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On the derivatives of bounded analytic functions

By Åke Samuelsson

1. Introduction

Let f be analytic in the unit disc, $|z|<1$ and suppose that $|f(z)|\leqslant 1$ for $|z|<1$. Then f ($f\neq 0$) admits a representation $f=B\cdot E$, where

$$B(z)=e^{i\theta}z^m\prod_k \frac{\bar{a}_k(a_k-z)}{|a_k|(1-\bar{a}_k z)} \tag{1.1}$$

is the normalized Blaschke product of f and where

$$E(z)=\exp\{-w(z)\},\ w(z)=\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t), \tag{1.2}$$

with a bounded and non-decreasing function μ defined on the interval $[0, 2\pi]$. If x is a point in the open interval $(0, 2\pi)$, such that

$$\frac{\mu(x+h)-\mu(x-h)}{h}\to+\infty \text{ as } h\to+0, \tag{1.3}$$

it is well known (cf. [1], p. 108) that

$$f(e^{ix})=\lim_{r\to 1-0} f(re^{ix})=0.$$

However, condition (1.3) does not imply the existence of the radial limits of the derivatives of f. In a previous paper [3] I proved that, *if*

$$\liminf_{h\to+0}\frac{\mu(x+h)-\mu(x-h)}{-h\log h}>1$$

and if $f(z)\neq 0$ *in the unit disc, then*

$$f'(e^{ix})=\lim_{r\to 1-0} f'(re^{ix})=0.$$

It is the primary object of the present paper to improve and generalize this result. We establish, *if*

$$\liminf_{h\to+0} \frac{\mu(x+h)-\mu(x-h)}{-h\log h} > \frac{n}{\pi},$$

where n is a natural number, then

$$f^{(k)}(e^{ix}) = \lim_{r\to 1-0} f^{(k)}(re^{ix}) = 0 \quad \textit{for} \quad 0 \leqslant k \leqslant n.$$

Moreover, local conditions on the function μ and on the Blaschke product, implying the existence of $f^{(n)}(e^{ix}) \neq 0$, will be given.

Throughout this paper we use the following notations and conventions.

The class of analytic functions f described above will be called $\mathfrak{F}$. The class of analytic functions w defined in the unit disc by

$$w(z) = \int_0^{2\pi} H_z(t)\, d\mu(t), \quad H_z(t) = \frac{e^{it}+z}{e^{it}-z}, \tag{1.4}$$

where μ is a function of bounded variation on the interval $[0, 2\pi]$, is denoted by $\mathfrak{W}$. It should be noted that $w \in \mathfrak{W}$ implies that Im $w(0) = 0$. Moreover, $\mathfrak{U}$ will be the class of harmonic functions u, which are the Poisson integral of a finite real measure on the unit circle $|z| = 1$, i.e.

$$u(re^{ix}) = \int_0^{2\pi} P_r(x-t)\, d\mu(t), \tag{1.5}$$

where $$P_r(t) = \frac{1-r^2}{1+r^2-2r\cos t}$$

is the Poisson kernel and where μ is a function of bounded variation on $[0, 2\pi]$.

If $u \in \mathfrak{U}$, $\tilde{u}$ is the conjugate harmonic function of u determined by $\tilde{u}(0) = 0$. The classes $\mathfrak{W}$ and $\mathfrak{U}$ are related to each other as follows; if $w \in \mathfrak{W}$, then $u =$ Re $w \in \mathfrak{U}$ and if $u \in \mathfrak{U}$, then $w = u + i\tilde{u} \in \mathfrak{W}$.

The function μ associated with the functions f, w and u in the representation formulas (1.2), (1.4) and (1.5) has a periodic extension denoted by μ^* and defined as follows; put $\mu^*(t) = \mu(t)$ for $0 \leqslant t < 2\pi$ and extend this function to a periodic function with the period 2π.

If $u \in \mathfrak{U}$ is the Poisson integral of the finite measure induced by μ, we put $|\mu|(t) =$ the total variation of μ on the interval $[0, t]$, and $|u|$ is then defined as the Poisson integral of the positive measure induced by $|\mu|$.

The point x is always in the open interval $(0, 2\pi)$.

For brevity's sake it is convenient to introduce the funtions φ and $\tilde{\varphi}$ defined by

$$\varphi(u;\, x, t) = \frac{\mu(x+t)-\mu(x-t)}{t}, \quad t > 0$$

and $$\tilde{\varphi}(\mu;\, x, t) = \frac{\mu(x+t)+\mu(x-t)-2\mu(x)}{t}, \quad t > 0.$$

The function μ is said to be smooth at the point x if and only if

$$\lim_{t\to+0} \tilde{\varphi}(\mu;\, x,\, t) = 0.$$

If g is any function analytic in the unit disc the point set $C(e^{ix}, g)$ is defined as follows; $\zeta \in C(e^{ix}, g)$ if and only if there is a sequence $\{r_k\}_{k=0}^{\infty}$ with $0 < r_k < 1$, such that

$$\lim_{k\to\infty} r_k = 1 \text{ and } \lim_{k\to\infty} g(r_k e^{ix}) = \zeta.$$

2. Some properties of the classes $\mathfrak{W}$ and $\mathfrak{U}$

The aim of this section is to connect the radial increase of the function $w \in \mathfrak{W}$ and its derivatives with the increase of the associated function μ. Of course, the results obtained can be formulated in terms of functions $u \in \mathfrak{U}$.

First, let us construct an auxiliary function $w_{\alpha\beta\gamma} \in \mathfrak{W}$. Put

$$w_1(z) = -\log(1 - ze^{-ix})$$

and

$$w_2(z) = -2i \log\left(\frac{e^{ix} - z}{z - 1}\right) - x + 2\pi, \quad |z| < 1,$$

where $\log \zeta$ is the principal branch of the logarithm function, defined in the region $\zeta + |\zeta| \neq 0$ and uniquely determined by $\log 1 = 0$. The auxiliary function $w_{\alpha\beta\gamma}$ is now defined by

$$w_{\alpha\beta\gamma}(z) = \alpha(w_1(z) - 1) + \beta - \frac{\gamma}{2}(w_2(z) - \pi),$$

and the measure associated with $w_{\alpha\beta\gamma}$ is induced by $\mu_{\alpha\beta\gamma}$, given by (cf. [2], p. 198)

$$2\pi\mu_{\alpha\beta\gamma}(t) = \lim_{r\to1-0}\int_0^t \operatorname{Re} w_{\alpha\beta\gamma}(re^{iy})\,dy = \alpha \lim_{r\to1-0}\int_0^t \operatorname{Re} w_1(re^{iy})\,dy$$

$$-\frac{\gamma}{2}\lim_{r\to1-0}\int_0^t \operatorname{Re} w_2(re^{iy})\,dy - \left(\alpha - \beta - \frac{\gamma}{2}\pi\right) t.$$

Obviously
$$\lim_{r\to1-0}\int_0^t \operatorname{Re} w_1(re^{iy})\,dy = -\int_0^t \log\left|2\sin\left(\frac{x-y}{2}\right)\right| dy$$

and, observing that $\operatorname{Re} w_2(z) = 2\pi\,\omega(z, 0, x)$, where $\omega(z, 0, x)$ is the harmonic measure of the arc $\{e^{it};\ 0 \leqslant t \leqslant x\}$ (cf. [2], p. 7), we have

$$(2\pi)^{-1}\lim_{r\to1-0}\int_0^t \operatorname{Re} w_2(re^{iy})\,dy = t + (x - t)\,\varepsilon(t),$$

where $\varepsilon(t) = 0$ if $0 \leqslant t < x$ and $\varepsilon(t) = 1$ if $x \leqslant t \leqslant 2\pi$.

It is now easy to verify that the function $\mu_{\alpha\beta\gamma}$ has the properties

$$\lim_{h\to+0} \{\pi\varphi(\mu_{\alpha\beta\gamma};\ x, h) + \alpha \log h\} = \beta$$

and
$$\lim_{h\to+0} 2\tilde{\varphi}(\mu_{\alpha\beta\gamma};\ x, h) = \gamma.$$

We now state and prove a theorem connecting the increase of the real part of w with the increase of the associated function μ.

Theorem 2.1. *Let $u \in \mathfrak{U}$. Then, if α is any real number,*

$$\liminf_{h\to+0} \{\pi\varphi(\mu; x, h) + \alpha \log h\} \leqslant \alpha + \liminf_{r\to 1-0} \{u(re^{ix}) + \alpha \log (1-r)\}$$

and

$$\limsup_{h\to+0} \{\pi\varphi(\mu; x, h) + \alpha \log h\} \geqslant \alpha + \limsup_{r\to 1-0} \{u(re^{ix}) + \alpha \log (1-r)\}.$$

Proof. If $\alpha = 0$, Theorem 2.1 is nothing but a re-writing of Fatou's theorem on Abel summability (Zygmund [4], p. 99).

If $\alpha \neq 0$, we put $u_\alpha = \operatorname{Re} w_{\alpha 00}$. Since

$$\lim_{h\to+0} \{\pi\varphi(\mu_\alpha;\ x,\ h) + \alpha \log h\} = \alpha + u_\alpha(re^{ix}) + \alpha \log (1-r) = 0,$$

we may apply Fatou's theorem to the function $u - u_\alpha$ to obtain Theorem 2.1 in the general case.

Corollary 2.1. *If*

$$\lim_{h\to+0} \{\pi\varphi(\mu;\ x,\ h) + \alpha \log h\} = \beta$$

then
$$\lim_{r\to 1-0} \{u(re^{ix}) + \alpha \log (1-r)\} = \beta - \alpha.$$

The following two inequalities

$$\liminf_{h\to+0} \frac{\pi\varphi(\mu;\ x,\ h)}{-\log h} \leqslant \liminf_{r\to 1-0} \frac{u(re^{ix})}{-\log (1-r)} \tag{2.1}$$

and
$$\limsup_{h\to+0} \frac{\pi\varphi(\mu;\ x,\ h)}{-\log h} \geqslant \limsup_{r\to 1-0} \frac{u(re^{ix})}{-\log (1-r)}$$

are immediate consequences of Theorem 2.1. It should be noted that there are harmonic functions, such that the sign of equality does not hold in these inequalities. For instance the function u associated with the singular positive measure induced by

$$\mu(t)=\begin{cases} e^{-1} & \text{if } x+e^{-2}<t\leqslant 2\pi, \\ ne^{-n} & \text{if } x+e^{-n-1}<t\leqslant x+e^{-n},\ n\geqslant 2, \\ 0 & \text{if } 0\leqslant t\leqslant x \end{cases}$$

is such a function

Theorem 2.1 shows that the radial increase of u depends on the behaviour of φ. Likewise, the increase of the conjugate harmonic function $\tilde{u}$ is connected with the behaviour of $\tilde{\varphi}$. This connection, however, is more intricate and the only thing we prove is the following analogue of Corollary 2.1.

Theorem 2.2. *Let $u\in\mathfrak{U}$ and suppose that*

$$\lim_{h\to+0} 2\tilde{\varphi}(\mu;\, x,\, h)=\gamma.$$

Then

$$\lim_{r\to 1-0}\left\{\tilde{u}(re^{ix})-\gamma\log(1-r)+\int_{1-r}^{\pi}\left(\tilde{\varphi}(\mu^*;\, x,\, t)-\frac{\gamma}{2}\right)\frac{t}{2\sin^2 t/2}\,dt\right\}$$
$$=u(0)\cot\frac{x}{2}-\gamma\log 2.$$

In particular $$\lim_{r\to 1-0}\{\tilde{u}(re^{ix})-\gamma\log(1-r)\}$$

exists if and only if the integral

$$\int_{0}^{\pi}\left(\tilde{\varphi}(\mu^*;\, x,\, t)-\frac{\gamma}{2}\right)\frac{t}{\sin^2 t/2}\,dt$$

converges.

Proof. If $\gamma=0$, Theorem 2.2 is in Zygmund ([4], p. 102). Putting $u_\gamma=\operatorname{Re} w_{00\gamma}$ we see that

$$\lim_{r\to 1-0}\{\tilde{u}_\gamma(re^{ix})-\gamma\log(1-r)\}=-\gamma\log\left|\sin\frac{x}{2}\right|-\gamma\log 2.$$

On the other hand, elementary calculations yield

$$\tilde{\varphi}(\mu_\gamma^*;\, x,\, t)=\begin{cases}\dfrac{\gamma}{2} & \text{if } 0<t<\min(x,\, 2\pi-x) \\[2mm] \dfrac{\gamma\pi}{2t} & \text{if } \min(x,\, 2\pi-x)<t\leqslant\pi\end{cases}$$

and thus, if $1-r<\min(x,\, 2\pi-x)$,

$$\int_{1-r}^{\pi}\left(\tilde{\varphi}(\mu_\gamma^*;\, x,\, t)-\frac{\gamma}{2}\right)\frac{t}{2\sin^2 t/2}\,dt=\frac{\gamma}{4}\int_x^{\pi}\frac{\pi-t}{\sin^2 t/2}\,dt=u_\gamma(0)\cot\frac{x}{2}+\gamma\log\left|\sin\frac{x}{2}\right|,$$

and it follows that the theorem is true for the special function u_γ.

The proof is now completed applying the theorem in the special case when $\gamma=0$ to the function $u-u_\gamma$.

Observing that

$$\int_{1-r}^{\pi} \frac{t}{\sin^2 t/2}\, dt = O(\log(1-r)) \text{ as } r \to 1-0$$

we have the following corollary.

Corollary 2.2. *If*

$$\lim_{h\to+0} 2\tilde{\varphi}(\mu;\, x,\, h) = \gamma,$$

then

$$\lim_{r\to 1-0} \frac{\tilde{u}(re^{ix})}{\log(1-r)} = \gamma.$$

Before studying the increase of $w^{(k)}$, let us again return to the auxiliary function $w_{\alpha\beta\gamma}$. It is easy to verify that

$$\lim_{r\to 1-0} (1-r)^k\, w_{\alpha\beta\gamma}^{(k)}(re^{ix}) = (k-1)!\ (\alpha - i\gamma)\, e^{-ikx}, \text{ for } k \geqslant 1.$$

However, this is true for any function $w \in \mathfrak{W}$, such that

$$\lim_{h\to+0} \{\pi\varphi(\mu;\, x,\, h) + \alpha \log h\} = \beta \tag{2.2}$$

and

$$\lim_{h\to+0} 2\tilde{\varphi}(\mu;\, x,\, h) = \gamma. \tag{2.3}$$

Theorem 2.3. *Let* $w \in \mathfrak{W}$ *and suppose that* (2.2) *and* (2.3) *hold. Then, if* k *is any natural number,*

$$\lim_{r\to 1-0} (1-r)^k\, w^{(k)}(re^{ix}) = (k-1)!\ (\alpha - i\gamma)\, e^{-ikx}$$

Proof. Since the theorem is true for $w = w_{\alpha\beta\gamma}$ we may assume that $\alpha=\beta=\gamma=0$. Derivation of (1.4) yields

$$(1-r)^k\, e^{ikx}\, w^{(k)}(re^{ix}) = \int_0^{2\pi} K_r(t-x)\, d\mu(t), \tag{2.4}$$

where

$$K_r(t) = 2\,k!\, \frac{(1-r)^k\, e^{it}}{(e^{it}-r)^{k+1}}$$

is a complex kernel with the following properties;

$$\overline{K_r(t)} = K_r(-t),$$

$$\lim_{r\to 1-0}\ \sup_{\delta\leqslant t\leqslant\pi} |t\, K_r'(-t)| = 0 \text{ for } 0 < \delta \leqslant \pi$$

and

$$\int_0^{\pi} |t\, K_r'(t)|\, dt = O(1) \text{ as } r \to 1-0.$$

The first two of these properties are trivial. Observe that, if $0 \leqslant t \leqslant \pi$, $0 < r < 1$, we have

$$|t K_r'(t)| \leqslant \pi r^{-\frac{1}{2}} k! \ (1 + (1+k) P_r(t)),$$

where P_r is the Poisson kernel. Hence the last property follows by integration.

Using the first property of K_r the integral in (2.4) may be written

$$\int_0^{2\pi} K_r(t-x)\, d\mu(t) = K_r(-x)\ w(0)$$

$$-\int_0^{\pi} \varphi(\mu^*;\ x,\ t)\, t \operatorname{Re} K_r'(t)\, dt - i \int_0^{\pi} \tilde{\varphi}(\mu^*;\ x,\ t)\, t \operatorname{Im} K_r'(t)\, dt,$$

whence, if $0 < \delta < \pi$,

$$\left| \int_0^{2\pi} K_r(t-x)\, d\mu(t) \right| \leqslant |w(0)|\ |K_r(x)| + C_\delta \int_0^{\pi} |t K_r'(t)|\, dt + \pi C_\pi \sup_{\delta \leqslant t \leqslant \pi} |t K_r'(t)|,$$

where

$$C_\theta = \sup_{0 \leqslant t \leqslant \theta} |\varphi(\mu^*;\ x,\ t)| + \sup_{0 \leqslant t \leqslant \theta} |\tilde{\varphi}(\mu^*;\ x,\ t)|.$$

Since $C_\delta \to 0$ as $\delta \to +0$ we have, by the properties of K_r,

$$\lim_{r \to 1-0} \int_0^{2\pi} K_r(t-x)\, d\mu(t) = 0,$$

establishing the theorem.

Remark. *If* (2.2) *and* (2.3) *are replaced by* $\pi\varphi(\mu;\ x,\ h) + \alpha \log h = O(1)$ *and* $\tilde{\varphi}(\mu;\ x,\ h) = O(1)$ *as* $h \to +0$, *we may conclude that* $w^{(k)}(re^{ix}) = O((1-r)^{-k})$ *as* $r \to 1-0$.

Writing $w = u + i\tilde{u}$ we have

$$e^{ikx} w^{(k)}(re^{ix}) = \frac{\partial^k u(re^{ix})}{\partial r^k} + i \frac{\partial^k \tilde{u}(re^{ix})}{\partial r^k}.$$

A closer examination of the proof of Theorem 2.3 shows that (2.2) alone implies

$$\lim_{r \to 1-0} \operatorname{Re} \{(1-r)^k e^{ikx} w^{(k)}(re^{ix})\} = (k-1)!\ \alpha,$$

while (2.3) implies

$$\lim_{r \to 1-0} \operatorname{Im} \{(1-r)^k e^{ikx} w^{(k)}(re^{ix})\} = -(k-1)!\ \gamma,$$

and thus Theorem 2.3 may be transformed into the following two theorems concerning harmonic functions in the class $\mathcal{U}$.

Theorem 2.4. *Let* $u \in \mathcal{U}$ *and suppose that* (2.2) *holds. Then, if* k *is any natural number,*

$$\lim_{r\to 1-0}(1-r)^k \frac{\partial^k u(re^{ix})}{\partial r^k} = (k-1)!\,\alpha.$$

Theorem 2.5. *Let $u \in \mathcal{U}$ and suppose that* (2.3) *holds. Then, if k is any natural number,*

$$\lim_{r\to 1-0}(1-r)^k \frac{\partial^k \tilde{u}(re^{ix})}{\partial r^k} = -(k-1)!\,\gamma.$$

The result of Theorem 2.5 in the case $k=1$ is in Zygmund ([4], p. 108).

We point out another consequence of the representation formula (2.4). Since

$$|K_r(t)| \leqslant \frac{2\,k!}{1+r} P_r(t),$$

we have, if $w = u + i\tilde{u}$.

$$(1-r)^k |w^{(k)}(re^{ix})| \leqslant \frac{2\,k!}{1+r}\int_0^{2\pi} P_r(t-x)\,d|\mu|(t) = \frac{2\,k!}{1+r}|u|(re^{ix}),$$

establishing the following theorem.

Theorem 2.6. *Let $w \in \mathcal{W}$. Then, if k is any natural number,*

$$(1-r)^k |w^{(k)}(re^{ix})| \leqslant \frac{2\,k!}{1+r}|u|(re^{ix}).$$

Theorem 2.6, for $k=1$, is in Zygmund ([4], p. 258) in the special case when μ is absolutely continuous.

3. Boundary behaviour of $f^{(n)}$

In this section we transfer the results of section 2 to the functions $f \in \mathfrak{F}$, defined in section 1. We denote by B, E and w the functions defined by (1.1) and (1.2). Unless otherwise stated $u = \operatorname{Re} w$ throughout this section.

Theorem 3.1. *Let $f \in \mathfrak{F}$ and suppose that*

$$\liminf_{h\to +0} \frac{\pi\varphi(\mu;\,x,\,h)}{-\log h} > n,$$

where n is a natural number. Then

$$f^{(k)}(e^{ix}) = \lim_{r\to 1-0} f^{(k)}(re^{ix}) = 0$$

for $0 \leqslant k \leqslant n$.

The proof of this theorem is based on the following lemma.

Lemma. *Let u be a non-negative, harmonic function defined in the unit disc and let C_z be a circle with center z, $|z|<1$, and radius $\beta(1-|z|)$, $0<\beta<1$. Then*

$$\frac{1-\beta}{1+\beta}u(z)\leqslant u(\zeta)\leqslant\frac{1+\beta}{1-\beta}u(z)$$

for every $\zeta\in C_z$.

Proof. If $\zeta\in C_z$, Harnack's inequalities applied to the function u restricted to a disc concentric to C_z and with radius $\gamma(1-|z|)$, where $\beta<\gamma<1$, yields

$$\frac{\gamma-\beta}{\gamma+\beta}u(z)\leqslant u(\zeta)\leqslant\frac{\gamma+\beta}{\gamma-\beta}u(z).$$

The lemma follows as γ tends to 1.

Proof of Theorem 3.1. Under the assumption of the theorem we have, by (2.1)

$$\liminf_{r\to 1-0}\frac{u(re^{ix})}{-\log(1-r)}>n$$

and thus there is an $\alpha>1$ such that

$$u(re^{ix})+\alpha n\log(1-r)\to+\infty \text{ as } r\to 1-0.$$

Put $$S_\theta=\{z;\ |z|<1,\ |z-e^{ix}|<1,\ |\arg(1-ze^{-ix})|\leqslant\theta\},$$

where $\theta=\arcsin\beta$ and $(1+\beta)/(1-\beta)=\alpha$.

Then $$\lim_{z\to e^{ix}}\frac{f(z)}{(z-e^{ix})^n}=0, \tag{3.1}$$

where the approach is uniform in S_θ. To prove this let T be the mapping of S_θ onto the segment $0<|\zeta|<1$, $\arg\zeta=x$, defined as follows. Given $z\in S_\theta$ let Tz be the point closest to e^{ix}, such that $\arg(Tz)=x$ and $|z-Tz|=\beta(1-|Tz|)$. Obviously $(1-\beta)(1-|Tz|)\leqslant|e^{ix}-z|$ and thus $Tz\to e^{ix}$ as $z\to e^{ix}$, where the approach is uniform in S_θ. According to our lemma we have

$$\frac{|f(z)|}{|z-e^{ix}|^n}\leqslant\frac{e^{-u(z)}}{|z-e^{ix}|^n}\leqslant\frac{\exp\{-(1-\beta)u(Tz)/(1+\beta)\}}{(1-\beta)^n(1-|Tz|)^n}$$
$$=(1-\beta)^{-n}\exp\{-\alpha^{-1}(u(Tz)+\alpha n\log(1-|Tz|))\}$$

from which (3.1) follows uniformly in S_θ.

We now take $z=re^{ix}$ and use Cauchy's integral formula

$$f^{(k)}(z)=\frac{k!}{2\pi i}\int_{C_z}\frac{f(\zeta)}{(\zeta-z)^{k+1}}d\zeta,$$

where C_z is the circle defined in the lemma. Hence we obtain

$$|f^{(k)}(z)| \leqslant k! \sup_{\zeta \in C_z} \frac{|f(\zeta)|}{|\zeta - z|^k} \leqslant k! \ (1+\beta^{-1})^k \sup_{\zeta \in C_z} \frac{|f(\zeta)|}{|\zeta - e^{ix}|^k}$$

and thus, by (3.1), we have $f^{(k)}(e^{ix}) = 0$ for $0 \leqslant k \leqslant n$.

We remark that the assumption of Theorem 3.1 is just a sufficient condition. To illustrate this put $f = B \cdot E$, where B is a Blaschke product with the following properties;

$$B(re^{ix}) = O(1-r) \quad \text{and} \quad B'(re^{ix}) = O(1) \quad \text{as} \quad r \to 1-0.$$

For instance the Blaschke product

$$\prod_{k=1}^{\infty} \frac{(1-k^{-2}) - ze^{-ix}}{1-(1-k^{-2})ze^{-ix}}$$

introduced by Frostman ([1], p. 109) has these properties. Since E is bounded, $E(e^{ix}) = 0$ implies $E'(re^{ix}) = o((1-r)^{-1})$ as $r \to 1-0$ and it follows that $f'(e^{ix}) = 0$ independent of the increase of $\varphi(\mu; x, h)$. In this case the behaviour of $f'(re^{ix})$ depends on the zeros of f in the neigbourhood of e^{ix}, but even if f has no zeros it may happen that (cf. (2.1))

$$\liminf_{h \to +0} \frac{\pi \varphi(\mu; x, h)}{-\log h} < n$$

while

$$\liminf_{r \to 1-0} \frac{u(re^{ix})}{-\log(1-r)} > n$$

and since the proof of Theorem 3.1 starts from this inequality, we still have $f^{(k)}(e^{ix}) = 0$ for $0 \leqslant k \leqslant n$. However, if

$$\limsup_{h \to +0} \frac{\pi \varphi(\mu; x, h)}{-\log h} < n$$

the conclusion of Theorem 3.1 is false, provided the zeros of f are not too close to the point e^{ix}.

Theorem 3.2. *Let* $f = B \cdot E \in \mathfrak{J}$. Suppose that $f^{(k)}(e^{ix}) = 0$ *for* $0 \leqslant k \leqslant n$ *and*

$$\limsup_{r \to 1-0} |B(re^{ix})| > 0.$$

Then

$$\limsup_{h \to +0} \{\pi \varphi(\mu; x, h) + n \log h\} = +\infty. \tag{3.2}$$

Proof. If $0 < r < r_0 < 1$ we have

$$|f^{(k-1)}(r_0 e^{ix}) - f^{(k-1)}(re^{ix})| \leqslant \int_r^{r_0} |f^{(k)}(\varrho e^{ix})| \, d\varrho$$

whence we obtain as $r_0 \to 1-0$

$$|f^{(k-1)}(re^{ix})| \leqslant \int_r^1 |f^{(k)}(\varrho e^{ix})|\, d\varrho$$

for $1 \leqslant k \leqslant n$. Repeated use of this inequality yields

$$n!\, |f(re^{ix})| \leqslant (1-r)^n \sup_{r \leqslant \varrho \leqslant 1} |f^{(n)}(\varrho e^{ix})|.$$

Hence, since $f^{(n)}(e^{ix}) = 0$,

$$\liminf_{r \to 1-0} |E(re^{ix})|/(1-r)^n = \liminf_{r \to 1-0} \exp\{-u(re^{ix}) - n \log (1-r)\} = 0$$

and thus, using Theorem 2.1, (3.2) follows.

Let us consider $f \in \mathfrak{J}$ such that

$$\pi\varphi(\mu;\, x, h) + n \log h = O(1) \text{ as } h \to +0. \tag{3.3}$$

It follows from Theorems 3.1 and 3.2 that

$$\limsup_{r \to 1-0} |f^{(n)}(re^{ix})| > 0,$$

provided $B(e^{ix}) \neq 0$. However, (3.3) does not imply that $f^{(n)}(re^{ix})$ is bounded. For, if g is any integrable function, such that $g(y) = 0$ if $0 \leqslant y < x$, $g(y) \to +\infty$ as $y \to x+0$ and u is the Poisson integral of the finite measure induced by

$$\mu(t) = \int_0^t g(y)\, dy,$$

it is easy to verify that

$$(1-r)\left|\frac{\partial u(re^{ix})}{\partial x}\right| \to +\infty \text{ as } r \to 1-0$$

and of course we can choose g so that (3.3) with $n=1$ holds. Then, if $w = u + i\tilde{u}$ and $E = \exp\{-w\}$, we have

$$r\,|E'(re^{ix})| \geqslant (1-r)\left|\frac{\partial u(re^{ix})}{\partial x}\right| \exp\{-u(re^{ix}) - \log (1-r)\}$$

and thus, by Theorem 2.1, $|E'(re^{ix})| \to +\infty$ as $r \to 1-0$. However, if we assume in addition to (3.3) that $\tilde{\varphi}(\mu;\, x, h) = O(1)$ as $h \to +0$ we may conclude that $f^{(n)}(re^{ix}) = O(1)$ as $r \to 1-0$.

Before we prove this statement let us consider a function E, given by (1.2). Since

$$E^{(n)}(z) = -\sum_{k=0}^{n-1} \binom{n-1}{k} w^{(n-k)}(z)\, E^{(k)}(z), \quad n > 1,$$

we may write

$$E^{(n)}(z) = Q_n(w'(z),\ w''(z), \ldots, w^{(n)}(z))\, E(z) \quad \text{for} \quad n \geqslant 0 \tag{3.4}$$

where Q_n are polynomials of degree n defined by

$$Q_0 = 1,\ Q_n(x_1, x_2, \ldots, x_n) = -\sum_{k=0}^{n-1} \binom{n-1}{k} x_{n-k}\, Q_k(x_1, x_2, \ldots, x_k).$$

The polynomials Q_n have the homogenity property

$$Q_n(\lambda x_1, \lambda^2 x_2, \ldots, \lambda^n x_n) = \lambda^n Q_n(x_1, x_2, \ldots, x_n),$$

where λ is any complex number. It is convenient to introduce another sequence of polynomials P_n connected with Q_n by

$$P_n(z) = Q_n(0!\, z, 1!\, z, \ldots, (n-1)!\, z).$$

It follows from the recurrence formula of the polynomials Q_n that the polynomials P_n are determined by

$$P_0(z) = 1,\ P_n(z) + (z + 1 - n)\, P_{n-1}(z) = 0$$

whence we see that

$$P_n(z) = (-1)^n \prod_{k=0}^{n-1} (z-k).$$

The homogenity property of Q_k may be used to rewrite (3.4) as

$$\begin{aligned}(1-r)^{k-\alpha} & E^{(k)}(re^{ix})\, e^{i\tilde{u}(re^{ix})} \\ &= Q_k((1-r)e^{ix} w'(re^{ix}), \ldots, (1-r)^k e^{ikx} w^{(k)}(re^{ix})) \cdot \frac{|E(re^{ix})|}{(1-r)^{\alpha}}\, e^{-ikx},\end{aligned} \tag{3.5}$$

where α is any real number. By Theorems 2.1 and 2.3, this identity has the following two consequences;

if $\qquad \pi\varphi(\mu;\, x, h) + \alpha \log h = O(1)$ *and* $\tilde{\varphi}(\mu;\, x, h) = O(1)$ *as* $h \to +0$,

then for $k \geqslant 0$

$$E^{(k)}(re^{ix}) = O((1-r)^{\alpha-k}) \ \textit{as}\ r \to 1-0, \tag{3.6}$$

if $\qquad \lim_{h\to+0} \{(\pi\varphi(\mu;\, x, h) + \alpha \log h\} = \beta$ *and* $\lim_{h\to+0} 2\tilde{\varphi}(\mu;\, x, h) = \gamma$,

then for $k \geqslant 0$

$$\lim_{r\to 1-0} (1-r)^{k-\alpha} E^{(k)}(re^{ix})\, e^{i\tilde{u}(re^{ix})} = P_k(\alpha - i\gamma)\, e^{\alpha-\beta-ikx}. \tag{3.7}$$

We are now able to prove the statement concerning the boundedness of $f^{(n)}$.

Theorem 3.3. *Let $f \in \mathfrak{J}$ and suppose that*

$$\pi\varphi(\mu;\, x,\, h) + n \log h = O(1) \quad \textit{and} \quad \tilde{\varphi}(\mu;\, x,\, h) = O(1) \quad \textit{as}\ h \to +0.$$

Then $f^{(n)}(re^{ix}) = O(1)$ *as* $r \to 1-0$.

Proof. Since $|B(z)| \leqslant 1$ implies $B^{(n-k)}(re^{ix}) = O((1-r)^{k-n})$ as $r \to 1-0$, for $0 \leqslant k \leqslant n$, we obtain from (3.6), with $\alpha = n$

$$E^{(k)}(re^{ix})\, B^{(n-k)}(re^{ix}) = O(1) \quad \text{as } r \to 1-0, \text{ for } 0 \leqslant k \leqslant n$$

and thus
$$f^{(n)}(re^{ix}) = O(1) \quad \text{as } r \to 1-0.$$

Theorem 3.4. *Let E be given by (1.2) and suppose that*

$$\lim_{h\to+0} \{\pi\varphi(\mu;\, x,\, h) + n \log h\} = \beta \quad \textit{and} \quad \lim_{h\to+0} 2\,\tilde{\varphi}(\mu;\, x,\, h) = \gamma.$$

Then $C(e^{ix}, E^{(n)}) \subset C$, *where*

$$C = \{z;\, |z| = |P_n(n - i\gamma)|\, e^{n-\beta}\}.$$

If $\gamma \neq 0$, the two sets $C(e^{ix}, E^{(n)})$ and C are equal. If $\gamma = 0$ the set $C(e^{ix}, E^{(n)})$ reduces to one point if and only if the integral

$$\int_0^\pi \tilde{\varphi}(\mu^*;x,t)\, \frac{t}{\sin^2 t/2}\, dt \tag{3.8}$$

converges.

Proof. If we put $k = \alpha = n$ in (3.7) we obtain

$$\lim_{r\to1-0} |E^{(n)}(re^{ix})| = |P_n(n - i\gamma)|\, e^{n-\beta} \neq 0$$

and thus $C(e^{ix}, E^{(n)}) \subset C$. If $\gamma \neq 0$, Corollary 2.2 shows that $|\tilde{u}(re^{ix})| \to +\infty$ as $r \to 1-0$ and thus $C(e^{ix}, E^{(n)}) = C$. If $\gamma = 0$, the conclusion of the theorem follows from Theorem 2.2.

Let $f = B \cdot E \in \mathfrak{J}$ and suppose that the associated measure satisfies the conditions of Theorem 3.3. Then the existence of $B(e^{ix})$ implies (cf. the proof of Theorem 3.3)

$$C(e^{ix} f^{(n)}) = B(e^{ix})\, C(e^{ix}, E^{(n)})$$

and thus we have the following corollary of Theorem 3.4.

Corollary. *Let $f \in \mathfrak{J}$ and suppose that $B(e^{ix})$ exists. If μ is smooth at the point x, the integral (3.8) converges and*

$$\lim_{h\to+0} \{\pi\varphi(\mu;\, x,\, h) + n \log h\} = \beta$$

then $f^{(n)}(e^{ix})$ *exists.*

Remark. *Under the assumptions of the corollary we have*

$$f^{(n)}(e^{ix}) = (-1)^n n!\ B(e^{ix})\, e^{n-\beta-i(\tilde{u}(e^{ix})+nx)}$$

The identity (3.5) may also be used to prove Theorem 3.1. Using the assumption of Theorem 3.1 it follows from Theorem 2.6 and this identity that

$$\lim_{r\to 1-0} (1-r)^{k-m} E^{(m)}(re^{ix}) = 0 \quad \text{for} \quad 0 \leqslant m \leqslant k \leqslant n$$

and thus arguing as in the proof of Theorem 3.3 we may conclude that $f^{(k)}(e^{ix}) = 0$ for $0 \leqslant k \leqslant n$. This method was used in [3] in the case $n = 1$. In both proofs of Theorem 3.1 we really use the fact that μ is non-decreasing, while in Theorems 3.2, 3.3 and 3.4 it is enough to suppose that μ is of bounded variation. Actually, if the analytic function f is "beschränktartig" Theorem 3.1 is false. To see this put

$$w(z) = -(1+\varepsilon)\log(1-ze^{-ix}) + i\varepsilon^{-1}(\exp\{-\varepsilon \log(1-ze^{-ix}\}-1).$$

If $0 < \varepsilon < 1$, this function belongs to $\mathfrak{W}$ and some simple calculations yield

$$\lim_{h\to +0} \{\pi\varphi(\mu;\, x,\, h) + (1+\varepsilon)\log h\} = 1+\varepsilon.$$

However, if $f = \exp\{-w\}$ it is easy to verify that $C(e^{ix}, f')$ is equal to the unit circle $|z| = 1$.

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