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On the propagation of analyticity of solutions of differential equations with constant coefficients

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1. Introduction

Let $P(D)$ be a partial differential operator with constant complex coefficients, let Ω be an open set in R^n, and write

$$\Omega_d = \{x;\ x \in \Omega,\ x_n > d\}.$$

If E and F are sets, we let $E \backslash F$ denote the set $E \cap \complement F$. By $C^\infty(\Omega)$ we denote the set of infinitely differentiable complex valued functions in Ω. The following theorem is due to John [5] and Malgrange [6] (see also Hörmander [4], Ch. III, VIII).

Theorem 1. *Let the distribution u in Ω satisfy the equation $P(D)u=f$, where $f \in C^\infty(\Omega_d)$, and assume that $u \in C^\infty(\Omega_d \backslash F)$, where F is a compact subset of Ω, and d is a real number. Then $u \in C^\infty(\Omega_d)$.*

The main purpose of this paper is to prove the analogous result with analyticity instead of infinite differentiability, i.e.

Theorem 2. *Assume in addition to the hypotheses of Theorem 1 that u is real analytic in $\Omega_d \backslash F$ and that f is real analytic in Ω_d. Then u is real analytic in Ω_d.*

We also prove a more general result involving classes of C^∞ functions. Such classes are defined as follows. If $L = \{L_k\}_{k=1}^\infty$ is an increasing sequence of positive numbers and Ω an open subset of R^n, we denote by $C^L(\Omega) = C^L$ the set of functions $f \in C^\infty(\Omega)$ such that to every compact set $F \subset \Omega$ there exists a constant C such that

$$|D^\alpha f(x)| \leqslant C^k L_k^k, \quad \text{if} \quad |\alpha| = k, x \in F, \quad k = 1, 2, \ldots .$$

Here D^α denotes $(\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \Sigma \alpha_j$. Note that $f \in C^L(\Omega)$, if $f \in C^L$ in some neighbourhood of every point in Ω. In fact this follows by applying the Borel-Lebesgue lemma. If $L_k = k$ for every k, the class $C^L(\Omega)$ is equal to the class $A(\Omega)$ of all real analytic functions on Ω. Here we shall only consider classes which contain $A(\Omega)$. Every such class can be defined by a sequence satisfying

$$L_k \geqslant k \quad (k = 1, 2, \ldots). \tag{1.1}$$

Definition. *We say that the increasing sequence L is affine invariant, if for any positive integers a and b there exists a constant C such that $C^{-1} L_k \leqslant L_{ak+b} \leqslant C L_k$ for every k.*

It is obvious that an increasing sequence L is affine invariant if and only if there exists a constant C such that

$$L_{2k} \leqslant CL_k \quad (k=1, 2, \ldots). \tag{1.2}$$

The property of translation invariance is defined similarly. Clearly an increasing sequence L is translation invariant if and only if there exists a constant C such that

$$L_{k+1} \leqslant CL_k \quad (k=1, 2, \ldots). \tag{1.3}$$

We shall prove the following theorem, which contains Theorem 2 as a special case.

Theorem 3. *Assume in addition to the hypotheses of Theorem 1 that* $u \in C^L(\Omega_d \setminus F)$ *and that* $f \in C^L(\Omega_d)$, *where* L *is affine invariant and satisfies (1.1). Then* $u \in C^L(\Omega_d)$.

The situation is much simpler, if the set F is contained in the interior of Ω_d. The corresponding analogue of Theorem 1 is well known. Using our terminology we can formulate that result as follows.

Theorem 4. *Let* L *be a translation invariant sequence satisfying (1.1). Let* u *be a distribution in* $\Omega \subset R^n$, *such that* $P(D)u \in C^L(\Omega)$ *and* $u \in C^L(\Omega \setminus F)$, *where* F *is a compact subset of* Ω. *Then* $u \in C^L(\Omega)$.

Some results related to Theorem 4 have been given by Agranovič [1].

The basic tool in our proof of Theorem 3 is an inequality (3.1) between the partial Fourier transforms of v and $P(D)v$ with weight functions which depend on one space variable. The inequality is valid for functions with compact support. The usual technique is to apply the inequality to the function $v = \chi u$, where u is the solution of the equation $P(D)u = f$, and χ is a function in $C_0^\infty(\Omega)$ which is equal to 1 in a certain set. ($C_0^\infty(\Omega)$ denotes the set of functions in $C^\infty(\Omega)$, whose supports are compact subsets of Ω.) Then one obtains an estimate for the derivatives of u in the set $F \cap \Omega_d$ in terms of bounds for derivatives of u in the set $\Omega_d \setminus F$ and of f in the set Ω_d together with bounds for derivatives of χ. However, by this method one cannot prove that u is analytic, since the derivatives of χ grow too fast when the order of differentiation tends to infinity. Therefore, following an idea of Ehrenpreis [3], we use a sequence χ_k of functions in $C_0^\infty(\Omega)$, whose derivatives of order k have the same order of magnitude as the derivatives of an analytic function (see Lemma 1).

In the special case when C^L is non-quasianalytic, i.e., contains functions with compact support, we could simplify the proof by applying the above-mentioned inequality to the function χu where χ is a fixed function in C^L with compact support. The general case would then follow from the special case by means of a theorem on the intersection of non-quasianalytic classes, which is given in Boman [2]. However, we have preferred to give here the more direct proof outlined above.

I wish to thank professor Lars Hörmander for introducing me to the problem considered here and for suggesting several improvements of the manuscript.

2. Preliminary lemmas

We first construct the sequence χ_k of functions mentioned in the introduction. Take a non-negative function $\varphi \in C_0^\infty(R^n)$ such that $\int \varphi(x)\,dx = 1$, and define for any positive a the function $\varphi_{(a)}$ by $\varphi_{(a)}(x) = a^n \varphi(ax)$. If K is a compact subset of $\Omega \subset R^n$ take $\Psi \in C_0^\infty(\Omega)$ such that $0 \leqslant \Psi \leqslant 1$ and $\Psi = 1$ in a neigbourhood of K, and put

$$\chi_k = \varphi_{(k)} * \ldots * \varphi_{(k)} * \Psi, \tag{2.1}$$

where the convolution contains the function $\varphi_{(k)}$ k times.

Lemma 1. *Let K be a compact subset of $\Omega \subset R^n$. If* supp $\varphi \subset \{x; |x| \leqslant \varepsilon\}$ *and ε is small enough, then the function χ_k defined by (2.1) is in $C_0^\infty(\Omega)$ and is equal to 1 in K for every k. Moreover, if $\int |\partial\varphi/\partial x_i| dx \leqslant C$ for each i and $C \geqslant 1$, then there exist constants $C_0 = 1, C_1, C_2, \ldots$ such that*

$$|D^\alpha \chi_k| \leqslant C_{|\alpha|-j}(Ck)^j, \quad 0 \leqslant j \leqslant \min(|\alpha|, k). \tag{2.2}$$

We shall later use the two special cases of (2.2) which are obtained by taking $j = |\alpha| \leqslant k$ and $j = 0$ respectively:

$$|D^\alpha \chi_k| \leqslant (Ck)^{|\alpha|} \quad (|\alpha| \leqslant k), \tag{2.2'}$$

$$|D^\alpha \chi_k| \leqslant C_{|\alpha|}. \tag{2.2''}$$

Proof of Lemma 1. Denote the convolution of k functions equal to $\varphi_{(k)}$ by Φ_k. It follows from the hypotheses that supp $\varphi_{(k)} \subset \{x; |x| \leqslant \varepsilon/k\}$ and hence that supp $\Phi_k \subset \{x; |x| \leqslant \varepsilon\}$. Also, $\int \Phi_k dx = 1$, since $\int \varphi dx = \int \varphi_{(k)} dx = 1$. This proves that $\chi_k \in C_0^\infty(\Omega)$ and that $\chi_k = 1$ in K, if ε is small enough. It remains to prove the estimate (2.2). Set $\alpha = \alpha' + \alpha''$, where $|\alpha'| = j$ and $|\alpha''| = |\alpha| - j$. Then $D^\alpha \chi_k = D^{\alpha'} \Phi_k * D^{\alpha''} \Psi$. Putting $C_m = \sup_{|\beta| \leqslant m} |D^\beta \Psi|$ we obtain

$$|D^\alpha \chi_k| \leqslant C_{|\alpha|-j} \int |D^{\alpha'} \Phi_k| dx. \tag{2.3}$$

Since $|\alpha'| = j \leqslant k$ we can compute $D^{\alpha'} \Phi_k$ by letting at most one derivative act on each factor in the convolution. By the assumption we have $\int \varphi dx = 1$, and also

$$\int |(\partial/\partial x_i) \varphi_{(k)}| dx = k \int |\partial\varphi/\partial x_i| dx \leqslant Ck \tag{2.4}$$

for any i. Since L^1 is a normed ring under convolution, we thus obtain (2.2) from (2.3) and (2.4).

The use of the inequality (2.2′) is illustrated by the following lemma.

Lemma 2. *Let L be an increasing sequence such that $L_k \geqslant k$ for every k. Assume that $u \in C^L(\Omega)$ and that the functions $\chi_k \in C^\infty(\Omega)$ satisfy (2.2′) in Ω. Then for any compact set $K \subset \Omega$ there exists a constant C such that*

$$|D^\alpha(\chi_k u)| \leqslant C^k L_k^k, \quad \text{if} \quad x \in K, \ |\alpha| \leqslant k, \quad k = 1, 2, \ldots.$$

Proof. Applying Leibniz' formula and (2.2′) we obtain

$$|D^\alpha (\chi_k u)| \leqslant 2^k \sup_{0 \leqslant j \leqslant k} C_1^j k^j C_2^{k-j+1} L_{k-j}^{k-j} \quad (x \in K, |\alpha| \leqslant k).$$

Since L is increasing and $L_k \geqslant k$, this gives with a sufficiently large C

$$|D^\alpha (\chi_k u)| \leqslant 2^k \sup_{0 \leqslant j \leqslant k} C_1^j C_2^{k-j+1} L_k^j L_k^{k-j} \leqslant C^k L_k^k.$$

Next we give two lemmas on the well-known relation between the bounds of the derivatives of a function and the rate of growth of its Fourier transform at infinity. (See e.g. Paley and Wiener [8].) We define the Fourier transform $\hat{w}$ of a function $w \in C_0^\infty(R^s)$ by $\hat{w}(\xi) = \int w(x) e^{-i<x,\xi>} dx$, where $<x, \xi> = x_1\xi_1 + \ldots + x_s\xi_s$.

Lemma 3. *Let k be a fixed positive integer and assume that $w \in C_0^\infty(R^s)$ satisfies*

$$\int |w| dx \leqslant C \quad \textit{and} \quad \int |D^\alpha w| dx \leqslant C^k L_k^k, \quad \textit{if} \quad |\alpha| = k.$$

Then there exist positive constants a and B which depend on C but are independent of k and w, such that

$$|\hat{w}(\xi)| (1 + (a|\xi|/L_k)^k) \leqslant B \quad (\xi \in R^s).$$

Proof. Since $|\xi|^2 \leqslant s \cdot \sup_j |\xi_j|^2$, we have

$$|\xi|^k \leqslant s^{k/2} \sup_j |\xi_j|^k \quad (k \geqslant 1). \tag{2.5}$$

Combining (2.5) with the formula

$$(i\xi_j)^k \hat{w} = \int D_j^k w(x) e^{-i<x,\xi>} dx$$

and with the assumption gives

$$|\hat{w}| \leqslant C \quad \text{and} \quad |\xi|^k |\hat{w}| \leqslant s^{k/2} C^k L_k^k.$$

With $a = (\sqrt{s} \cdot C)^{-1}$ this gives

$$|\hat{w}| (1 + (a|\xi|/L_k)^k) \leqslant C + 1.$$

Lemma 4. *Assume that $W \in L^\infty(R^s)$ and that*

$$|W(\xi)| (1 + (a|\xi|/L_k)^k) \leqslant B \quad (\xi \in R^s), \tag{2.6}$$

where k is an integer $\geqslant s+1$. Then there exists a constant C depending on a and B but independent of W and k, such that

$$|D^\alpha \hat{W}| \leqslant C^k L_k^k, \quad \textit{if} \quad |\alpha| \leqslant k - s - 1.$$

Proof. Since $D^\alpha \hat{W}(x) = \int (-i\xi)^\alpha W(\xi) e^{-i<x,\xi>} d\xi$,

we have by (2.6)

$$|D^\alpha \hat{W}(x)| \leqslant B \int |\xi|^{|\alpha|} (1 + (a|\xi|/L_k)^k)^{-1} d\xi \leqslant B (L_k/a)^{|\alpha|+s} \int |\xi|^{|\alpha|} (1 + |\xi|^k)^{-1} d\xi.$$

When $|\xi| \leqslant 1$ the integrand is bounded by 1, and when $|\xi| \geqslant 1$ it is bounded by $|\xi|^{|\alpha|-k} \leqslant |\xi|^{-1-s}$. This proves the statement.

Lemma 5. *Assume that the sequence L is affine invariant. Then there exists a constant $b>0$ such that*

$$(1+(br/L_k)^k)^2 \leqslant 2(1+(r/L_{2k})^{2k}), \quad \text{if} \quad r>0, \quad k=1,2,\ldots. \tag{2.7}$$

Proof. By Cauchy's inequality

$$(1+(br/L_k)^k)^2 \leqslant 2(1+(br/L_k)^{2k}).$$

Now (2.7) follows, if b is chosen so that $bL_{2k} \leqslant L_k$ for every k, which is possible by (1.2).

Lemma 6. *If L is translation invariant, then for any fixed s we have with a constant C_s*

$$L_{k+s}^{k+s} \leqslant C_s^k L_k^k \quad (k=1,2,\ldots).$$

Proof. Using (1.3) we obtain

$$L_{k+s}^{k+s} = L_{k+s}^k L_{k+s}^s \leqslant (C^s L_k)^k (C^{k+s-1} L_1)^s \leqslant C_s^k L_k^k \quad (k=1,2,\ldots).$$

It is obvious that the class C^L is closed with respect to differentiation, if L is translation invariant.

3. The basic inequality

We shall make use of the partial Fourier transform of functions $v \in C_0^\infty(R^n)$ with respect to the variables $x'=(x_1,\ldots,x_{n-1})$

$$\hat{v}(\xi', x_n) = \int v(x)\, e^{-i\langle x', \xi'\rangle}\, dx'.$$

Lemma 7. *Assume that the plane $x_n=0$ is non-characteristic with respect to $P(D)$, that Ω is a bounded subset of R^n and that γ is a positive continuous function defined for $\xi' \in R^{n-1}$. Then there exists a constant C, which is independent of v and γ, such that*

$$\sup_{\xi', x_n} |\hat{v}(\xi', x_n)|\,(\gamma(\xi'))^{x_n} \leqslant C \sup_{\xi'} \int |P(i\xi', D_n)\hat{v}(\xi', x_n)|\,(\gamma(\xi'))^{x_n}\, dx_n, \; v \in C_0^\infty(\Omega). \tag{3.1}$$

Proof. Let Q denote an arbitrary polynomial in one variable with leading coefficient 1. It is known (see e.g. Nirenberg [7]) that the following inequality holds for all $w \in C_0^\infty(R^1)$ which vanish outside a fixed inverval $-T \leqslant t \leqslant T$

$$\sup_t |w(t)| \leqslant C \int |Q(D)\, w(t)|\, dt. \tag{3.2}$$

Here C depends on the number T and the degree of Q, but is independent of the coefficients for the lower order terms of Q and the function w. In fact it is easy to prove (3.2) if one first reduces the general case to the special case when $Q(D) = D+a$, where a is a complex number. Applying (3.2) to the function $w_1(t)=w(t)e^{bt}$ and the polynomial $Q_1(\tau)=Q(\tau-b)$, where b is a constant, one obtains with the same constant C the inequality

$$\sup_t |e^{bt} w| \leqslant C \int |e^{bt} Q(D)\, w|\, dt, \; w \in C_0^\infty(-T, T). \tag{3.3}$$

Now we can apply (3.3) to the function $w(x_n)=\hat{v}(\xi', x_n)$ and the polynomial $Q(\xi_n)=P(i\xi', \xi_n)$, since the leading coefficient of $Q(\xi_n)$ is different from zero and independent of ξ', if the plane $x_n=0$ is non-characteristic with respect to $P(D)$. Choosing $b=\log \gamma(\xi')$ and taking supremum with respect to ξ' gives (3.1).

In order to illustrate the technique used in the proof of Theorem 3 we now give a proof of Theorem 4.

4. Estimates for the derivatives of the solution

Proof of Theorem 4. Take bounded open sets ω and ω' such that $F\subset\omega\subset\bar{\omega}\subset\omega\subset\bar{\omega}'\subset\Omega$. By Lemma 1 we can take functions χ_k of the form (2.1) such that $\chi_k\in C_0^\infty(\omega')$, $\chi_k=1$ in ω, and χ_k satisfies (2.2). Then $P(D)(\chi_k u)\in C_0^\infty(\omega')$, if the distribution u satisfies the assumptions of the theorem. Moreover, with a constant C independent of k we have

$$|D^\alpha P(D)(\chi_k u)|\leqslant C^k L_k^k, \quad \text{if} \quad |\alpha|+m\leqslant k, \tag{4.1}$$

where m is the order of $P(D)$. In fact, $\chi_k=1$ in ω, so that $P(D)(\chi_k u)$ is there equal to the function $P(D)u\in C^L$. Moreover, in an arbitrary compact set $K\subset\Omega\backslash F$, (4.1) must hold by virtue of Lemma 2 and the fact that $u\in C^L(\Omega\backslash F)$ by assumption. This shows that (4.1) holds for every x, since we can take K such that $K\cup\omega\supset\omega'$. Now, let E be a fundamental solution of $P(D)$, which is a distribution of order p. Since $D^\alpha(\chi_k u)=E*D^\alpha P(D)(\chi_k u)$ for any α, we then obtain

$$|D^\alpha(\chi_k u)|\leqslant C\cdot\sup_{|\beta|\leqslant k-m}|D^\beta P(D)(\chi_k u)|\leqslant C^k L_k^k, \quad \text{if} \quad |\alpha|+p+m\leqslant k.$$

Since $\chi_k=1$ in ω and L is translation invariant (Lemma 6), this gives with $|\alpha|=j$

$$|D^\alpha u|\leqslant C^{j+p+m}(L_{j+p+m})^{j+p+m}\leqslant C_1^j L_j^j\,(x\in\omega;\ j=1,2,\dots),$$

which proves that $u\in C^L(\omega)$ and hence that $u\in C^L(\Omega)$.

Proof of Theorem 3. Let $\bar{d}$ be the infimum of all $\delta>d$ such that $u\in C^L(\Omega_\delta)$. Then it is clear that $u\in C^L(\Omega_{\bar{d}})$, so that what we have to prove is that $\bar{d}=d$. To do so we shall assume that $\bar{d}>d$ and prove that $u\in C^L(\Omega_\delta)$ when $\delta=(d+3\bar{d})/4$ in contradiction with the definition of $\bar{d}$. In the first step we shall prove this using the additional assumption that the plane $x_n=0$ is non-characteristic with respect to $P(D)$.

Let ω and ω' be two bounded open sets such that

$$\{x; x\in F, (3d+\bar{d})/4\leqslant x_n\leqslant\bar{d}\}\subset\omega\subset\bar{\omega}\subset\omega'\subset\bar{\omega}'\subset\{x; x\in\Omega, d<x_n<(5\bar{d}-d)/4\}$$

(see Fig. 1). Applying Lemma 1 we construct functions $\chi_k\in C_0^\infty(\omega')$, such that $\chi_k=1$ in ω and the derivatives of χ_k satisfy the estimate (2.2). The main step of the proof will be applying the inequality (3.1) with suitable weight functions γ to the functions $v_k=\chi_{k+m}u$, where u is the solution of the equation $P(D)u=f$ which we are studying, and m is the order of $P(D)$. Note that Theorem 1 shows that $u\in C^\infty(\Omega_d)$, so that $v_k\in C_0^\infty(\omega')$ for every k. Using the assumptions that $u\in C^L(\Omega_d\backslash F)$ and that $u\in C^L(\Omega_{\bar{d}})$ we shall first prove that v_k satisfies the estimates

$$|D^\alpha P(D)v_k|\leqslant C_1^k L_k^k, \quad \text{if} \quad x_n>(3d+\bar{d})/4,\ |\alpha|=k, \quad k=1,2,\dots, \tag{4.2}$$

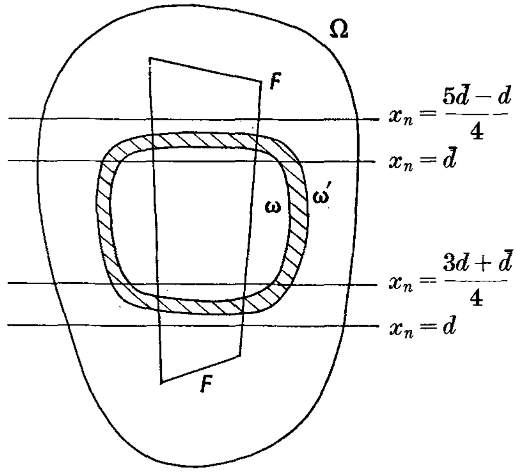

Fig. 1.

$$|P(D)v_k| \leqslant C_2, \quad \text{if} \quad x \in R^n, \quad k = 1, 2, \ldots. \tag{4.3}$$

That (4.3) holds follows immediately from (2.2″) and Leibniz' formula. We now prove (4.2). Take a compact set $K \subset \Omega_{\bar{d}} \cup (\Omega_d \setminus F)$ such that $K \cup \omega \supset \omega'_{(3d+\bar{d})/4}$. Since v_k vanishes outside ω', it is clearly enough to prove (4.2) for $x \in K \cup \omega$. However, for $x \in K$, (4.2) follows from Lemma 2, Lemma 6 and the assumptions that $u \in C^L(\Omega_d \setminus F)$ and that $u \in C^L(\Omega_{\bar{d}})$. On the other hand, (4.2) is trivial for $x \in \omega$, since $P(D)v_k$ is there equal to a fixed function $f \in C^L(\Omega_d)$.

From (4.3), (4.2) applied to derivatives with respect to $x_1, \ldots, x_{n-1}$, and from Lemma 3 we can now infer that there exist positive constants a and B which are independent of k such that

$$|P(i\xi', D_n)\hat{v}_k(\xi', x_n)|(1 + (a|\xi'|/L_k)^k) \leqslant B,$$

if $\qquad x_n > (3d + \bar{d})/4, \ \xi' \in R^{n-1}, \quad k = 1, 2, \ldots.$

Since L is affine invariant we can apply Lemma 5 and obtain with a new constant $a > 0$

$$|P(i\xi', D_n)\hat{v}_{2k}(\xi', x_n)|(1 + (a|\xi'|/L_k)^k)^2 \leqslant 2B,$$

if $\qquad x_n > (3d + \bar{d})/4, \ \xi' \in R^{n-1}, \quad k = 1, 2, \ldots. \qquad (4.4)$

From (4.3) we obtain

$$|P(i\xi', D_n)\hat{v}_k(\xi', x_n)| \leqslant C_3, \quad \text{if} \quad x_n \in R^1, \ \xi' \in R^{n-1}, \quad k = 1, 2, \ldots. \tag{4.5}$$

Put $h(x_n) = 2(x_n - (3d + \bar{d})/4)/(\bar{d} - d)$. Then $h(x_n) \leqslant 2$ when $x \in \omega'$. Applying (4.5) when $x_n < (3d + \bar{d})/4$ and (4.4) when $x_n > (3d + \bar{d})/4$ we obtain

$$\int |P(i\xi', D_n)\hat{v}_{2k}(\xi', x_n)|(1 + (a|\xi'|/L_k)^k)^{h(x_n)} dx_n \leqslant (\bar{d} - d)(C_3 + 2B) = B_1,$$

if $\qquad \xi' \in R^{n-1}, \quad k = 1, 2, \ldots.$

Using our provisional assumption that the plane $x_n=0$ is non-characteristic we can apply Lemma 7 (after a translation in the x_n coordinate) and infer that

$$|\hat{v}_{2k}(\xi', x_n)|(1+(a|\xi'|/L_k)^k)^{h(x_n)} \leqslant CB_1, \quad \text{if} \quad (\xi', x_n)\in R^n, \quad k=1, 2, \ldots.$$

Since $h(x_n)>1$ when $x_n>(d+3\bar{d})/4$ this implies that

$$|\hat{v}_{2k}(\xi', x_n)|(1+(a|\xi'|/L_k)^k) \leqslant CB_1, \quad \text{if} \quad x_n>(d+3\bar{d})/4, \quad \xi'\in R^{n-1}, \quad k=1, \ 2, \ldots.$$

By Lemma 4 and Fourier's inversion formula we obtain with a new C

$$|D'^{\alpha}v_{2k}| \leqslant C^k L_k^k, \quad \text{if} \quad x_n>(d+3\bar{d})/4, \ |\alpha| \leqslant k-n.$$

Here D'^{α} denotes an arbitrary derivative with respect to $x'=(x_1,\ldots, x_{n-1})$. Since $\chi_k=1$ in ω and L is translation invariant, we now obtain with still another C

$$|D'^{\alpha}u| \leqslant C^k L_k^k, \quad \text{if} \quad x\in\omega, \ x_n>(d+3\bar{d})/4, \ |\alpha| \leqslant k, \quad k=1, 2, \ldots. \tag{4.6}$$

In order to estimate arbitrary derivatives of u including derivatives with respect to x_n we shall again make use of the assumption that $x_n=0$ is a non-characteristic plane.

Lemma 8. *Assume that the plane $x_n=0$ is non-characteristic with respect to $P(D)$ and that $P(D)u\in C^L(\Omega)$. Assume further that $u\in C^{\infty}(\Omega)$ and that for every compact set $F\subset\Omega$ there exists a constant C, such that, if m denotes the order of $P(D)$,*

$$|D^{\alpha}u| \leqslant C^k L_k^k, \quad \textit{when} \quad x\in F, \ |\alpha| \leqslant k, \ \alpha_n\leqslant m-1, \quad k=1, 2, \ldots. \tag{4.7}$$

Then $u\in C^L(\Omega)$.

Proof. Since the plane $x_n=0$ is non-characteristic, the equation $P(D)u=f$ can be written

$$D_n^m u=\Sigma a_{\alpha} D^{\alpha}u+f, \tag{4.8}$$

where $\alpha_n<m$ and $|\alpha|\leqslant m$ in every term in the right-hand side. Take an arbitrary compact set $F\subset\Omega$, choose C_1 such that $|D^{\alpha}f| \leqslant C_1^k L_k^k$ when $x\in F$ and $|\alpha| \leqslant k$, and take $B=1+\Sigma|a_{\alpha}|$. Differentiating (4.8) with respect to the x' variables and applying (4.7) then gives

$$|D^{\alpha}u| \leqslant (BC^k+C_1^k)L_k^k, \text{ if } |\alpha| \leqslant k, \ \alpha_n\leqslant m. \tag{4.9}$$

Again differentiating (4.8) and applying (4.9) gives

$$|D^{\alpha}u| \leqslant (B^2C^k+BC_1^k+C_1^k)L_k^k, \quad \text{if} \quad |\alpha| \leqslant k, \ \alpha_n\leqslant m+1.$$

Continuing this procedure we arrive at the estimate

$$|D^{\alpha}u| \leqslant (B^kC^k+C_1^k(B^k-1)/(B-1))L_k^k \leqslant C_2^k L_k^k, \quad \text{if} \quad x\in F, \quad |\alpha| \leqslant k, \quad k=1, 2,\ldots,$$

where C_2 is a new constant. This proves the lemma.

End of proof of Theorem 3. We first prove the assertion of the theorem using the additional assumption that the plane $x_n=0$ is non-characteristic. Since the sequence L is translation invariant, the class C^L is closed with respect to differentiation. Then it is clear that the functions $D_n^j u$ satisfy the assumptions of Theorem 3 for arbitrary j. Hence formula (4.6) must be valid with $D_n^j u$ instead of u and possibly a new C. Thus the assumptions of Lemma 8 are fulfilled, and we conclude that $u \in C^L(\Omega_{(d+3\bar{d})/4})$. In view of the definition of $\bar{d}$ this contradicts the assumption that $\bar{d} > d$. This proves Theorem 3 in the special case when the plane $x_n = 0$ is non-characteristic.

Finally, if the plane $x_n = 0$ is characteristic, there are non-characteristic planes forming arbitrarily small angles with the plane $x_n = 0$. Applying the result just proved to regions bounded by such planes instead of the plane $x_n = 0$ we obtain the same result even if $x_n = 0$ is characteristic. Thus the proof of Theorem 3 is complete.

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