1, 2, \ldots, n-1$. When $n > 2$ $\partial\Theta_0$ is assumed to possess piecewise continuous curvature. Θ_x is is obtained from Θ_0 by the mapping $y \to k^{-1}(x)y$, $k(0) = 1$, $k(x) > 0$, $k(x)$ non-decreasing and twice continuously differentiable, $-\infty < x < \infty$. Under these assumptions we shall establish the following

Theorem 7.1. *Let λ be the principal eigevalue of Θ_0. Then*

$$\max_{z \in \Theta_x} \omega(z;\, \Theta_\xi;\, D_\xi) \geq (k(\xi)/k(x))^{1-n} \exp\left(-\lambda^{\frac{1}{2}} \int_x^\xi k(t)\, dt\right).$$

We start by proving some lemmata. We use the following notation. Let $\{v_n\}_1^\infty$ be the normed eigenfunctions of $\Delta v + \lambda v = 0$ in Θ_0, $v = 0$ on $\partial\Theta_0$, and $\{\lambda_n\}_1^\infty$ the corresponding eigenvalues. Let $\{V_{x,n}\}_1^\infty$ and $\{\Lambda_{x,n}\}_1^\infty$ be defined in the same way with respect to Θ_x. Then

$$\Lambda_{x,n} = k^2(x)\,\lambda_n,\quad V_{x,n}(y) = k^{(n-1)/2}(x)\, v_n(k(x)\,y). \tag{7.1}$$

In the following we write λ and v instead of λ_1 and v_1.

Lemma 7.1. *Let $m(x)$ be the Nevanlinna mean of $u(x, y) = \omega(z;\, \Theta_\xi;\, D_\xi)$ over Θ_x, $x \leq \xi$,*

$$m(x) = \int_{\Theta_x} u(x, y)\, v(k(x)\, y)\, dy. \tag{7.2}$$

Let $y \cdot \mathrm{grad}\, v$ denote the scalar product in Θ_0. Then

$$m'(x) = \int_{\Theta_x} u_x(x, y)\, v(k(x)\, y)\, dy + k'(x) \int_{\Theta_x} u(x, y)\, (y \cdot \mathrm{grad}\ v)\, (k(x)\, y)\, dy,\ x < \xi. \tag{7.3}$$

Proof. Under our assumptions about k and $\partial\Theta_0$, u possesses continuous first derivatives up to the boundary at points where $\partial\Theta_0$ is of continuous curvature (in the case $n > 2$) for $x_1 < \xi$ [7.3, p. 635]. The same is true for v, since $\exp(\lambda^{\frac{1}{2}} x_1)v(y)$ is harmonic in a right cylinder with base Θ_0. The lemma then follows.

Lemma 7.2. $\int_{\Theta_x} u_x(x, y)\, v(k(x)\, y)\, dy \leq \lambda\, k(x)\, m(x)$, $x < \xi$.

Proof. Let x be fixed and set $G = \{z = (s, t) \,|\, s < x,\ t \in \Theta_x\}$. We write Γ instead of ∂G and define $\gamma = \Gamma - \Theta_x$. Let U be harmonic in G with boundary values $u(x, y)$ on Θ_x and 0 on γ. $G \subset D_x$ since Θ_0 is Steiner symmetric with respect to the coordinate hyperplanes and k is non-decreasing. By the maximum principle the inner normal derivatives of U and u satisfy the inequality $\partial U/\partial n \leq \partial u/\partial n$ on Θ_x. We write $V(y)$ instead of $v(k(x)y)$. Then

$$\int_{\Theta_x} u_x V\,dy \leqslant -\int_{\Theta_x} V \frac{\partial U}{\partial n}\,dy = -\int_{\Gamma} V \frac{\partial U}{\partial n}\,d\sigma, \tag{7.4}$$

where $d\sigma$ is the area element on Γ. We now apply Green's formula to the last integral and note that $U=0$ on γ and $\partial V/\partial n=0$ on Θ_x. (We first consider finite subdomains of G and then use some majorant of $\partial U/\partial n$ in the limiting process.) Taking (7.1) into account we obtain

$$\int_{\Gamma} V \frac{\partial U}{\partial n}\,d\sigma = \int_G U\Delta V\,dz = -\lambda k^2(x)\int_G UV\,dz.$$

Hence by (7.4),

$$\int_{\Theta_x} u_x(x,y)\,v(k(x)y)\,dy \leqslant \lambda k^2(x)\iint_G U(s,t)\,v(k(x)t)\,ds\,dt. \tag{7.5}$$

Now U can be represented in the following way in the cylinder G:

$$U(s,t) = \int_{\Theta_x} u(x,y)\,P(s,t;\,x,y)\,dy,\ s<x, \tag{7.6}$$

where
$$P(s,t;\,x,y) = \sum_{n=1}^{\infty} \exp\left((s-x)\Lambda_{x,n}^{\frac{1}{2}}\right) V_{x,n}(y)\,V_{x,n}(t).$$

We shall need the relation

$$\int_{\Theta_x} P(s,t;\,x,y)\,v(k(x)t)\,dt = \exp\left(\lambda^{\frac{1}{2}}(s-x)\,k(x)\right) v(k(x)y),\ s<x. \tag{7.7}$$

By (7.6) and (7.7), after first considering subdomains

$$\{z=(s,t)\,|\,z\in G,\ -\infty<a<s<b<x\}$$

of G, we obtain

$$\iint_G U(s,t)\,v(k(x)t)\,ds\,dt = \lambda^{\frac{1}{2}} m(x)\,k^{-1}(x).$$

By (7.5), this proves our lemma.

Lemma 7.3. *Under the assumption that Θ_0 is Steiner symmetric with respect to the coordinate hyperplanes, the scalar product $y\cdot\operatorname{grad} v$ is non-negative.*

Proof. We want to prove that v is symmetrically decreasing with respect to $y_i=0$, $i=1,2,\ldots,n-1$. Let us assume that this is false for some i. We then symmetrize v with respect to $y_i=0$ (cf. § 1) and denote the symmetrized function by v^*. By [7.2, pp. 184–186]

$$\int_{\Theta_0} v^2\,dy = \int_{\Theta_0} v^{*2}\,dy,\quad \int_{\Theta_0} |\operatorname{grad} v|^2\,dy \geqslant \int_{\Theta_0} |\operatorname{grad} v^*|^2\,dy.$$

By **B** in § 6 v minimizes the Rayleigh quotient and thus $v=v^*$, and the lemma is proved.

Proof of Theorem 7.1. Since k' is non-negative it follows from the lemmata 7.1, 7.2, and 7.3 that

$$m'(x) \leqslant \lambda^{\frac{1}{2}} k(x)\, m(x),\ x<\xi.$$

Hence
$$m(x) \geqslant m(\xi) \exp\left(-\lambda^{\frac{1}{2}} \int_x^{\xi} k(t)\, dt\right).$$

However,
$$m(\xi)=\int_{\Theta_\xi} v(k(\xi)\, y)\, dy = k^{1-n}(\xi) \int_{\Theta_0} v(y)\, dy = c_0\, k^{1-n}(\xi)$$

and v being positive in Θ_0,

$$m(x) \leqslant \max_y\, u(x,y) \int_{\Theta_x} v(k(x)\, y)\, dy = c_0\, k^{1-n}(x) \max_y\, u(x,y).$$

Hence
$$\max_y\, u(x,y) \geqslant (k(\xi)/k(x))^{1-n} \exp\left(-\lambda^{\frac{1}{2}} \int_x^{\xi} k(t)\, dt\right)$$

and Theorem 7.1 is proved.

Remark. Theorem 7.1 is of interest in connection with the estimate (6.15) in Theorem 6.3. Also cf. Theorem 2.1.

8. *Harmonic measures and probability theory*

Harmonic measures have a probabilistic interpretation in the theory of Brownian motion. A standard work of reference for this theory is that of Lévy [8.3]. Later works of Doob and others are not referred to here. Let the domain D in R^n be bounded by a finite number of closed surfaces. $p(z;\alpha;D)$ is the probability that the Brownian motion particle which starts from the point z in D first reaches ∂D on a subdomain α of ∂D. Then [8.3, p. 62]

$$\omega(z;\alpha;D)=p(z;\alpha;D). \tag{8.1}$$

This interpretation is useful for heuristic argument. For instance, the choice of ϱ in the proof of Theorem 3.2 appears reasonable since it takes into account those segments of Θ_x that the Brownian motion particle starting from z_0 (and the curves in Γ) must pass through to reach θ_ξ. It appears difficult to obtain majorants of harmonic measures by a study of Brownian motion or of the corresponding random walk. We can, however, prove symmetrization results for harmonic measures by considering independent components of a Brownian motion. Definitions of the different kinds of symmetrization are given in § 1.

Theorem 8.1. (*The two-dimensional case.*) *Let D_ξ be bounded by a finite number of simple closed curves; $\vartheta_x=\cup_{i=1}^{m(x)}\vartheta_x^i$, $m(x)\leqslant M$. Assume that the ϑ_x^i vary continuously;*

it is allowed that at a finite number of points two segments come together or one splits or one vanishes. * *denotes symmetrization with respect to the* x_1-*axis. Then*

$$\max_{z \in \vartheta_x} \omega(z;\ \theta_\xi;\ D_\xi) \leqslant \omega(x;\ \theta_\xi^*;\ D_\xi^*).$$

Proof. By (8.1) is is sufficient to prove that

$$\max_{z \in \vartheta_x} p(z;\ \theta_\xi;\ D_\xi) \leqslant p(x;\ \theta_\xi^*;\ D_\xi^*). \tag{8.2}$$

Consider a two-dimensional Brownian motion $\{(X(\tau),\ Y(\tau)) = Z(\tau),\ 0 \leqslant \tau < \infty\}$, starting from a point $z = (x, y)$, where $\{X(\tau), 0 \leqslant \tau < \infty\}$ and $\{Y(\tau), 0 \leqslant \tau < \infty\}$ denote one-dimensional Brownian motions and the components are independent. $X(\tau)$ and $Y(\tau)$ are continuous with probability one [8.3, p. 10], so we assume continuity of $X(\tau)$ and $Y(\tau)$ in the following.

Let $M_X(t)$ denote $\max X(\tau)$ when $0 \leqslant \tau \leqslant t$. M_X has an inverse function T_X indicating the first passage time [8.3, p. 31]. We now write

$$p(z;\theta_\xi;D_\xi) = \int_{t=0}^{\infty} P\{t \leqslant T_X(\xi) < t + dt,\ Z(\tau) \in D_\xi \quad \text{when} \quad 0 \leqslant \tau < t\}$$

$$= \int_{t=0}^{\infty} P\{t \leqslant T_X(\xi) < t+dt\}\, P\{Y(\tau) \in \vartheta_{X(\tau)} \quad \text{when} \quad 0 \leqslant \tau < t \,|\, t \leqslant T_X(\xi) < t+dt\}$$

and $\quad p(z;\ \theta_\xi^*;\ D_\xi^*)$

$$= \int_{t=0}^{\infty} P\{t \leqslant T_X(\xi) < t+dt\}\, P\{Y(\tau) \in \vartheta^*_{X(\tau)} \quad \text{when} \quad 0 \leqslant \tau < t \,|\, t \leqslant T_X(\xi) < t+dt\}.$$

Given a sample function $X(\tau)$, $0 \leqslant \tau \leqslant t$, we now consider

$$P\{Y(\tau) \in \vartheta_{X(\tau)} \quad \text{when} \quad 0 \leqslant \tau \leqslant t\}.$$

Some segments of $\vartheta_{X(\tau)}$ may be inaccessible to the Y-particle that is to reach $\vartheta_{X(t)}$. Later in the proof we require that the domain accessible to the Y-particle be bounded by curves continuous in τ. By translation (perpendicular to the τ-axis) of any inaccessible segments we obtain a larger accessible domain δ with accessible segments $\vartheta'_{X(\tau)}$ of total length $\vartheta'(X(\tau)) = \vartheta(X(\tau))$. We need only consider such components of the complement of δ that possess positive area. They can be enumerated according to the length of their projections on the τ-axis.

Thus, to prove (8.2) it suffices to prove the following inequality:

$$P\{Y(\tau) \in \alpha_\tau \quad \text{when} \quad 0 \leqslant \tau \leqslant t\} \leqslant P\{Y(\tau) \in \alpha_\tau^* \quad \text{when} \quad 0 \leqslant \tau \leqslant t\}, \tag{8.3}$$

where $Y(0) = 0$, and α_τ is a finite union of accessible open line-segments varying continuously in τ, so that two components of α_τ are separated by one point for at most a finite number of values of τ. * denotes symmetrization with respect to the τ-axis.

Now choose $\{\tau_\nu^{(k)}\}_0^k$, so that $0 = \tau_0^{(k)} < \tau_1^{(k)} < \ldots < \tau_k^{(k)} = t$ and $\{\tau_\nu^{(k)}\}_0^k$ becomes dense in $0 \leqslant \tau \leqslant t$, when $k \to \infty$. We shall prove that for any k

$$P\{Y(\tau_\nu^{(k)}) \in \alpha_{\tau_\nu^{(k)}},\ \nu = 1, 2, \ldots, k\} \leqslant P\{Y(\tau_\nu^{(k)}) \in \alpha^*_{\tau_\nu^{(k)}},\ \nu = 1, 2, \ldots, k\}. \tag{8.4}$$

Thanks to our assumptions about the α_τ (8.3) follows from (8.4) by e.g. [8.1, Theorem 2.2, p. 54]. Now, omitting the upper index k

$$P\{Y(\tau_\nu)\in\alpha_{\tau_\nu},\ \nu=1,2,\dots,k\}$$
$$=(2\pi)^{-k/2}(\tau_1(\tau_2-\tau_1)\dots(\tau_k-\tau_{k-1}))^{-\frac{1}{2}}\int_{\alpha_{\tau_1}}\dots\int_{\alpha_{\tau_k}}\exp\left(-\frac{y_1^2}{2\tau_1}-\frac{(y_2-y_1)^2}{2(\tau_2-\tau_1)}-\right.$$
$$\left.\dots-\frac{(y_k-y_{k-1})^2}{2(\tau_k-\tau_{k-1})}\right)dy_1\dots dy_k. \tag{8.5}$$

Thus (8.4) will follow from an inequality of the following type:

$$\int_{\alpha_1}\dots\int_{\alpha_k}\exp\left(-c_1y_1^2-c_2(y_2-y_1)^2-\dots-c_k(y_k-y_{k-1})^2\right)dy_1\dots dy_k$$
$$\leqslant\int_{\alpha_1^*}\dots\int_{\alpha_k^*}\exp\left(-c_1y_1^2-c_2(y_2-y_1)^2-\dots-c_k(y_k-y_{k-1})^2\right)dy_1\dots dy_k$$

where the c_ν are positive constants and the α_ν are finite unions of intervals and $\alpha_\nu^*=\{y_\nu\,|\,|y_\nu|<l(\alpha_\nu)/2\}$, $l(\alpha_\nu)$ being the total length of $\alpha_\nu=1,2,\dots,k$. Such an inequality follows from Lemma 8.1 below and thereby our theorem is proved.

Lemma 8.1. *Let* $\{a_\nu^{(i)}\}$ *be symmetrically decreasing sequences of numbers with* $a_0^{(i)}\geqslant a_1^{(i)}=a_{-1}^{(i)}\geqslant a_2^{(i)}=a_{-2}^{(i)}\geqslant\dots\geqslant 0$, $i=1,2,\dots,k$. *We assume that*

$$b_\nu^{(i)}=\begin{cases}1 \text{ for } 2s_i+1 \text{ values of } \nu\\ 0 \text{ otherwise}\end{cases},\quad i=1,2,\dots,k.$$

Then

$$\sum_{\nu_1}\dots\sum_{\nu_k}a_{\nu_1}^{(1)}b_{\nu_1}^{(1)}a_{\nu_1-\nu_2}^{(2)}b_{\nu_2}^{(2)}\dots a_{\nu_{k-1}-\nu_k}^{(k)}b_{\nu_k}^{(k)}\leqslant\sum_{|\nu_1|\leqslant s_1}\dots\sum_{|\nu_k|\leqslant s_k}a_{\nu_1}^{(1)}a_{\nu_1-\nu_2}^{(2)}\dots a_{\nu_{k-1}-\nu_k}^{(k)}.$$

Proof. Let $\{a_\nu\}$ be a finite set of numbers. The rearranged set $\{a_\nu^+\}$ is defined by $a_0^+\geqslant a_1^+\geqslant a_{-1}^+\geqslant\dots$. We write $A_\nu=a_\nu^{(1)}b_\nu^{(1)}$. Then $A_\nu^+\neq 0$ and $A_\nu^+\leqslant a_\nu^{(1)}$ for $\nu=-s_1,\dots,s_1$. Hence it is sufficient to prove that

$$\sum_{\nu_1}\dots\sum_{\nu_k}A_{\nu_1}a_{\nu_1-\nu_2}^{(2)}b_{\nu_2}^{(2)}\dots a_{\nu_{k-1}-\nu_k}^{(k)}b_{\nu_k}^{(k)}\leqslant\sum_{|\nu_1|\leqslant s_1}\dots\sum_{|\nu_k|\leqslant s_k}A_{\nu_1}^+a_{\nu_1-\nu_2}^{(2)}\dots a_{\nu_{k-1}-\nu_k}^{(k)}. \tag{8.6}$$

When $k=2$ (8.6) follows from Theorem 373 [8.2, p. 265]. To prove (8.6) when $k>2$, the induction method of Theorem 374 [8.2, p. 273–274] is used. We write

$$B_{\nu_2}=\sum_{\nu_1}A_{\nu_1}a_{\nu_1-\nu_2}^{(2)}b_{\nu_2}^{(2)},\quad c_{\nu_2}=\sum_{|\nu_3|\leqslant s_3}\dots\sum_{|\nu_k|\leqslant s_k}a_{\nu_2-\nu_3}^{(3)}\dots a_{\nu_{k-1}-\nu_k}^{(k)}.$$

Under the assumption that (8.6) is true for $k-1$

$$\sum_{\nu_2}\dots\sum_{\nu_k}B_{\nu_2}a_{\nu_2-\nu_3}^{(3)}b_{\nu_3}^{(3)}\dots a_{\nu_{k-1}-\nu_k}^{(k)}b_{\nu_k}^{(k)}\leqslant\sum_{|\nu_2|\leqslant s_2}B_{\nu_2}^+c_{\nu_2}.$$

Let φ be a permutation function for which $B^{+}_{\varphi(\nu)}=B_\nu$. We write $c_{\varphi(\nu)}=C_\nu$ and $d_\nu=b^{(2)}_\nu C_\nu$. By Theorem 373 [8.2, p. 265]

$$\sum_{|\nu_1|\leqslant s_1} B^{+}_{\nu_2} c_{\nu_2}=\sum_{\nu_2} B_{\nu_2} C_{\nu_2}=\sum_{\nu_1}\sum_{\nu_2} A_{\nu_1} d_{\nu_2} a^{(2)}_{\nu_1-\nu_2}\leqslant \sum_{|\nu_1|\leqslant s_1}\sum_{\nu_2} A^{+}_{\nu_1} d^{+}_{\nu_2} a^{(2)}_{\nu_1-\nu_2}.$$

However by Theorem 375 [8.2, p. 273] $c_0\geqslant c_1=c_{-1}\geqslant c_2=c_{-2}\geqslant\dots$. Hence $d^{+}_{\nu_2}\leqslant c_{\nu_2}$ when $|\nu_2|\leqslant s_2$ and

$$\sum_{|\nu_2|\leqslant s_2} B^{+}_{\nu_2} c_{\nu_2}\leqslant \sum_{|\nu_1|\leqslant s_1}\sum_{|\nu_2|\leqslant s_2} A^{+}_{\nu_1} c_{\nu_2} a^{(2)}_{\nu_1-\nu_2}=\sum_{|\nu_1|\leqslant s_1}\dots\sum_{|\nu_k|\leqslant s_k} A^{+}_{\nu_1} a^{(2)}_{\nu_1-\nu_2}\dots a^{(k)}_{\nu_{k-1}-\nu_k}.$$

The general result (8.6) now follows by induction and thereby the lemma is proved.

For the sake of simplicity we formulate the n-dimensional result for domains D_ξ such that D_ξ is the restriction to $\{z\,|\,x_1<\xi\}$ of a finite union of spheres. By an exhaustion process we can extend the result to harmonic measures in more general domains. Let $\{D^{(\nu)}_\xi\}$ be a monotone sequence of subdomains of D_ξ converging to D_ξ, such that each $D^{(\nu)}_\xi$ is the restriction to $\{z\,|\,x_1<\xi\}$ of a finite union of spheres. Then by Theorem 8.2 below and the maximum principle

$$\max_{y_i}\ \omega(z;\theta^{(\nu)}_\xi;D^{(\nu)}_\xi)\leqslant\omega(z_i;\theta^{(\nu)*}_\xi;D^{(\nu)*}_\xi)\leqslant\omega(z_i;\theta^{*}_\xi;D^{*}_\xi)$$

and thus
$$\max_{y_i}\ \omega(z;\theta_\xi;D_\xi)\leqslant\omega(z_i;\theta^{*}_\xi;D^{*}_\xi).$$

Theorem 8.2. (*The n-dimensional case.*) *Let D_ξ be the restriction to $\{z\,|\,x_1<\xi\}$ of a finite union of n-dimensional spheres. * denotes symmetrization with respect to a coordinate hyperplane $y_i=0$. Given z, z_i has the same coordinates as z except that the ith coordinate of z_i is zero. Then*

$$\max_{y_i}\ \omega(z;\theta_\xi;D_\xi)\leqslant\omega(z_i;\theta^{*}_\xi;D^{*}_\xi).$$

Proof. Consider an n-dimensional Brownian motion $\{(X_1(\tau),\ Y_1(\tau),\dots,Y_{n-1}(\tau))=(X(\tau),\ Y(\tau))=Z(\tau),\ 0\leqslant\tau<\infty\}$ starting from a point $z=(x,y)$. $\{X_1(\tau),\ 0\leqslant\tau<\infty\}$, $\{Y_i(\tau),\ 0\leqslant\tau<\infty\}$, $i=1,2,\dots,n-1$, denote one-dimensional Brownian motions and the components are mutually independent. The proof is analogous to that of Theorem 8.1.

When considering $P\{Y(\tau)\in\vartheta_{X(\tau)}$ when $0\leqslant\tau\leqslant t\}$ we now translate any inaccessible line-segments on straight lines perpendicular to the hyperplane $y_i=0$ in the (τ,y)-space. We then proceed as in the proof of Theorem 8.1 up to (8.5).

Instead of (8.5) we now have, with $Y(0)=y$,

$$P\{Y(\tau_\nu)\in\alpha_{\tau_\nu},\ \nu=1,2,\dots,k\}=(2\pi)^{-k(n-1)/2}(\tau_1(\tau_2-\tau_1)\dots(\tau_k-\tau_{k-1}))^{-(n-1)/2}$$
$$\times\int_{\alpha_{\tau_1}}\dots\int_{\alpha_{\tau_k}}\exp\left(-\frac{|y-y^{(1)}|^2}{2\tau_1}-\frac{|y^{(1)}-y^{(2)}|^2}{2(\tau_2-\tau_1)}-\dots-\frac{|y^{(k-1)}-y^{(k)}|^2}{2(\tau_k-\tau_{k-1})}\right)dy^{(1)}\dots dy^{(k)}.$$

We now consider the integrand as a product, one factor being

$$\exp\left(-\frac{(y_i-y_i^{(1)})^2}{2\tau_1}-\frac{(y_i^{(1)}-y_i^{(2)})^2}{2(\tau_2-\tau_1)}-\dots-\frac{(y_i^{(k-1)}-y_i^{(k)})^2}{2(\tau_k-\tau_{k-1})}\right).$$

We first integrate with respect to $dy_i^{(1)} \dots dy_i^{(k)}$ and use Lemma 8.1. Our theorem now follows in the same way as Theorem 8.1.

Remark 1. The method of proof used above is not suitable for discussing the case of equality in the theorems. Cf. Theorem 4.1.

Remark 2. It seems reasonable that Theorem 4.2 can be generalized to higher dimensions, but a probabilistic proof does not appear easy.

Remark 3. In connection with the results of this paragraph we note that the principal eigenvalue occurring in **B** in § 6 and in Theorem 6.2 is decreased by symmetrization, according to **B**.

Remark 4. By a limiting process we can establish the result of Theorem 8.2 for more general domains. By an infinite sequence of symmetrizations with respect to hyperplanes through the x_1-axis, we can thus obtain the following result, * denoting symmetrization with respect to the x_1-axis ($n > 2$),

$$\max_{z \in \vartheta_x} \omega(z; \theta_\xi; D_\xi) \leqslant \omega(x; \theta_\xi^*; D_\xi^*).$$

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Tryckt den 13 april 1965

Uppsala 1965. Almqvist & Wiksells Boktryckeri AB