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The remainder in Tauberian theorems

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1. Introduction

In his paper "Tauberian Theorems" of 1932, N. WIENER stated a Tauberian theorem which was sufficiently general to include many of the various earlier Tauberian theorems. In a form convenient for our purposes this theorem may be stated as follows (cf. PITT [2]).

Wiener's general Tauberian theorem

$K(x)$ and $\Phi^*(x)$ are functions of the real variable x, and we suppose

$$\Phi^*(x) \text{ real and bounded}$$

$$\Phi^*(x+h) - \Phi^*(x) \geq w(h), \qquad w(h) \to 0 \text{ when } h \to +0.$$

$$K(x) \in L, \; k(t) = \int_{-\infty}^{\infty} K(u) e^{itu} du \neq 0 \text{ for } t \text{ real}.$$

Then

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} \Phi^*(x-u) K(u) du = A \int_{-\infty}^{\infty} K(u) du$$

implies

$$\lim_{x\to\infty} \Phi^*(x) = A.$$

Let

$$\Psi^*(x) = \int_{-\infty}^{\infty} \Phi^*(x-u) K(u) du.$$

Then a Tauberian theorem yields an asymptotic estimation of the function $\Phi^*(x)$, if we know the asymptotic behaviour of the function $\Psi^*(x)$. The question arises whether it is possible to estimate the "remainder" $\Phi(x) = \Phi^*(x) - A$ when we know the behaviour of the analoguous "remainder" $\Psi(x) = \Psi^*(x) - A \int_{-\infty}^{\infty} K(x) dx$. Wiener's Tauberian theorem merely changes the imposed condition $\Phi(x) = O(1)$ to $\Phi(x) = o(1)$, $x \to \infty$. Wiener himself considered, in the paper quoted, that it is impossible to reach better results with his methods.

Since then, however, theorems have appeared which prove that under more restricted conditions on the kernel $K(x)$, it is actually possible to estimate the remainder $\Phi(x)$. (See BEURLING [1] page 22.) The present paper is confined to Tauberian relations for which such an estimation of the remainder is possible.

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2. The class of Tauberian relations considered

Let V denote the class of functions of bounded variation, and let E be the class of functions $\Phi(x)$ defined as follows.

Definition

$\Phi(x) \in E$ if $\Phi(x)$ is real and bounded and $\Phi(x) + e^{\varepsilon x}$ is non-decreasing for every $\varepsilon > 0$ and $x > x_\varepsilon$.

Let $F(x)$ and $\Phi(x)$ be functions of the real variable x.
Consider a Tauberian relation of the form

$$\lim_{x \to \infty} \int_{-\infty}^{\infty} \Phi(x-u)\, dF(u) = 0,$$

where $F(x) \in V$, $\Phi(x) \in E$. Let all functions $F(x) \in V$ be divided into two classes:

Definition

$F(x)$ belongs to class I if two numbers $\theta > 0$ and $\alpha > 0$ can be found, such that for every $\Phi(x) \in E$ the relation

$$\int_{-\infty}^{\infty} \Phi(x-u)\, dF(u) = O(e^{-\gamma x})$$

with $0 < \gamma < \alpha$ implies

$$\Phi(x) = O(e^{-\theta \gamma x}),$$

where θ and α are constants depending only on $F(x)$.
$F(x)$ belongs to class II if no such positive numbers θ and α can be found.

If $F(x)$ belongs to class I it is possible to estimate the remainder $\Phi(x)$. It is this case which will be studied here.
Class II yields a pure Tauberian problem, which is of no interest in connection with the remainder problem studied here.

3. Some sufficiency theorems

The following theorems are modifications of a theorem of BEURLING (see loc. cit.).

Theorem 1

Let $\Phi(u) \in E$, $F(u) \in V$, $f(x) = \int_{-\infty}^{\infty} e^{ixu}\, dF(u)$, and suppose that $\frac{1}{f(t)}$ can be analytically continued into the strip $-a \le \Im(t) \le b$ containing the real axis,

$$(\mathrm{s}_1) \qquad \left|\frac{1}{f(t)}\right| < \text{const. } (1+|t|)^p \text{ in } -a \le \Im(t) \le b.$$

Then the relation

$$\int_{-\infty}^{\infty} \Phi(x-u)\, dF(u) = O(e^{-\gamma x})$$

implies

$$\Phi(x) = O(e^{-\theta c x}) \qquad c = \min\{\gamma, a\}$$

for every $\theta < \dfrac{2}{2p+3}$.

Let n be an integer $> p+1$ and r a real number. Introduce the function

$$w(t) = w_{n,r}(t) = \frac{(1-e^{-irt})^n}{(it)^n f(t)}.$$

Throughout the paper $w(t) = w_{n,r}(t)$ will denote a function of this type.

$w(t)$ is analytic in $-a \le \Im(t) \le b$, and $|w(t)| < \text{const.}\,(1+|t|)^{-(n-p)}$ in the same region. Introduce its Fourier transform

$$W(u) = W_{n,r}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w(t)\, e^{iut}\, dt.$$

We can change the line of integration into any line $\Im(t) = \beta$, $-a \le \beta \le b$, thus obtaining $w(x+i\beta)$ and $e^{-\beta u} W(u)$ as Fourier transforms. Since $w(x+i\beta) \in L^2$, $-a \le \beta \le b$, we have

$$e^{au} W(u) \in L^2 \quad \text{and} \quad e^{-bu} W(u) \in L^2.$$

Hence

$$W(u) \in L.$$

Let

$$\Psi(x) = \int_{-\infty}^{\infty} \Phi(x-u)\, dF(u),$$

and introduce the notation

$$J(\Phi, x) = J_{n,r}(\Phi, x) = \int_x^{x+r} du_1 \int_{u_1}^{u_1+r} du_2 \ldots \int_{u_{n-1}}^{u_{n-1}+r} \Phi(u)\, du.$$

We shall now prove the fundamental formula

$$J(\Phi, x) = \int_{-\infty}^{\infty} \Psi(x-u)\, W(u)\, du. \tag{1}$$

It is easy to see that

$$\int_{-\infty}^{\infty} W(y-u)\, dF(u)$$

is Fourier transform to

$$\frac{(1-e^{-irt})^n}{(it)^n}.$$

Hence

$$J(\Phi, x) = \int_{-\infty}^{\infty} \Phi(x-y)\, dy \int_{-\infty}^{\infty} W(y-u)\, dF(u),$$

the y-integral being in reality taken between finite limits, since

$$\int_{-\infty}^{\infty} W(y-u)\,dF(u) \text{ is zero outside } (-nr,\,0).$$

Since $W(u)\in L$, inversion is justified by absolute convergence and yields (1). Hence (1) is proved.

Let us consider $J(\Phi, x)$ when $x\to\infty$ and $r=r(x)\to 0$. (1) gives

$$(2)\quad |J(\Phi,x)|\le \int_{-\infty}^{\infty}|\Psi(x-u)\,W(u)|\,du = O\Big\{e^{-\gamma x}\int_{-\infty}^{0}|e^{\gamma u}\,W(u)|\,du + \\ + e^{-\gamma x}\int_{0}^{x} e^{(\gamma-a)u}\,|e^{a u}\,W(u)|\,du+\int_{x}^{\infty}e^{-a u}\,|e^{a u}\,W(u)|\,du\Big\}.$$

Using the notation

$$M_p\{f\}=\Big\{\int_{-\infty}^{\infty}|f(x)|^p\,dx\Big\}^{\frac{1}{p}},\ M\{f\}=M_1\{f\},$$

Schwarz's inequality gives

$$(3)\qquad J(\Phi,x)=O\left(x^{\frac{1}{2}}\,e^{-c x}\left[M_2\{e^{\frac{1}{2}c u}\,W(u)\}+M_2\{e^{a u}\,W(u)\}\right]\right),\qquad c=\min\{\gamma,\,a\}.$$

Let us study this expression for small r. Since $M_2\{e^{a u}\,W(u)\}$ and $M_2\{e^{\frac{1}{2}c u}\,W(u)\}$ may be treated in exactly the same way, it is enough to consider $M_2\{e^{a u}\,W(u)\}$. Parseval's relation gives

$$\sqrt{2\pi}\,M_2\{e^{a u}\,W(u)\}=M_2\{w(x-ia)\}.$$

For the sake of brevity, let

$$v(t)=\frac{1}{t^n\,f(t)}.$$

This notation will be used throughout the present paper. For small r

$$i^n\,w(x-ia)=(1-e^{-irx-ra})^n\,v(x-ia)=\sum_{q=0}^{n}O(r^{n-q})\,(1-e^{-irx})^q\,v(x-ia).$$

Minkowski's inequality gives

$$(4)\qquad M_2\{w(x-ia)\}=\sum_{q=0}^{n}O(r^{n-q})\,M_2\{(1-e^{-irx})^q\,v(x-ia)\}.$$

It follows from condition (s_1) that

$$|v(x-ia)|<\text{const.}\ (1+|x|)^{-(n-p)}.$$

Using this relation it is easy to verify that

$$(5)\qquad M_2\{(1-e^{-irx})^q\,v(x-ia)\}=\begin{cases}O(r^q+r^{n-p-\frac{1}{2}}), & q\neq n-p-\frac{1}{2}\\ O(r^q\,(\log r)^{\frac{1}{2}}), & q=n-p-\frac{1}{2}.\end{cases}$$

Inserting in (4) we find

$$(6)\qquad M_2\{w(x-ia)\}=O(r^{n-p-\frac{1}{2}}).$$

Similarly, we obtain

$$M_2\left\{w\left(x - i\frac{c}{2}\right)\right\} = O(r^{n-p-\frac{1}{2}}).$$

Using Parseval's relation and substituting in (3) we have proved

$$(7) \qquad J_{n,r}(\Phi, x) = O(x^{\frac{1}{2}} e^{-cx} r^{n-p-\frac{1}{2}}), \qquad c = \min\{\gamma, a\}$$

for every integer $n > p+1$.

Formula (7) was derived with the single assumption about $\Phi(x)$ that it is bounded. Using the condition $\Phi(x) \in E$, we may now proceed to prove the theorem. If for every $\eta > 0$

$$(8) \qquad |\Phi(x)| < e^{-\frac{2cx}{2p+3}+\eta x}, \qquad x > x_\eta,$$

then the theorem is a trivial result.

If this is not the case, then for arbitrarily large values of x and some fixed number $\alpha < \dfrac{2c}{2p+3}$ either

$$(9\,\text{a}) \qquad \Phi(x) > e^{-\alpha x}$$

or

$$(9\,\text{b}) \qquad \Phi(x) < -e^{-\alpha x}.$$

By the definition of E we have for every $\varepsilon > 0$, if $h > 0$, $x > x_\varepsilon$

$$\Phi(x+h) \geq \Phi(x) - e^{\varepsilon x}(e^{\varepsilon h} - 1).$$

Choose an $\varepsilon > 0$, and suppose $x > x_\varepsilon$ to be a point where (9 a) holds. Then

$$\Phi(x+h) > \tfrac{1}{2} e^{-\alpha x}$$

if

$$0 \leq h \leq \frac{1}{\varepsilon} \log\left(1 + \tfrac{1}{2} e^{-(\alpha+\varepsilon)x}\right) = h(x).$$

Let $r = r(x) = \dfrac{h(x)}{n}$ where n is an integer $> p+1$. Then

$$r^n \tfrac{1}{2} e^{-\alpha x} < \int_x^{x+r} du_1 \int_{u_1}^{u_1+r} du_2 \ldots \int_{u_{n-1}}^{u_{n-1}+r} \Phi(u)\, du = J_{n,r}(\Phi, x).$$

Using (7) we get

$$r^{p+\frac{1}{2}} e^{-\alpha x} = O(x^{\frac{1}{2}} e^{-cx})$$

or, inserting r,

$$e^{-\alpha x}\left\{\frac{1}{\varepsilon} \log\left(1 + \tfrac{1}{2} e^{-(\alpha+\varepsilon)x}\right)\right\}^{p+\frac{1}{2}} = O(x^{\frac{1}{2}} e^{-cx}).$$

Letting $x \to \infty$ we find

$$\alpha \geq \frac{2c}{2p+3} - \frac{2p+1}{2p+3} \cdot \varepsilon.$$

Since ε may be taken arbitrarily small, (9 a) is impossible.

In the same way, considering an interval to the left of x, we prove that (9 b) is impossible.

Hence (8) must hold, i.e. theorem 1 is proved.

The following three theorems are similar to theorem 1. We only replace the condition (s_1) for $\frac{1}{f(t)}$ by other similar conditions, still sufficient for $F(u)$ to belong to the class I.

In theorem 1 the condition (s_1) was imposed because of its simplicity. It can be generalised as follows.

Theorem 2

Replace the condition (s_1) *in theorem 1 by*

$$(s_2) \qquad \int_{-\infty}^{\infty} \frac{dx}{|f(x+i\beta)|^2(1+|x|)^{2p+1}} < \text{const.} \qquad -a \le \beta \le b.$$

Then theorem 1 thus modified will still hold.

Introduce $w(t)$ as on page 577. Condition (s_2) gives $w(x) \in L^2$. Hence its Fourier transform $W(u)$ certainly exists in the L^2-sense.

The conditions further imply that

$$w(t) \text{ is analytic in the strip } -a \le \Im(t) \le b, \text{ and}$$

$$\int_{-\infty}^{\infty} |w(x+i\beta)|^2 dx < \text{const.} \qquad -a \le \beta \le b.$$

Then it is a well-known conclusion that

$$w(x+i\beta) \to 0 \text{ when } x \to \infty, \qquad -a < \beta < b.$$

We may therefore move the line of integration in

$$W(u) = \frac{1}{2\pi} \operatorname*{l.i.m.}_{T\to\infty} \int_{-T}^{T} w(t)\, e^{-iut}\, dt$$

in the same way as before. Hence $W(u) \in L$, and formula (1) still holds. (3) and (4) may be obtained as before.

For the function

$$v(t) = \frac{1}{t^n f(t)}$$

condition (s_2) implies

$$v(x-ia) \in L^2 \quad \text{and} \quad \int_{|x|>T} |v(x-ia)|^2 dx = O(T^{1-2(n-p)}).$$

Using these relations, it is easy to derive (5). In the same way as in theorem 1 this yields

$$(6) \qquad M_2\{w(x-ia)\} = O(r^{n-p-\frac{1}{2}}),$$

and the proof may be completed in the same way as in theorem 1.

In the following two theorems $\frac{1}{f(t)}$ is required to be analytic in the strip $-a \le \Im(t) < 0$ only. It seems likely that this would also be sufficient in theorems 1–2. It may be shown that it is sufficient with the additional condition $\int_{-\infty}^{\infty} |u|^{\frac{1}{2}} |dF(u)| < \infty$.

Theorem 3

Let $\Phi(u) \in E$, $F(u) \in V$, $f(x) = \int_{-\infty}^{\infty} e^{ixu} dF(u)$, *and suppose that* $\frac{1}{f(x)}$ *are limiting values of a function* $\frac{1}{f(t)}$, *analytic in* $-a \le \Im(t) < 0$, $a > 0$,

$$(s_3) \qquad \left| \frac{d}{dx} \frac{1}{f(x+i\beta)} \right| < \text{const.}\ (1+|x|)^{p-1}, \qquad -a \le \beta \le 0.$$

If $p > 0$, *then the relation*

$$\int_{-\infty}^{\infty} \Phi(x-u)\, dF(u) = O(e^{-\gamma x})$$

implies

$$\Phi(x) = O(e^{-\theta c x}), \qquad c = \min\{\gamma, a\}$$

for every $\theta < \frac{1}{p+1}$.

Condition (s_3) implies condition (s_1) in the strip $-a \le \Im(t) \le 0$. We may therefore introduce the function $w(t)$ and its Fourier transform $W(u)$ as in theorem 1, and obtain

$$W(u) \in L^2, \quad e^{au} W(u) \in L^2$$

in the same way as before.

(s_3) and its consequence (s_1) also give us

$$w'(x) = O(x^{-(n-p)}).$$

Hence $w'(x) \in L^2$, and Parseval's relation yields for its Fourier transform $iu\,W(u)$

$$u\,W(u) \in L^2.$$

Hence

$$W(u) \in L.$$

It should be observed that in the proof of theorem 1 the function $e^{-bu} W(u)$ was merely used to prove $W(u) \in L$ so as to derive (1). The rest of the proof was independent of the conditions on $\frac{1}{f(t)}$ in the strip $0 < \Im(t) < b$. Hence, having proved $W(u) \in L$, we can now derive the main formula (1) and formula (6) by the same argument as in theorem 1.

From (1) we find, using the estimation (2) on page 578,

$$(10) \qquad J(\Phi, x) = O(e^{-cx}[M\{e^{cu} W(u)\} + M\{e^{au} W(u)\}]), \qquad c = \min\{\gamma, a\}.$$

Using Parseval's relation and (6) on page 578 we find

$$M_2\{e^{au}W(u)\}=O(r^{n-p-\frac{1}{2}}).$$

Schwarz's inequality gives

$$\int_{-r}^{r} e^{au}|W(u)|\,du=O(r^{\frac{1}{2}}M_2\{e^{au}W(u)\})=O(r^{n-p}). \tag{11}$$

For the rest of the interval, if

$$s>0,\ s'>0,\ \frac{1}{s}+\frac{1}{s'}=1,$$

Hölder's inequality gives

$$\int_{|u|>r} e^{au}|W(u)|\,du\le\left\{2\int_{r}^{\infty}\frac{du}{u^s}\right\}^{\frac{1}{s}}M_{s'}\{ue^{au}W(u)\}=O(r^{\frac{1}{s}-1}M_{s'}\{ue^{au}W(u)\}). \tag{12}$$

According to (s_3) and its consequence (s_1),

$$|w'(x-ia)|<\text{const. }(1+|x|)^{-(n-p)}.$$

Hence $w'(x-ia)\in L^s$ for every $s\ge 1$, since $n>p+1$. Plancherel's theorem yields for its Fourier transform $iue^{au}W(u)$

$$\text{if } 1<s\le 2,\quad \frac{1}{s}+\frac{1}{s'}=1,$$

$$M_{s'}\{ue^{au}W(u)\}\le C_s M_s\{w'(x-ia)\}.$$

Let us consider this expression when $r\to 0$. By definition (p. 578)

$$i^n w(x-ia)=(1-e^{-irx-ra})^n v(x-ia).$$

Hence

$$i^n w'(x-ia)=-irne^{-irx-ra}(1-e^{-irx-ra})^{n-1}v(x-ia)+(1-e^{-irx-ra})^n v'(x-ia)=$$
$$=\sum_{q=0}^{n-1}O(r^{n-q})(1-e^{-irx})^q v(x-ia)+\sum_{q=0}^{n}O(r^{n-q})(1-e^{-irx})^q v'(x-ia).$$

Minkowski's inequality gives us

$$M_s\{w'(x-ia)\}=\sum_{q=0}^{n-1}O(r^{n-q})M_s\{(1-e^{-irx})^q v(x-ia)\}+ \tag{13}$$
$$+\sum_{q=0}^{n}O(r^{n-q})M_s\{(1-e^{-irx})^q v'(x-ia)\}.$$

According to condition (s_3), we have

$$|v(x-ia)|<\text{const. }(1+|x|)^{-(n-p)},\ |v'(x-ia)|<\text{const. }(1+|x|)^{-(n+1-p)}.$$

Applying these relations it is easy to derive

$$(14)\qquad M_s\{(1-e^{-irx})^q v(x-ia)\} = \begin{cases} O\left(r^q + r^{n-p-\frac{1}{s}}\right), & q \neq n-p-\frac{1}{s}, \\ O\left(r^q (\log r)^{\frac{1}{s}}\right), & q = n-p-\frac{1}{s}, \end{cases}$$

and

$$(15)\qquad M_s\{(1-e^{-irx})^q v'(x-ia)\} = \begin{cases} O\left(r^q + r^{n+1-p-\frac{1}{s}}\right), & q \neq n+1-p-\frac{1}{s}, \\ O\left(r^q (\log r)^{\frac{1}{s}}\right), & q = n+1-p-\frac{1}{s}. \end{cases}$$

Substituting in (13) we obtain

$$(16)\qquad M_s\{w'(x-ia)\} = O(r^n) + O\left(r^{n+1-p-\frac{1}{s}}\right), \qquad \frac{1}{s} \neq 1-p,$$

i.e., we have proved that for $1 < s \leq 2$, $\frac{1}{s} \neq 1-p$,

$$M_{s'}\{u e^{au} W(u)\} = O(r^n) + O\left(r^{n+1-p-\frac{1}{s}}\right).$$

Substituting in (12) and using (11), we obtain

$$M\{e^{au} W(u)\} = O\left(r^{n-1+\frac{1}{s}}\right) + O(r^{n-p}).$$

From this we conclude that

$$M\{e^{au} W(u)\} = O(r^{n-p}),$$

since we have assumed $p > 0$ and can choose $\frac{1}{s} > 1-p$. Similarly, we obtain

$$M\{e^{cu} W(u)\} = O(r^{n-p}).$$

Hence (10) gives us

$$J_{n,r}(\Phi, x) = O(e^{-cx} r^{n-p}), \qquad c = \min\{\gamma, a\}.$$

From this we derive the result stated in exactly the same way as in theorem 1.

Theorem 3 may be generalized in an analogous way to theorem 2 from theorem 1, i.e., we replace our O-condition for $\frac{1}{f(t)}$ by an integrability condition. We may state this in the following form

Theorem 4

Replace condition (s_3) *in theorem 3 by*

$$(s_4)\qquad \int_{-\infty}^{\infty} \left| \frac{d}{dx} \frac{1}{f(x+i\beta)} \right|^2 \frac{dx}{(1+|x|)^{2p-1}} < \text{const.} \qquad -a \leq \beta \leq 0.$$

Then theorem 3 thus modified will still hold.

Condition (s_4) implies condition (s_2) in the relevant strip $-a \le \beta \le 0$ according to a known inequality (HARDY–LITTLEWOOD–PÓLYA: Inequalities, p. 245). We may therefore introduce the function $w(t)$ and its Fourier transform $W(u)$ as in theorem 2, and obtain

$$W(u) \in L^2, \qquad e^{a u} W(u) \in L^2$$

in the same way as before. According to condition (s_4) and its consequence (s_2), we have $w'(t) \in L^2$. Hence

$$u\, W(u) \in L^2 \quad \text{and} \quad W(u) \in L.$$

Observing that in theorem 2 we merely used the conditions on $\frac{1}{f(t)}$ in the strip $0 < \Im(t) < b$ in order to prove that $W(u) \in L$, we may now derive the main formula (1) and formula (6) in the same way as in theorem 2.

The proof then follows the same lines as in theorem 3. We use formula (10) for the estimation of $J(\Phi, x)$. From (6) we obtain (11) as on page 582. For the estimation of $M_s\{w'(x-ia)\}$, however, we have to use condition (s_4) instead of (s_3), in order to obtain (14) and (15) on page 583. Let us show that this is possible. By definition,

$$v(t) = \frac{1}{t^n f(t)}.$$

Hence

$$v'(x-ia) = \frac{1}{(x-ia)^n}\left(\frac{d}{dx}\,\frac{1}{f(x-ia)} - \frac{n}{(x-ia)\,f(x-ia)}\right) = \frac{k(x)}{(x-ia)^n},$$

where

$$\frac{k(x)}{(1+|x|)^{p-\frac{1}{2}}} \in L^2,$$

according to (s_4) and its consequence (s_2). Let

$$1 < s < 2.$$

Hölder's inequality yields

$$\int\limits_{|x|>T} |v'(x-ia)|^s\, dx < \left\{ \int\limits_{|x|>T} \frac{|k(x)|^2\, dx}{|x|^{2p-1}} \right\}^{\frac{s}{2}} \left\{ 2 \int\limits_T^{\infty} \frac{dx}{x^{\frac{s(2n-2p+1)}{2-s}}} \right\}^{\frac{2-s}{2}} = O\left(T^{1-s(n+1-p)}\right).$$

Using this and $v'(x-ia) \in L^s$, it is easy to derive (15). Similarly we obtain

$$\int\limits_{|x|>T} |v(x-ia)|^s\, dx = O\left(T^{1-s(n-p)}\right).$$

Using this and $v(x-ia) \in L^s$ it is easy to prove (14). Inserting in (13), we obtain (16), and the proof may be completed in the same way as in theorem 3.

Let us now study the case where we assume a knowledge of the zeros or of the poles of $\frac{1}{f(t)}$ in the strip $-a \le \Im(t) < 0$. The following two theorems deal with this case.

Theorem 5

Impose on $F(u)$ the conditions from any of the theorems 1-4. Furthermore, let $\frac{1}{f(t)}$ have a zero of order λ_ν at t_ν, $-a < \Im(t_\nu) < 0$, $\nu = 1, 2 \ldots N$.

Then the assumption

$$\int_{-\infty}^{\infty} \Phi(x-u)\, dF(u) = O(e^{-\gamma x})$$

can be replaced by

$$\int_{-\infty}^{\infty} \Phi(x-u)\, dF(u) = \sum_{\nu=1}^{N} e^{-it_\nu x} P_\nu(x) + O(e^{-\gamma x}),$$

where $P_\nu(x)$ denotes an arbitrary polynominal of degree $< \lambda_\nu$, and the result of the theorem in question will still hold.

Introduce $w(t)$ and its Fourier transform $W(u)$ as before. Let

$$\Psi(x) = \int_{-\infty}^{\infty} \Phi(x-u)\, dF(u),$$

$$\Psi_0(x) = \sum_{\nu=1}^{N} e^{-it_\nu x} P_\nu(x).$$

Formula (1) can be derived in the same way as in the previous theorems. As before, we have

$$W(u) \in L^2, \quad e^{au} W(u) \in L^2.$$

If $-a < \Im(t) < 0$, then

$$\int_{-\infty}^{\infty} u^m W(u) e^{itu}\, du = (-i)^m w^{(m)}(t), \qquad m = 0, 1, 2 \ldots,$$

since the integral converges absolutely in this region. Now

$$\frac{d^m}{dt^m} \frac{1}{f(t)} = 0 \quad \text{at} \quad t = t_\nu, \qquad m < \lambda_\nu.$$

This implies

$$w^{(m)}(t_\nu) = 0, \qquad m < \lambda_\nu,$$

and

$$\int_{-\infty}^{\infty} u^m W(u) e^{it_\nu u}\, du = (-i)^m w^{(m)}(t_\nu) = 0, \qquad m < \lambda_\nu.$$

Let

$$P_\nu(x) = \sum_{m=0}^{\lambda_\nu - 1} C_{\nu,m} x^m.$$

Then

$$\int_{-\infty}^{\infty} \Psi_0(x-u) W(u)\, du = \sum_{\nu=1}^{N} \sum_{m=0}^{\lambda_\nu - 1} C_{\nu,m} e^{-it_\nu x} \int_{-\infty}^{\infty} e^{it_\nu u} (x-u)^m W(u)\, du = 0,$$

and (1) may be written

$$J(\Phi, x) = \int_{-\infty}^{\infty} \{\Psi(x-u) - \Psi_0(x-u)\} W(u)\, du,$$

where

$$\Psi(x) - \Psi_0(x) = O(e^{-\gamma x}), \qquad x \to \infty,$$

by assumption, and

$$\Psi(x) - \Psi_0(x) = O(e^{-\varkappa x}), \qquad x \to -\infty, \qquad \max |\Im(t_\nu)| < \varkappa < a.$$

Hence

$$|J(\Phi, x)| \le \int_{-\infty}^{\infty} |\Psi(x-u) - \Psi_0(x-u)|\, |W(u)|\, du = O(e^{-\gamma x} \int_{-\infty}^{0} |e^{\gamma u} W(u)|\, du +$$

$$+ e^{-\gamma x} \int_{0}^{x} e^{(\gamma - a) u} |e^{a u} W(u)|\, du + e^{-\varkappa x} \int_{x}^{\infty} e^{-(a-\varkappa) u} |e^{a u} W(u)|\, du).$$

From this expression it is clear, that we can estimate $J(\Phi, x)$ and with it $\Phi(x)$ in the same way and obtain the same result as in the previous theorems.

For the sake of simplicity the following theorem is stated as a generalisation of theorem 1. It is obvious how similar generalisations of theorems 2–4 may be obtained.

Theorem 6

Let $F(u) \in V$, $\Phi(u) \in E$, $f(x) = \int_{-\infty}^{\infty} e^{ixu}\, dF(u)$.

Suppose that

$\frac{1}{f(t)}$ *is meromorphic in the strip* $-a \le \Im(t) \le b$ *containing the real axis, and has in this strip poles only at* t_ν, $\nu = 1, 2 \ldots N$, *of order* λ_ν *resp.*, $-a < \Im(t_\nu) < 0$,

$$\frac{1}{f(x + i\beta)} = O(1 + |x|)^p, \qquad |x| \to \infty, \qquad -a \le \beta \le b.$$

Then the relation

$$\int_{-\infty}^{\infty} \Phi(x-u)\, dF(u) = O(e^{-\gamma x})$$

implies

$$\Phi(x) = \sum_{-\theta c < \Im(t_\nu) < 0} e^{-i t_\nu x} P_\nu(x) + O(e^{-\theta c x}), \qquad c = \min\{\gamma, a\}$$

for every

$$\theta < \frac{2}{2p+3}$$

where $P_\nu(x)$ *is a polynominal of degree* $< \lambda_\nu$.

Introduce $w(t)$ and its Fourier transform $W(u)$ as before. Formula (1) still holds, by the same argument as in theorem 1, since $\frac{1}{f(t)}$ is analytic in

$$-\varepsilon < \Im(t) < b, \text{ some } \varepsilon > 0.$$

Denote the principal part of $\frac{1}{f(t)}$ at t_ν by $q_\nu(t)$, and let

$$q(t) = \sum_{\nu=1}^{N} q_\nu(t).$$

It is easy to see that $q(t)$ is Fourier transform to

$$Q(u) = \begin{cases} \sum_{\nu=1}^{N} e^{-i t_\nu u} P_\nu^*(u), & u > 0, \\ 0 & u < 0, \end{cases}$$

where $P_\nu^*(u)$ is a polynominal of degree $< \lambda_\nu$.

Let

$$g(t) = \frac{(1 - e^{-irt})^n}{(it)^n},$$

and call its Fourier transform $G(u)$.

Then

$$w(t) = \frac{g(t)}{f(t)},$$

and

$$W(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} w(t) e^{-iut} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) \left\{ \frac{1}{f(t)} - q(t) \right\} e^{-iut} dt + \\ + \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) q(t) e^{-iut} dt = W_1(u) + W_2(u).$$

(1) gives

$$J(\Phi, x) = \int_{-\infty}^{\infty} \Psi(x-u) W_1(u) du + \int_{-\infty}^{\infty} \Psi(x-u) W_2(u) du. \tag{17}$$

The function $\frac{1}{f(t)} - q(t)$ satisfies the same conditions in the strip $-a \le \Im(t) \le b$ as $f(t)$ did in theorem 1. Hence we can use the same argument as in theorem 1 to prove

$$\int_{-\infty}^{\infty} |\Psi(x-u) W_1(u)| du = O(x^{\frac{1}{2}} e^{-cx} r^{n-p-\frac{1}{2}}), \qquad c = \min\{\gamma, a\}.$$

Since $Q(u) \in L^2$, $G(u) \in L^2$, Parseval's relation gives

$$W_2(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) q(t) e^{-iut} dt = \int_{-\infty}^{\infty} G(u-v) Q(v) dv.$$

Hence

$$\int_{-\infty}^{\infty} \Psi(x-u) W_2(u) du = \int_{-\infty}^{\infty} G(x-u) du \int_{-\infty}^{\infty} \Psi(u-v) Q(v) dv, \tag{18}$$

the inversion being justified by absolute convergence, since $G(u)$ is zero outside a finite interval.

Let

$$\int_{-\infty}^{\infty} \Psi(u-v)\,Q(v)\,dv = \int_{-\infty}^{\infty} - \int_{-\infty}^{0} \Psi(u-v)\{\sum_{-c<\Im\{t_\nu\}<0} e^{-it_\nu v} P_\nu^*(v)\}\,dv +$$

$$+\int_{0}^{\infty} \Psi(u-v)\{\sum_{-a<\Im\{t_\nu\}\le -c} e^{-it_\nu v} P_\nu^*(v)\}\,dv = H(u) + R_1(u) + R_2(u),$$

where $R_2(u) \equiv 0$ if $a < \gamma$. Then it is easy to verify that

$$R_1(u) = O(e^{-\gamma u}), \qquad u \to \infty,$$

$$R_2(u) = O(u^\lambda e^{-cu}), \qquad u \to \infty, \qquad \lambda = \max \lambda_\nu,$$

and

$$H(u) = \sum_{-c<\Im\{t_\nu\}<0} e^{-it_\nu u} P_\nu(u), \tag{19}$$

where $P_\nu(u)$ is a polynominal of degree $< \lambda_\nu$, whose coefficients depend on $f(t)$ and $\Psi(u)$. Substituting in (18), and applying the definition of $G(u)$, we obtain

$$\int_{-\infty}^{\infty} \Psi(x-u)\,W_2(u)\,du = \int_{-\infty}^{\infty} G(x-u)\{H(u)+R_1(u)+R_2(u)\}\,du =$$

$$= J(H+R_1+R_2,\, x) = J(H,\, x) + O(x^\lambda e^{-cx} r^n).$$

Thus (17) gives

$$J(\Phi - H,\, x) = O(x^{\frac{1}{2}} e^{-cx} r^{n-p-\frac{1}{2}}) + O(x^\lambda e^{-cx} r^n),$$

where $H(u)$ is given by (19).

Since $\frac{d}{du} H(u)$ is bounded if $u > 0$, we can argue as on page 8 to show that this implies

$$\Im\{H(x)\} = O(e^{-\theta c x}), \quad \Phi(x) - \Re\{H(x)\} = O(e^{-\theta c x})$$

for every $\theta < \frac{2}{2p+3}$.

Hence

$$\sum_{-\theta c<\Im\{t_\nu\}<0} e^{-it_\nu x} P_\nu(x)$$

is real, and

$$\Phi(x) - \sum_{-\theta c<\Im\{t_\nu\}<0} e^{-it_\nu x} P_\nu(x) = O(e^{-\theta c x})$$

for every

$$\theta < \frac{2}{2p+3},$$

which proves theorem 6.

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