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Likelihood ratios of Gaussian processes[1]

By T. S. Pitcher[2]

1. Introduction

Let $x(t)$, $A \leqslant t \leqslant B$ be a real Gaussian stochastic process with autocorrelation function $R(s,t)$. Each choice of a mean value function $f(t)$ for the process establishes a measure m_f on the set of sample functions made into a measure space in the usual way [1]. In statistical applications one often wishes to know when m_f and m_g are totally singular and when they are absolutely continuous with respect to each other, i.e., when the likelihood ratio exists. In the latter case it is desirable to be able to compute $(dm_f/dm_g)(x)$ in terms of the sample function $x(t)$.

The transformation on the space of sample functions which carries $x(t)$ into $x(t)+f(t)$ preserves measurability and carries m_g into m_{f+g}, i.e., if $a(x)$ is a measurable function so is $a(x+f)$ and we have

$$\int a(x)\, d m_{f+g} = \int a(x+f)\, d m_g.$$

The following lemma shows that it is sufficient to consider the case $g=0$.

Lemma 1.1. m_f and m_g are totally singular if and only if m_{f-g} and m_0 are. m_f is absolutely continuous with respect to m_g if and only if m_{f-g} is absolutely continuous with respect to m_0 and in this case $(dm_f/dm_g)(x) = (dm_{f-g}/dm_0)\,(x-g)$.

Proof. If $m_f(A)=1$ and $m_g(A)=0$ then $m_{f-g}(A+g)=1$ and $m_0(A+g)=0$ which proves the first assertion. If dm_f/dm_g exists then

$$\int a(x)\,\frac{dm_f}{dm_g}\,(x+g)\,dm = \int a(x-g)\,\frac{dm_f}{dm_g}\,(x)\,dm_g = \int a(x-g)\,dm_f = \int a(x)\,dm_{f-g}$$

which proves the second assertion.

From now on we shall assume that R is continuous and bounded, and that the process is separable. We will write m for m_0, $m_f \| m$ if m_f and m are totally singular, and $m_f \equiv m$ if m_f and m are mutually absolutely continuous.

Lemma 1.2. If A and B are finite $x(t)$ is in L_2 with m probability one. If f is not in L_2, $m_f \| m$.

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Proof.
$$\int_A^B dt \int x^2(t)\,dm = \int_A^B R(t,t)\,dt < \infty,$$

so by Fubini's theorem
$$\int_A^B x^2(t)\,dt$$

is m measurable and finite almost everywhere. The set L_2 of sample functions having
$$\int_A^B x^2(t)\,dt < \infty$$

is measurable and $m(L_2)=1$. If f is not in L_2, $m_f(L_2)=m(L_2-f)=0$ since L_2-f is contained in the compliment of L_2.

If A and B are finite let (ϕ_n) and (λ_n) be the eigenfunctions and eigenvalues of the integral operator with kernel R, so that
$$\lambda_n \phi_n(t) = \int_A^B R(t,s)\,\phi_n(s)\,ds.$$

The following theorem due to Grenander [2] settles most of the questions in this case.

Theorem 1.1. Suppose A and B are finite and f is in L_2 on (A, B) and let
$$f_n = \int_A^B f(t)\,\phi_n(t)\,dt.$$

Then $m_f \equiv m$ if
$$\Sigma f_n^2/\lambda_n < \infty$$

and $m_f \| m$ if
$$\Sigma f_n^2/\lambda_n = \infty,$$

so that these are the only cases which occur. Moreover we can write
$$\log \frac{d\,m_f}{d\,m}(x) = \int_A^B \left(x(t) - \frac{1}{2} f(t)\right) h(t)\,dt$$

for some h in L_2 if and only if
$$\Sigma f_n^2/\lambda_n^2 < \infty.$$

2. Some general results

In this section A and B may be finite or infinite.

Theorem 2.1. If $m_f \equiv m$ then $m_{\lambda f} \equiv m$ for all $0 \leqslant \lambda < \infty$, and $\log\,(d\,m_{\lambda f}/d\,m)(x) = \lambda\phi(x) - \frac{1}{2}\lambda^2 C$ where $C \geqslant 0$, for any real number a, $\phi(ax) = a\phi(x)$ almost everywhere and for almost all pairs (x, y), $\phi(x+y) = \phi(x) + \phi(y)$.

Proof. Let S_n be the subfield generated by the coordinate functions $x(k/n)$ for $\max\,(A, -n) \leqslant k/n \leqslant \min\,(B, n)$. The S_n are increasing and the original measure field is the smallest field containing them all. Let $x(k_1), \dots x(k_N)$ be a non-degenerate set which spans the set $x(k/n)$, R^n its correlation matrix, σ_n the determinant of R^n and $A^n=(R^n)^{-1}$. If G is an S_n measurable function then $G(x)=g(x(k_1), \dots x(k_N))$ for some Baire function g and

$$\int G(x)\,d\,m=\frac{1}{(2\pi)^{N/2}}\frac{1}{\sigma_n}\int d\,t_1 \dots \int d\,t_n\,g(t_1,\dots,t_N)\exp\left(-\frac{1}{2\sigma_n}\Sigma A^n_{ij}t_i t_j\right).$$

If for some f we set

$$\phi_n(x)=\frac{2}{\sigma_n}\Sigma A^n_{ij}x(k_i)f(k_j) \quad \text{and} \quad C_n=\frac{2}{\sigma_n}\Sigma A^n_{ij}f(k_i)f(k_j),$$

then $C_n \geqslant 0$ and

(*) $\int G(x)\exp(\lambda\phi_n(x)-\frac{1}{2}\lambda^2 C_n)\,d\,m$

$$=\frac{1}{(2\pi)^{N/2}}\frac{1}{\sigma_n}\int d\,t_1 \dots \int d\,t_n\,g(t_1,\dots t_N)\exp\left(-\frac{1}{2\sigma_n}\Sigma A^n_{ij}(t_i-\lambda f(k_i))(t_j-\lambda f(k_j))\right)$$

$$=\frac{1}{(2\pi)^{N/2}}\frac{1}{\sigma_n}\int d\,t_1 \dots \int d\,t_n\,g(t_1+\lambda f(k_1),\dots t_N+\lambda f(k_N))\exp\left(-\frac{1}{2\sigma_n}\Sigma A^n_{ij}t_i t_j\right)$$

$$=\int G(x+\lambda f)\,d\,m=\int G(x)\,d\,m_{\lambda f}.$$

Setting $\lambda=1$ in (*) shows that $\exp(\phi_n(x)-\frac{1}{2}C_n)$ is the conditional expectation with respect to m of $d\,m_f/d\,m$ on S_n. By the martingale convergence theorem (Theorem 4.1, page 319 of [3]) $\exp(\phi_n(x)-\frac{1}{2}C_n)$ converges to $(d\,m_f/d\,m)(x)$ for almost all x. Since $x\to -x$ preserves m-measure, there is at least one x for which both $\exp(\phi_n(x)-\frac{1}{2}C_n)$ and $\exp(\phi_n(-x)-\frac{1}{2}C_n)=\exp(-\phi_n(x)-\frac{1}{2}C_n)$ converge to a finite limit and hence $-(\phi_n(x)-\frac{1}{2}C_n)-(-\phi_n(x)-\frac{1}{2}C_n)=C_n$ converges to some finite non-negative limit C. Thus $\phi_n(x)$ converges almost everywhere to some finite limit $\phi(x)$ and $(d\,m_f/d\,m)(x)=\exp(\phi(x)-\frac{1}{2}C)$ almost everywhere.

If $H(x)$ is an S_p measurable function for some $p<n$ then by (*)

$$\int H(x)\exp(\lambda\phi_p(x)-\tfrac{1}{2}\lambda^2 C_p)\,d\,m=\int H(x+\lambda f)\,d\,m$$
$$=\int H(x)\exp(\lambda\phi_n(x)-\tfrac{1}{2}\lambda^2 C_n)\,d\,m,$$

so the sequence

$\exp(\lambda\phi_n(x)-\frac{1}{2}\lambda^2 C_n)$ is a martingale with limit $\exp(\lambda\phi(x)-\frac{1}{2}\lambda^2 C)$.

Applying (*) with $G(x)=1$ and λ replaced by 2λ we get

$$\int \exp^2(\lambda\phi_n(x)-\tfrac{1}{2}\lambda^2 C_n)\,d\,m=\exp(\lambda^2 C_n)\int\exp(2\lambda\phi_n(x)-2\lambda^2 C_n)\,d\,m$$
$$=\exp(\lambda^2 C_n)\to\exp(\lambda^2 C),$$

so by Theorem 4.1, page 319 of [3] the conditional expectation of

$\exp(\lambda\phi(x)-\frac{1}{2}\lambda^2 C)$ on S_n is $\exp(\lambda\phi_n(x)-\frac{1}{2}\lambda^2 C_n)$.

If H is the characteristic function of a set in S_n

$$\int H(x)\exp\left(\lambda\phi(x)-\tfrac{1}{2}\lambda^2 C\right)dm=\int H(x)\exp\left(\lambda\phi_n(x)-\tfrac{1}{2}\lambda^2 C_n\right)dm=\int H(x+\lambda f)\,dm.$$

Since this relation holds for sets in any S_n and by the dominated convergence theorem for monotone limits it holds for all measurable sets by Theorem 1.2, page 599, of [3]. Hence $m_{\lambda f}\equiv m$ and $\log\,(d\,m_{\lambda f}/d\,m)(x)=\lambda\phi(x)-\tfrac{1}{2}\lambda^2 C$.

Theorem 2.2. ϕ is in L_p for all p.

$$\int x(t)\phi(x)\,dm=f(t),\quad \int\phi(x)\,dm=0,\quad \text{and}\int\phi^2(x)\,dm=C.$$

For any n and $0<\lambda<1$

$$\frac{d\,m_{\lambda f}}{d\,m}(x)=\sum_{k=0}^{n}\frac{(\lambda\phi(x)-\tfrac{1}{2}\lambda^2 C)^k}{k!}+0_n,$$

where
$$|0_n|\leqslant\lambda^{n+1}e^C\left(\frac{d\,m_f}{d\,m}(x)+\frac{d\,m_{-f}}{d\,m}(x)\right).$$

Proof.

$$\begin{aligned}|\phi|^p&\leqslant\exp\,(p|\phi|)\leqslant\exp\,(p\phi)+\exp\,(-p\phi)\\&=(\exp\,(p\phi-\tfrac{1}{2}p^2C)+\exp\,(-p\phi-\tfrac{1}{2}p^2C))\exp\,(\tfrac{1}{2}p^2C)\\&=\left(\frac{d\,m_{pf}}{d\,m}+\frac{d\,m_{-pf}}{d\,m}\right)\exp\,(\tfrac{1}{2}p^2C),\end{aligned}$$

so $\int|\phi|^p\,dm\leqslant 2\exp\,(\tfrac{1}{2}p^2C)$ which proves the first assertion. For $0<\lambda<1$

$$\begin{aligned}|0_n|&\leqslant\sum_{n+1}^{\infty}\frac{|\lambda\phi-\tfrac{1}{2}\lambda^2C|^k}{k!}\\&\leqslant\lambda^{n+1}\sum_{n+1}^{\infty}\frac{|\phi-\tfrac{1}{2}\lambda C|^k}{k!}\\&\leqslant\lambda^{n+1}\exp|\phi-\tfrac{1}{2}\lambda C|\leqslant\lambda^{n+1}(\exp\,(\phi-\tfrac{1}{2}\lambda C)+\exp\,(-\phi+\tfrac{1}{2}\lambda C))\\&=\lambda^{n+1}\left(\frac{d\,m_{\lambda f}}{d\,m}\exp\tfrac{1}{2}(1-\lambda)C+\frac{d\,m_{-\lambda f}}{d\,m}\exp\tfrac{1}{2}(1+\lambda)C\right)\\&\leqslant\lambda^{n+1}e^C\frac{d\,m_{\lambda f}}{d\,m}+\frac{d\,m_{-\lambda f}}{d\,m}.\end{aligned}$$

Now

$$\begin{aligned}1&=\int\frac{d\,m_{\lambda f}}{d\,m}(x)\,d\,m\\&=\int\left(1+\lambda\phi(x)-\tfrac{1}{2}\lambda^2C+\frac{(\lambda\phi(x)-\tfrac{1}{2}\lambda^2C)^2}{2}+0_2\right)d\,m\\&=1+\lambda\int\phi(x)\,d\,m+\frac{\lambda^2}{2}\int\phi^2(x)\,d\,m-C-\frac{\lambda^3}{2}C\int\phi(x)\,d\,m+\frac{1}{8}\lambda^4C^2+\int 0_2\,d\,m.\end{aligned}$$

The previous inequality shows that the last term is of order λ^3 so the coefficients of λ and λ^2 are zero giving $\int \phi(x)\,dm = 0$ and $\int \phi^2(x)\,dm = C$. Also

$$\begin{aligned}\lambda f(t) &= \int (x(t) + \lambda f(t))\,dm \\ &= \int x(t)\,dm_{\lambda f} \\ &= \int x(t)\frac{dm_{\lambda f}}{dm}(x)\,dm \\ &= \int x(t)(1 + \lambda\phi - \tfrac{1}{2}\lambda^2 C + 0_1)\,dm.\end{aligned}$$

The last term is of order λ^2 since

$$\begin{aligned}\int x(t)\,0_1(x)\,dm &\leq \lambda^2 e^C \int |x(t)| \left(\frac{dm_{\lambda f}}{dm} + \frac{dm_{-\gamma f}}{dm}\right) dm \\ &= \lambda^2 e^C \int (|x(t) + f(t)| + |x(t) - f(t)|)\,dm \\ &\leq 2\lambda^2 e^C (|f(t)| + \int |x(t)|\,dm),\end{aligned}$$

so the coefficients of λ on the two sides of the preceding equality are equal giving $f(t) = \int x(t)\phi(t)\,dm$.

Theorem 2.3. If F has bounded variation on (A, B) and $f(s) = \int_A^B R(s,t)\,dF(t)$,

then $$m \equiv m_f \cdot \text{ and } \log \frac{dm_f}{dm}(x) = \int_A^B x(t)\,dF(t) - \tfrac{1}{2}\int_A^B f(t)\,dF(t).$$

Conversely if $$\log \frac{dm_f}{dm}(x) = \int_A^B x(t)\,dF(t) - \tfrac{1}{2}d$$

then $$d = \int_A^B f(t)\,dF(t) \text{ and } f(s) = \int_A^B R(s,t)\,dF(t).$$

Proof. Since $\int \left[\int_A^B x(t)\,dF(t)\right]^2 dm = \int_A^B dF(s)\int_A^B dF(t)\,R(s,t) < \infty$, $\int_A^B x(t)\,dF(t)$ exists almost everywhere and is in L_2. We now define the subfield S_n, the function ϕ_n, and the constant C_n as in Theorem 2.1. We have

$$\int \phi_n(x)\,x(t_k)\,dm = \Sigma A_{ij}^n f(t_j)\,R(t_i, t_k) = f(t_k)$$

and $$\int \left[\int_A^B x(t)\,dF(t)\right] x(t_k)\,dm = \int_A^B R(t_k, t)\,dF(t) = f(t_k),$$

so $\phi_n(x)$ is the projection of $\int_A^B x(t)\,dF(t)$ on the linear manifold spanned by the $x(t_i)$. Since $\int_A^B x(t)\,dF(t)$ and the $x(t_i)$ are Gaussian, this implies that $\phi_n(x)$ is the conditional expection of $\int_A^B x(t)\,dF(t)$ on S_n, and by the martingale theorem (Theorem 4.1, page 319 of [3]) that $\phi_n(x) \to \int_A^B x(t)\,dF(t)$ almost everywhere and in mean. In particular $C_n = \int \phi_n^2(x)\,dm \to \int \left[\int_A^B x(t)\,dF(t)\right]^2 dm = \int_A^B f(t)\,dF(t)$. It only remains to show that $\exp(\phi_n - \frac{1}{2}C_n)$ converges to dm_f/dm.

$$\int \exp^2(\phi_n(x) - \tfrac{1}{2}C_n)\,dm = e^{C_n} \int \exp(2\phi_n(x) - 2C_n)\,dm = e^{C_n}$$

since $\exp(2\phi_n - 2C_n)$ differs from $\exp(\phi_n - \frac{1}{2}C_n)$ only in having f replaced by $2f$. The e^{C_n} are uniformly bounded so by the martingale theorem $\exp(\phi_n - \frac{1}{2}C_n)$ converges almost everywhere and in mean to a limiting function which is easily shown as in the proof of Theorem 2.1 to be dm_f/dm.

Conversely, suppose

$$\frac{dm_f}{dm} = \exp(\phi - \tfrac{1}{2}C) = \exp\left(\int_A^B x(t)\,dF(t) - \tfrac{1}{2}d\right).$$

Then $\phi(x) - \int_A^B x(t)\,dF(t)$ is a constant and integrating shows this constant to be zero.

$$f(s) = \int x(s)\,\phi(x)\,dm = \int x(s) \int_A^B x(t)\,dF(t) = \int_A^B R(s,t)\,dF(t)$$

and

$$d = C = \int \phi^2(x)\,dm = \int \left[\int_A^B x(t)\,dF(t)\right]^2 dm = \int_A^B dF(t) \int_A^B dF(s)\,R(s,t) = \int f(t)\,dF(t).$$

3. The infinite case

In this section the results of section 1 are extended to the infinite case.

Let S_n be the subfield generated by the $s(t)$'s with

$$A_n = \max(A, -n) \leqslant t \leqslant \min(B, n) = B_n.$$

Let $m'_{n,f}$, S'_n be the measure and the field of measurable sets associated with the Gaussian stochastic process $x(t)$, $A_n \leqslant t \leqslant B_n$ with autocorrelation function $R(s,t)$ and mean $f(s)$ defined for $A_n \leqslant s, t \leqslant B_n$. If we write α_n for the map which takes every sample function of the original process into the restriction of that function to (A_n, B_n) then α_n sets up a $1:1$ correspondence between S_n and S'_n for which $m_f(A) = m'_{n,f}(\alpha_n(A))$.

Lemma 3.1. m_f and m are mutually absolutely continuous over the subfield S_n if and only if $m_n' \equiv m_{n,f}'$ and in this case

$$\frac{d\,m_f}{d\,m}(x) = \frac{d\,m_n'}{d\,m_{n,f}'}(\alpha_n(x)).$$

Proof. Let C_A be the characteristic function of a set in S_n, then

$$\int \frac{d\,m_n'}{d\,m_{n,f}'}(\alpha_n(x))\,d\,m = \int \frac{d\,m_n'}{d\,m_{n,f}'}(\alpha_n(x))\,C_{\alpha_n(A)}(\alpha_n(x))\,d\,m$$

$$= \int \frac{d\,m_n'}{d\,m_{n,f}'}(x)\,C_{\alpha_n(A)}(x)\,d\,m_n'$$

$$= m_n'(\alpha_n(A) - f) = m_n'(\alpha_n(A - f)) = m(A - f).$$

The above lemma shows that for each S_n either m_f and m are totally singular over S_n and hence are totally singular or else m_f and m are mutually absolutely continuous over S_n with derivative equal to $\exp(\phi_n(x) - \frac{1}{2} C_n)$.

Theorem 3.1. The C_n are non-decreasing. If $C = \lim C_n < \infty$ then $m_f \equiv m$ and $(d\,m_f/d\,m)(x) = \exp(\phi(x) - \frac{1}{2} C)$, where $\phi_n(x) \to \phi(x)$ almost everywhere. If $\lim C_n = \infty$ then $m_f \| m$.

Proof.

$$\int \exp^2(\phi_n(x) - \tfrac{1}{2} C_n)\,d\,m = [\int \exp(2\phi_n(x) - 2C_n)\,d\,m]\,e^{C_n} = e^{C_n},$$

which proves that the C_n are non-decreasing since $\exp^2(\phi_n(x) - \frac{1}{2} C_n)$ is a semimartingale. If $\lim C_n < \infty$ the martingale theorem implies that $\phi_n(x) \to \phi(x)$ almost everywhere and it can be shown as in Theorem 2.1 that $(d\,m_f/d\,m)(x) = \exp(\phi(x) - \frac{1}{2} C)$.

Conversely, suppose $C \to \infty$. By the martingale theorem $\lambda\phi_n - \frac{1}{2}\lambda^2 C_n$ converges almost everywhere. We first prove that $\lambda\phi_n - \frac{1}{2}\lambda^2 C_n \to -\infty$ with probability one for any $\lambda \neq 0$. If not there is a set A, $m(A) > 0$ on which $\lim \lambda\phi_n - \frac{1}{2}\lambda^2 C_n \geqslant a > -\infty$ and for large enough n there are sets A_n on which $\lambda\phi_n - \frac{1}{2}\lambda^2 C_n \geqslant a - 1 > -\infty$. We have

$$1 = \int \exp((\lambda - \alpha_n)\phi_n - \tfrac{1}{2}(\lambda - \alpha_n)^2 C_n)\,d\,m$$

$$= \int \exp\left[(\lambda - \alpha_n)\left(\frac{\lambda\phi_n - \frac{1}{2}\lambda^2 C_n}{\lambda} + \tfrac{1}{2}\alpha_n C_n\right)\right] d\,m$$

$$\geqslant m(A_n)\exp\left[(\lambda - \alpha_n)\left(\frac{a-1}{\lambda} + \tfrac{1}{2}\alpha_n C_n\right)\right]$$

and choosing α_n so that $\alpha_n \to 0$ and $\lambda\alpha_n C_n \to \infty$ makes the right hand side go to $+\infty$ which is a contradiction. If B is the set where $\phi_n - \frac{1}{2} C_n^2 \to -\infty$ then $m(B) = 1$. $m_f(B) = m(B - f)$ and elements of $B - f$ are y's such that $\phi_n(y + f) - \frac{1}{2} C_n$

$=\phi_n(y)+\frac{1}{2}C_n \to -\infty$. If $m_f(B) \neq 0$ then the set where $\phi_n-\frac{1}{2}C_n$, $-\phi_n-\frac{1}{2}C_n$ and $\phi_n+\frac{1}{2}C_n$ all go to $-\infty$ has positive measure hence is not empty so for some x, $0=-\phi_n(x)-\frac{1}{2}C_n+\phi_n(x)+\frac{1}{2}C_n \to -\infty$. The set B separates m and m_f.

If C is the set of functions on (A, B) which are solutions of

$$f(s)=\int_A^B R(s, t)\, d F(t)$$

for some F of bounded variation, D is the set of functions f for which $m_f \equiv m$ and E is the set of all sample functions on (A, B) then by Theorem 2.3 $C \subset D \subset E$. D is, in a probability sense, very rare in E. In fact, if A and B are finite, Theorem 1.1 says that D consists of those x for which the series

$$\sum_{n=1}^{\infty}\left[\int_A^B x(t)\,\phi_n(t)\, dt\right]^2 / \lambda_n$$

converges to a finite limit and since the summands are independent identically distributed random variables, thus has probability 0. In the infinite case, using the notation established at the beginning of this section C is contained in the set of functions f for which $m_f \equiv m$ over S_n and this has probability 0.

For f in D write $$\frac{d m_f}{d m_g}=\exp\left(\phi_f-\frac{1}{2} C_f\right).$$

If f and g are in D,

$$\begin{aligned}
\frac{d m_{f+g}}{d m}(x) &= \frac{d m_{f+g}}{d m_f}(x)\,\frac{d m_f}{d m}(x) \\
&= \frac{d m_g}{d m}(x+f)\,\frac{d m_f}{d m}(x) \\
&= \exp\left(\phi_g(x+f)+\phi_f(x)-\frac{1}{2} C_f-\frac{1}{2} C_g\right) \\
&= \exp\left(\phi_f(x)+\phi_g(x)+(\phi_g(f)-\frac{1}{2} C_f-\frac{1}{2} C_g\right) \\
&= \exp\left(\phi_f(x)+\phi_g(x)+(\phi_g(f)-\frac{1}{2} C_f-\frac{1}{2} C_g\right),
\end{aligned}$$

so D is a linear space and $\phi_{f+g}=\phi_f+\phi_g$.

Theorem 3.2. D is a Hilbert space with the inner product $(f, g)=\int \phi_f(x)\,\phi_g(x)\, d m$. C is dense in this Hilbert space.

Proof. To prove that D is a Hilbert space we must show that it is complete in the norm established by the inner product. If f_i is a Cauchy sequence then ϕ_{f_i} is a Cauchy sequence in the ordinary L_2 so it converges in mean to a random variable ψ. If $f(t)=\int x(t)\,\psi(x)\, d m$, then $f(t)=\lim f_i(t)$ since

$$|f(t)-f_i(t)|^2=\left|\int x(t)(\psi(x)-\phi_{f_i}(x))\, d m\right|^2 \leqslant R(t, t) \int|\phi(x)-\phi_f(x)|^2\, d m \to 0.$$

Let S_n be the subfield generated by $x(k/n)$, $\max(A, -n) \leqslant k/n \leqslant \min(B, n)$. The conditional expectation of dm_{f_k}/dm on S_n has the form

$$\exp \Sigma A_{ij}^n (x(i/n) f_k(j/n) - \tfrac{1}{2} f_k(i/n) f_k(j/n))$$

and converges, pointwise and in mean, to

$$d_n = \exp \Sigma A_{ij}^n (x(i/n) f(j/n) - \tfrac{1}{2} f(i/n) f(j/n)).$$

d_n is a martingale,

$$\int d_n^2\, dm = \lim \exp \tfrac{1}{2} \Sigma A_{ij}^n f_k(i/n) f_k(j/n) \leqslant \lim \sup \exp C_{f_i} = \int \psi^2\, dm,$$

so d_n converges, pointwise and in mean, to dm_f/dm. It remains to show that $\|f_i - f\| \to 0$, i.e., that ϕ_i converges to ϕ in mean, which is equivalent to $\phi = \psi$. Now $\int x(t)(\phi(x) - \psi(x))\, dm = f(t) - f(t) = 0$ so $\phi - \psi$ is orthogonal to the manifold M spanned by the $x(t)$'s. However ϕ and all the ϕ_{f_i}'s are in M, from their method of construction and so is ψ which is a limit of the ϕ_f's so $\psi = \phi$ and the space D is complete.

C is clearly a linear subspace of D. We need to show that if $(f, g) = 0$ for every g in C then $f = 0$. We will prove below that if

$$g(s) = \int_A^B R(s, t)\, dG(t),$$

then

$$(f, g) = \int_A^B f(s)\, dG(s).$$

Choosing $G(s) = 0$ for $s < s_0$ and $G(s) = 1$ for $s \geqslant s_0$ will then give

$$0 = \int_A^B f(s)\, dG(s) = f(s_0) \text{ for all } s_0$$

which will complete the proof.

Lemma 3.2. If $g(s) = \int_A^B R(s, t)\, dG(t)$ is in C, then for any f in D

$$(f, g) = \int_A^B f(s)\, dG(s).$$

Proof.

$$\begin{aligned}(f, g) &= \int \phi_f(x) \phi_g(x)\, dm \\ &= \int \phi_f(x) \int_A^B x(t)\, dG(t)\, dm \\ &= \int_A^B \left(\int \phi_f(x) x(t)\, dm \right) dG(t) \\ &= \int_A^B f(t)\, dG(t).\end{aligned}$$

The following example shows that C is not all of D. Let $R(s, t) = r(s-t)$ where r has two continuous derivatives and

$$F_n(t) = \begin{cases} 0 \text{ if } t \leqslant -1/n, \\ -2n \text{ if } |t| < 1/n, \\ 0 \text{ if } t \geqslant 1/n, \end{cases}$$

then
$$f_n(s) = \int_A^B R(s, t)\, dF_n(t) = 2n\,(r(s+1/\mathrm{n}) - r(s-1/\mathrm{n})).$$

f_n is a Cauchy sequence from C and its limit is dr/ds. If dr/ds is in D we have

$$\frac{dr}{ds}(s) = \int_A^B r(s-t)\, dF(t)$$

which is not true in general. In fact if $r(s) = \exp(-s^2)$ we have

$$-s \exp(-s^2) = \exp(-s^2) \int_A^B \exp(2st - t^2)\, dF(t)$$

and multiplying by $\exp s^2$ and differentiating twice gives

$$0 = \int_A^B t^2 \exp(2st - t^2)\, dF(t).$$

Since the integrand is positive this means F is constant except for a jump at 0 and this gives $-s\exp(-s^2) = C\exp(-s^2)$ which is impossible.

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