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On the probabilities that a random walk is negative

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1. Introduction, notations and summary

Let $X_1, X_2, \dots$ be independent copies of a random variable X with distribution function $F(x)$. The successive partial sums are denoted $S_n = X_1 + X_2 + \dots + X_n$, $n = 1, 2, \dots$. We define $a_n = P(S_n < 0)$, $n = 1, 2, \dots$. To every distribution function we get an associated sequence $\{a_n\}_1^\infty$. We list two immediate relations between the existence of moments of X and the asymptotic behavior of $\{a_n\}_1^\infty$.

A. The law of large numbers implies that $\lim_{n\to\infty} a_n = 0$ if $EX > 0$ and that $\lim_{n\to\infty} a_n = 1$ if $EX < 0$.

B. From the central limit theorem follows that if $EX^2 < \infty$ and $EX = 0$ then

$$\lim_{n\to\infty} a_n = \tfrac{1}{2}. \tag{1.1}$$

The main aim of this paper is to answer the following question raised by F. Spitzer in [3], p. 337. Does there exist a distribution $F(x)$ for which the sequence $\{a_n\}_1^\infty$ fails to have a $(C, 1)$-limit?

In Theorem 1 we show that there is a distribution such that $E|X|^{2-\delta} < \infty$ for every $\delta > 0$, for which $\{a_n\}_1^\infty$ does not possess a $(C, 1)$-limit. In Theorem 2 we discuss the limitability of $\{a_n\}_1^\infty$ for general limitation methods, and show that for any regular linear limitation method there exists a distribution for which $\{a_n\}_1^\infty$ cannot be limited.

According to A, B and the result in Theorem 1, the condition $EX^2 < \infty$ and $EX = 0$ is a weakest possible sufficient condition in terms of moments only for (1.1) to hold. In Theorem 3 we give a more general sufficient condition for (1.1). The essence of this theorem is that (1.1) holds if $F(x)$ does not deviate too much from a distribution which is symmetric around zero.

I wish to thank Professor L. Carleson for having suggested the theme of this paper and for valuable guidance.

2. Existence of distributions for which $\{a_n\}_1^\infty$ cannot be limited

Theorem 1. *There exists a distribution $F(x)$ with $E|X|^{2-\delta} < \infty$ for every $\delta > 0$ for which upper and lower $(C, 1)$-limits of $\{a_n\}_1^\infty$ are respectively 1 and 0.*

Proof. We show the existence by an explicit exampel. We define a discrete distribution with mass points $\{c_\nu\}_1^\infty$ and corresponding probabilities

$$p_\nu = P(X = c_\nu) = [(e-1)\,\nu\,!]^{-1}, \quad \nu = 1, 2, \ldots .$$

The essential feature of this choice of the probabilities is that $p_{\nu+1}/p_\nu \to 0$ when $\nu \to \infty$. We first determine the c's with odd indices. Let

$$c_{2\nu-1} = (-1)^{\nu+1}\, p_{2\nu-1}^{-(\frac{1}{2}+\lambda(2\nu-1))}, \quad \nu = 1, 2, \ldots,$$

where

$$\lambda(2\nu-1) = \lambda(2\nu) = (\log 2\nu)^{-\frac{1}{2}}. \tag{2.1}$$

The essential property of $\lambda(\nu)$ is that it tends to 0, but not too fast, when $\nu \to \infty$. We observe that c_ν is alternatively positive and negative when ν runs through odd indices. For even indices we define $c_{2\nu}$ through the relation

$$p_{2\nu-1}\, c_{2\nu-1} + p_{2\nu}\, c_{2\nu} = 0, \quad \nu = 1, 2, \ldots \tag{2.2}$$

which yields

$$c_{2\nu} = (-1)^\nu \cdot p_{2\nu-1}^{\frac{1}{2}-\lambda(2\nu)}\, p_{2\nu}^{-1}. \tag{2.3}$$

The distribution is now completely specified and we derive some of its properties. It is easily checked that

$$\sum_N^\infty p_\nu \sim p_N \quad \text{when} \quad N \to \infty \tag{2.4}$$

and that

$$|c_{2\nu-1}| < |c_{2\nu}|, \;\; \nu = 1, 2, \ldots . \tag{2.5}$$

Note. Throughout the paper the symbol $\sim$ means that the ratio of the quantity to the right and to the left of $\sim$ tends to 1.

Next we show that $E\,|X|^{2-\delta} < \infty$ for every $\delta > 0$. Let ν be even and $\delta > 0$. As $\lambda(n) \to 0$ when $n \to \infty$ the following inequality holds when ν is sufficiently large

$$p_\nu\,|c_\nu|^{2-\delta} = p_\nu^{\frac{1}{2}\delta - \lambda(\nu)(2-\delta)} \leqslant p_\nu^{\frac{1}{4}\delta} < (\nu\,!)^{-\frac{1}{4}\delta}.$$

For ν odd and sufficiently large we have

$$p_\nu\,|c_\nu|^{2-\delta} = p_\nu^{\delta-1}\, p_{\nu-1}^{1-\frac{1}{2}\delta-\lambda(\nu)(2-\delta)} \leqslant \left(\frac{p_{\nu-1}}{p_\nu}\right)^{1-\delta} \cdot p_{\nu-1}^{\frac{1}{4}\delta} \leqslant \nu\,[(\nu-1)\,!]^{-\frac{1}{4}\delta}.$$

Thus

$$E\,|X|^{2-\delta} = \sum_{\nu=1}^\infty p_\nu\,|c_\nu|^{2-\delta} < \infty. \tag{2.6}$$

In passing we make the following observations. We are going to show that for the distribution we have constructed it holds that $\{a_n\}_1^\infty$ has not a $(C, 1)$-limit and *a fortiori* that $\{a_n\}_1^\infty$ has not a limit. From A and B in § 1, it follows that such a distribution must satisfy

(i) $EX^2 = \infty$,

(ii) $EX = 0$ if the mean of X exists.

For the above distribution (i) is easily verified and (ii) follows from (2.2). We shall need the following estimate later.

$$\sum_1^N p_\nu c_\nu^2 \leqslant H p_N c_N^2, \quad N \text{ even}, \tag{2.7}$$

where H is a constant independent of N. From (2.2) if follows that $p_{2\nu}|c_{2\nu}| = p_{2\nu-1}|c_{2\nu-1}|$ and (2.5) gives $p_{2\nu} c_{2\nu}^2 > p_{2\nu-1} c_{2\nu-1}^2$. Thus

$$\sum_{\nu=1}^N p_\nu c_\nu^2 \leqslant 2 \sum_{\nu=1}^{N/2} p_{2\nu} c_{2\nu}^2 = 2 p_N c_N^2 \sum_{\nu=1}^{N/2} p_{2\nu} c_{2\nu}^2 p_N^{-1} c_N^{-2}.$$

Estimates with Stirling's formula yield

$$\lim_{\nu \to \infty} p_{2(\nu-1)} c_{2(\nu-1)}^2 p_{2\nu}^{-1} c_{2\nu}^{-2} = 0$$

and thus $$p_{2(\nu-1)} c_{2(\nu-1)}^2 p_{2\nu}^{-1} c_{2\nu}^{-2} \leqslant \tfrac{1}{2} \quad \text{for} \quad \nu \geqslant \nu_0.$$

This implies

$$\sum_1^{N/2} p_{2\nu} c_{2\nu}^2 \leqslant 2 p_N c_N^2 \left\{ 1 + \tfrac{1}{2} + (\tfrac{1}{2})^2 + \ldots + p_N^{-1} c_N^{-2} \sum_1^{\nu_0} p_{2\nu} c_{2\nu} \right\}$$

and now (2.7) follows as $p_N c_N^2 \to \infty$ when $N \to \infty$.

We introduce the events

$A(n, N)$: S_n and c_N have the same sign.

$B(n, N)$: $X_1, X_2, \ldots, X_n$ all attain their values among $(c_1, c_2, \ldots, c_N)$,

$C_k(n, N)$: Exactly k of $X_1, X_2, \ldots, X_n$ attain the value c_N, $k = 1, 2, \ldots, n$.

For simplicity, we shall sometimes suppress the indices n and N and we understand that they both are the same for A, B and C when these events occur simultaneously. The following inequalities are immediate

$$P(A) \geqslant P\Big(\bigcup_{k=0}^n ABC_k\Big) = \sum_{k=0}^n P(ABC_k)$$

$$\geqslant \sum_{k=K}^n P(B) \cdot P(C_k \mid B) \cdot P(A \mid BC_k),$$

where K is a non-negative integer $\leqslant n$. $P(A \mid BC_k)$ increases with k for $k \leqslant n$ and we get

$$P(A(n, N)) \geqslant P(B) \cdot \sum_{k=K}^n P(C_k \mid B) \cdot P(A \mid BC_K). \tag{2.8}$$

We shall let N tend to infinity and we consider the following choices of n and K as functions of N.

$$n(N) = N^{-\alpha\lambda(N)} p_{N+1}^{-1} \quad \text{for} \quad \tfrac{1}{2} \leqslant \alpha \leqslant 1,$$
$$K(N) = \tfrac{1}{2}\, n(N)\, p_N.$$

Our aim is to show that

$$P(A(n(N), N)) \to 1 \tag{2.9}$$

uniformly in α for $\frac{1}{2} \leqslant \alpha \leqslant 1$, when $N \to \infty$ through odd values. We do this by showing that all three factors to the right in (2.8) tend to 1 uniformly in α for $\frac{1}{2} \leqslant \alpha \leqslant 1$. We start by showing

$$\lim_{N\to\infty} P(B(n(N), N)) = 1 \tag{2.10}$$

and the convergence is uniform for $\frac{1}{2} \leqslant \alpha \leqslant 1$.

$$P(B(n(N), N)) = \left(\sum_1^N p_\nu\right)^{n(N)} = \left(1 - \sum_{N+1}^{\infty} p_\nu\right)^{n(N)} \sim \exp\left(-n(N)\, p_{N+1}\right)$$

according to (2.4). Thus

$$P(B(n(N), N)) \sim \exp\left(-N^{-\alpha\lambda(N)}\right)$$

and (2.10) follows. Next we prove

$$\lim_{N\to\infty} \sum_{k=K(N)}^{n(N)} P(C_k(n(N), N) \mid B) = 1 \tag{2.11}$$

and the convergence is uniform for $\frac{1}{2} \leqslant \alpha \leqslant 1$. We introduce the truncated random variable $X^{(N)}$ and the random variable $Y^{(N)}$.

$$P(X^{(N)} = c_\nu) = p_\nu \left(\sum_1^N p_\nu\right)^{-1}, \quad \nu = 1, 2, \ldots, N \tag{2.12}$$

$$Y^{(N)} = \begin{cases} 1 & \text{if } X^{(N)} = c_N \\ 0 & \text{otherwise.} \end{cases}$$

Then
$$P(C_k \mid B) = P\left(\sum_1^n Y_\nu^{(N)} = k\right)$$

and
$$\sum_{k=K(N)}^{n(N)} P(C_k(n(N), N) \mid B) = P\left(\sum_1^{n(N)} Y_\nu^{(N)} \geqslant K(N)\right), \tag{2.13}$$

where the $Y_\nu^{(N)}$'s are independent. The random variable $\sum_1^n Y_\nu^{(N)}$ has a binomial distribution with mean $n(N)\, p_N \left(\sum_1^N p_\nu\right)^{-1}$. Estimates with Tchebycheff's inequality give that the right hand side in (2.13) tends to 1 uniformly for $\frac{1}{2} \leqslant \alpha \leqslant 1$. Thus (2.11) is proved. Finally we show

$$P(A(n(N), N) \mid BC_{K(N)}) \to 1 \tag{2.14}$$

uniformly for $\frac{1}{2} \leqslant \alpha \leqslant 1$ when $N \to \infty$ through odd values.

The conditioned random variable $S_n \mid BC_{K(N)}$ is identical in distribution with the random variable

$$K(N) \cdot c_N + \sum_{1}^{n(N)-K(N)} X_\nu^{(N-1)},$$

where the $X_\nu^{(N-1)}$ are independent copies of the random variable $X^{(N-1)}$ defined in (2.12). Thus

$$P(A \mid BC_{K(N)}) \geqslant P\left(\left|\sum_{1}^{n(N)-K(N)} X_\nu^{(N-1)}\right| < K(N) \cdot |c_N|\right).$$

As N is assumed to be odd it follows from (2.2) that $EX^{(N-1)} = 0$. Tchebycheff's inequality now yields

$$P(A \mid BC^{K(N)}) \geqslant 1 - \frac{(n - K(N)) \sum_{1}^{N-1} p_\nu c_\nu^2}{K(N)^2 c_N^2 \sum_{1}^{N-1} p_\nu} \geqslant$$

and in virtue of (2.7)

$$\geqslant 1 - H \cdot \frac{n(N) p_{N-1} c_{N-1}^2}{K(N)^2 c_N^2 \sum_{1}^{N-1} p_\nu} = 1 - R(N).$$

By inserting the choices of $n(N)$ and $K(N)$ we get

$$R(N) \sim 4H \frac{p_{N+1} \cdot p_{N-2}}{p_N \cdot p_{N-1}} \frac{p_N^{2\lambda(N)}}{p_{N-2}^{2\lambda(N-2)}} \cdot N^{\alpha\lambda(N)}$$

$$\sim 4HN^{\alpha\lambda(N)} [N(N-1)]^{-2\lambda(N)} \cdot p_{N-2}^{2(\lambda(N)-\lambda(N-2))}.$$

An estimate with Stirling's formula gives that $p_{N-2}^{2(\lambda(N)-\lambda(N-2))} \sim 1$ and we get

$$R(N) \sim 4H \exp\left\{(\alpha - 4)\sqrt{\log N}\right\}.$$

Thus $P(A \mid BC_{K(N)}) \to 1$ uniformly for $\frac{1}{2} \leqslant \alpha \leqslant 1$ and (2.14) is proved. Now formulas (2.8), (2.10), (2.11), and (2.14) together imply (2.9).

For odd values of N, c_N is every second time positive and every second time negative. Thus we get as an immediate consequence of (2.9) that $\overline{\lim}_{n\to\infty} a_n = 1$ and $\underline{\lim}_{n\to\infty} a_n = 0$. We want to sharpen this to the result that also upper and lower $(C, 1)$-limits of $\{a_n\}_1^\infty$ are respectively 1 and 0. Choose $\varepsilon > 0$. From (2.9) it follows that if N is odd and sufficiently large and $c_N < 0$, then

$$a_n \geqslant 1 - \varepsilon \text{ for } n_1(N) \leqslant n \leqslant n_2(N),$$

where $n_1(N) = N^{-\lambda(N)} p_{N+1}^{-1}$ and $n_2(N) = N^{-\frac{1}{2}\lambda(N)} p_{N+1}^{-1}$. Thus

$$\frac{1}{n_2(N)}\sum_{\nu=1}^{n_2(N)} a_\nu \geqslant \frac{1}{n_2(N)}\sum_{n_1(N)}^{n_2(N)} a_\nu \geqslant (1-\varepsilon)\left(1-\frac{n_1(N)}{n_2(N)}\right).$$

Now $n_1(N)/n_2(N)\to 0$ when $N\to\infty$. Thus

$$\overline{\lim_{n\to\infty}}\,\frac{1}{n}\sum_{\nu=1}^{n} a_\nu = 1.$$

In the same manner, it follows that

$$\underline{\lim_{n\to\infty}}\,\frac{1}{n}\sum_{\nu=1}^{n} a_\nu = 0$$

and Theorem 1 is proved.

Concluding remark. As the a_n's are probabilities, they lie between 0 and 1. It is well known that Abel and $(C, 1)$-limitability are equivalent for bounded sequences (see e.g. [1] Theorem 92). Thus for the distribution constructed above it holds that $\{a_n\}_1^\infty$ cannot be Abel limited. In fact, it is not hard to show directly that $\{a_n\}_1^\infty$ has upper and lower Abel limits respectively 1 and 0.

The part of Theorem 1 which concerns non-limitability of $\{a_n\}_1^\infty$ holds for general linear limitation methods. We consider a regular limitation matrix $[\gamma_{mn}]$, m, $n=1,2,\ldots$, i.e. we assume

(i) $\sum_n |\gamma_{mn}| \leqslant C$,

(ii) $\lim_{m\to\infty} \gamma_{mn}=0$ for all n,

(iii) $\sum_n \gamma_{mn}\to 1$ when $m\to\infty$.

Theorem 2. *For every regular limitation matrix* $[\gamma_{mn}]$ *there exists a distribution* $F(x)$ *for which the sequence* $\{a_n\}_1^\infty$ *satisfies*

$$\left.\begin{aligned}&\overline{\lim_{m\to\infty}}\,\sum_n \gamma_{mn}\,a_n = 1\\ \textit{and}\qquad &\underline{\lim_{m\to\infty}}\,\sum_n \gamma_{mn}\,a_n = 0.\end{aligned}\right\} \tag{2.15}$$

Remark. We do not know any general relation between $[\gamma_{mn}]$ and the order of the moments that $F(x)$ can possess when (2.15) holds.

Proof. The main idea in the proof is the same as in the proof of Theorem 1 and therefore we make the proof somewhat brief. We construct a discrete distribution with points of mass $\{c_\nu\}_1^\infty$ and corresponding probabilities $\{p_\nu\}_1^\infty$. The successive signs of $c_1, c_2\ldots$ are chosen $+-+-+-\ldots$. Let $\{\varepsilon_\nu\}_1^\infty$ be a sequence of positive numbers which tend to 0. We determine $\{p_\nu\}_1^\infty$, $\{c_\nu\}_1^\infty$ a sequence $\{m_\nu\}_1^\infty$ of integers and a sequence $\{I_\nu\}_1^\infty$ of intervals of integers $I_\nu=[i_\nu, j_\nu]$ recursively. We assume that p_ν, c_ν, m_ν, and I_ν are determined for $\nu=1, 2, \ldots, N-1$. We consider p_N as a function of the parameter λ_N given by the relation

$$p_N = \lambda_N \left(1 - \sum_1^{N-1} p_\nu\right)$$

and we shall determine p_N by determining λ_N, $\frac{1}{2} \leqslant \lambda_N < 1$. First we choose c_N so that $|c_N| > |c_{N-1}|$ and $\operatorname{sgn}(\sum_1^N p_\nu c_\nu) = \operatorname{sgn}(c_N)$ when $\lambda_N = \frac{1}{2}$. Let $X_\nu^{(N)}(\lambda_N)$, $\nu = 1, 2, \ldots$, be independent random variables with distribution

$$P(X_\nu^{(N)}(\lambda_N) = c_k) = p_k \left(\sum_1^N p_\nu\right)^{-1}, \quad k = 1, 2, \ldots, N.$$

Tchebycheff's inequality implies the existence of a number $i_N = i(\varepsilon_N)$ such that $i_N > j_{N-1}$ and

$$P\left(\sum_{\nu=1}^{n} X_\nu^{(N)}(\lambda_N) \text{ and } c_N \text{ have the same sign}\right) \geqslant 1 - \varepsilon_N \tag{2.16}$$

when $n \geqslant i_N$ and $\frac{1}{2} \leqslant \lambda_N < 1$. Now choose $m_N > m_{N-1}$ so large that $\sum_{n=1}^{i_N} |\gamma_{m_N n}| \leqslant \varepsilon_N$ and j_N large enough for $\sum_{n=j_N+1}^{\infty} |\gamma_{m_N n}| \leqslant \varepsilon_N$ to hold. These choices are clearly possible. Finally, we fix λ_N and thus p_N by the condition

$$\left(\sum_1^N p_\nu\right)^{j_N} \geqslant 1 - \varepsilon_N. \tag{2.17}$$

The distribution is now completely determined. Let $X_1, X_2, \ldots$ be independent random variables with this distribution and $\varrho(n, N) = P(S_n$ and c_N have the same sign). Then

$$\varrho(n, N) = P(S_n \text{ and } c_N \text{ have the same sign } |\max_{1 \leqslant \nu \leqslant n} |X_\nu| \leqslant c_N)\; P(\max_{1 \leqslant \nu \leqslant n} |X_\nu| \leqslant c_N).$$

In virtue of (2.16) and (2.17) we get

$$\varrho(n, N) \geqslant (1 - \varepsilon_N)^2 \quad \text{when} \quad n \in I_N$$

and thus $a_n \geqslant (1 - \varepsilon_N)^2$ when $n \in I_N$ and N is even, while $a_n \leqslant 1 - (1 - \varepsilon_N)^2$ when $n \in I_N$ and N is odd. The theorem now follows.

3. A sufficient condition for $\lim a_n = \frac{1}{2}$

According to B in § 1 $EX^2 < \infty$ and $EX = 0$ is a sufficient condition for $\lim a_n = \frac{1}{2}$. However, this condition is not necessary. This follows immediately from the fact that if $F(x)$ is continuous and symmetric around 0 then $a_n = \frac{1}{2}$ for all n and thus $\lim a_n = \frac{1}{2}$. In the next theorem we show that the assumptions about symmetry and the existence of a finite second moment and zero mean can be combined to get a more general sufficient condition.

Theorem 3. *If $F(x)$ is non-degenerate and can be decomposed $F(x)=H(x)+G(x)$, where H and G are of bounded variation and satisfy*

(1) *$H(x)$ is symmetric around* 0, *i.e.*

$H(-x)-H(-\infty)=H(\infty)-H(x)$ *for* all $x \geqslant 0$ which are continuity points of $H(x)$.

(2) $\int_{-\infty}^{\infty} x^2 |dG(x)| < \infty$ *and* $\int_{-\infty}^{\infty} x\, dG(x)=0$,

hen $\lim_{n\to\infty} a_n = \frac{1}{2}$.

Proof. The proof will be based on the following formula from [2], p. 331.

$$|a_n - \tfrac{1}{2}| \leqslant \frac{n}{\pi}\int_0^{\delta} \frac{|\varphi(t)|^n}{t}\,|\arg \varphi(t)|\,dt + R(n,\delta), \tag{3.1}$$

where $\varphi(t)$ is the characteristic function of $F(x)$ and where $R(n,\delta)\to 0$ when $n\to\infty$ for every $\delta>0$.

As $H(x)$ is symmetric around 0, we have

$$\operatorname{Im}\{\varphi(t)\} = \int_{-\infty}^{\infty} \sin\, xt\, dF(x) = \int_{-\infty}^{\infty} \sin\, xt\, dG(x)$$

and from (2) it follows that

$$\lim_{t\to 0} t^{-2}\int_{-\infty}^{\infty} \sin\, xt\, dG(x) = 0.$$

Thus
$$|\arg \varphi(t)| \leqslant t^2 h(t), \tag{3.2}$$

where $h(t)\to 0$ when $t\to 0$.

We shall also use the fact that there are positive numbers δ_0 and C such that

$$|\varphi(t)| \leqslant 1 - Ct^2 \quad \text{for} \quad |t| \leqslant \delta_0. \tag{3.3}$$

For a proof of (3.3) see e.g. Lemma 1 in [2].

By inserting the estimates (3.2) and (3.3) into (3.1), we obtain, for $0<\delta\leqslant\delta_0$,

$$|a_n - \tfrac{1}{2}| \leqslant \frac{1}{\pi}\sup_{0\leqslant t\leqslant\delta} h(t)\int_0^{\delta} nt\,e^{-nCt^2}\,dt + R(n,\delta)$$
$$\leqslant \frac{1}{2\pi C}\sup_{0\leqslant t\leqslant\delta} h(t) + R(n,\delta).$$

Thus
$$\overline{\lim_{n\to\infty}}\, |a_n - \tfrac{1}{2}| \leqslant (2\pi C)^{-1}\sup_{0\leqslant t\leqslant\delta} h(t)$$

and by letting $\delta\to 0$, we obtain the desired result.

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