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Non-associative normed algebras and Hurwitz' problem

By LARS INGELSTAM

1. Introduction

We will deal here with algebras over the real or complex numbers. All algebras are supposed to have identity element (denoted e) but are not assumed associative or finite-dimensional in general. A *norm* is a real-valued function $x \to \|x\|$ on the algebra such that $\|\alpha x\| = |\alpha| \cdot \|x\|$ for any scalar α, $\|x+y\| \leq \|x\| + \|y\|$ for arbitrary x and y and $\|x\| > 0$ for all $x \neq 0$. We further consider the following conditions on the norm:

(i) A positive definite inner product (x, y) is defined on the algebra, so that $\|x\|^2 = (x, x)$.

(ii) $\|xy\| = \|x\| \cdot \|y\|$ for all x, y.

(iii) $\|e\| = 1$ and $\|xy\| \leq \|x\| \cdot \|y\|$.

The term *normed algebra* is usually reserved for algebras (not necessarily with identity) with norm satisfying (iii). A normed algebra satisfying (i) and (iii) will be called a *prehilbert algebra with identity* and one satisfying (ii) an *absolute-valued algebra*.

The classical result by A. Hurwitz [3] is that a finite-dimensional real normed algebra that satisfies (i) and (ii) must be isomorphic to the real numbers (R), the complex numbers (C), the quaternions (Q) or the Cayley numbers (D).

More recent results that can be regarded as generalizations of Hurwitz' theorem have been mainly along two lines:

I. Other scalar fields than the reals and no restriction on the dimension. N. Jacobsson [6] has obtained a complete analogue of Hurwitz' result for arbitrary fields of characteristic $\neq 2$. Largely the same result was earlier obtained by I. Kaplansky [7].

II. No inner product is assumed, i.e. the algebras considered are real absolute-valued. A. A. Albert [1, Theorem 2] has proved that an algebraic (see below) such algebra is isomorphic to R, C, Q or D. A related result is due to F. B. Wright [11]: An absolute-valued division algebra is isomorphic to R, C, Q or D.

We recall some definitions for (non-associative) algebras. Let A denote an algebra and A_x the subalgebra generated by the identity and an element x. A is called

quadratic if A_x is a field of order 2 over the ground field for all x except scalar multiples of e,
algebraic if A_x has finite dimension,
alternative if $x(xy)=(xx)y$ and $(yx)x=y(xx)$ for all x, y in A,
power associative if A_x is associative and commutative for all x. If A is alternative or quadratic it is also power associative.

Our first result will go in the direction II. We prove (Theorem 2.2) that a power associative real normed algebra satisfying (ii) must be isomorphic to R, C, Q or D. The second main result in this paper generalizes Hurwitz' theorem in another direction. We keep condition (i), replace (ii) by the weaker (iii) and are then able to prove that an alternative such algebra is isomorphic to R, C, Q or D (Theorem 3.1). (Since alternativity in fact follows almost immediately from (i) and (ii), see [6, p. 58] or [7], we are justified in calling our results generalizations of Hurwitz' theorem.) Both proofs use Banach algebra methods and in the latter we rely on a previously published result [4] by the author on the associative case. Some results by E. Strzelecki, announced in [10] without proofs, deal with situations similar to ours.

For algebras over the complex field we get, under the same assumptions, that the only possible algebra is the complex field itself (Corollaries 2.3 and 3.2). In the last section we construct a class of algebras, generalizing Q and D. In this class we can find examples that show the natural limitations of the kind of results obtained or cited in the paper.

An algebra A over the real numbers is said to be of *complex type* [5] if the scalar multiplication can be extended to the complex numbers so that A becomes a complex algebra.

Lemma 1.1. *Of the real algebras R, C, Q and D only C is of complex type.*

Proof. It is easy to see that an algebra with identity e is of complex type only if there exists an element j in its center satisfying $j^2=-e$, see [5, p. 29]. Since the centers of R, Q and D are all isomorphic to R the conclusion follows.

2. Absolute-valued algebras

We begin by quoting a special case of the announced theorem.

Lemma 2.1. *A commutative and associative real absolute-valued algebra with identity is isomorphic to R or C.*

Proof. An absolute-valued algebra cannot have any non-zero topological divisors of 0. Hence a result by I. Kaplansky [8, Theorem 3.1] applies and gives the conclusion.

Theorem 2.2. *A power associative real absolute-valued algebra with identity is isomorphic to R, C, Q or D.*

Proof. We first show that the algebra, which we call A, is quadratic. Let x be an element which is not a scalar multiple of e and B the algebra generated by x and e. According to the assumption B is associative and commutative.

From Lemma 2.1 it now follows that B is isomorphic to C (since it is at least two-dimensional). Hence A is quadratic, in particular algebraic of order 2. But an algebraic absolute-valued algebra with identity is isomorphic to R, C, Q or D; a result by A. A. Albert [1, Theorem 2].

The following corollary for complex scalars generalizes a well-known result for commutative associative Banach algebras [9, p. 129].

Corollary 2.3. *A power associative complex absolute-valued algebra with identity is isomorphic to the complex numbers.*

Proof. If scalar multiplication is restricted to real numbers the algebra, A, satisfies the conditions of Theorem 2.2. and A is isomorphic to R, C, Q or D. But A is also of complex type and Lemma 1.1 tells that A is isomorphic to C.

3. Prehilbert algebras with identity

We recall that a prehilbert algebra with identity is an algebra with identity (e) on which is defined a positive definite bilinear (in the complex case sesquilinear) form (x, y), such that the norm $\|x\| = (x, x)^{\frac{1}{2}}$ satisfies

$$\|xy\| \leqslant \|x\| \cdot \|y\|,$$

$$\|e\| = 1.$$

Theorem 3.1. *An alternative real prehilbert algebra with identity is isomorphic to R, C, Q or D.*

Proof. We first prove that the algebra, A, is quadratic. Let x be an element which is not a scalar multiple of e. The algebra spanned by x and e is an associative commutative prehilbert algebra with identity and dimension $\geqslant 2$. It follows from [4, Theorem 2 and Remark] that it is isomorphic to C. Hence A is quadratic and also [1, Theorem 1] finite-dimensional, since it is alternative.

For two given elements x and y we study the algebra A_0, generated by e, x and y. We distinguish two cases:

1. The set $\{e, x, y\}$ is linearly dependent. Then A_0 is generated by e and one element and is isomorphic to R or C.

2. The set $\{e, x, y\}$ is linearly independent. Since A is quadratic, x and y satisfy equations

$$(x - \alpha e)^2 + \gamma^2 e = 0,$$

$$(y - \beta e) \ \ + \delta^2 e = 0,$$

with α, β, γ, δ scalars, γ and $\delta \neq 0$. With $a = \gamma^{-1}(x - \alpha e)$ and $b = \delta^{-1}(y - \beta e)$ we have $a^2 = -e$ and $b^2 = -e$. If we further put $u = a + b$, $v = a - b$ we get $uv + vu = 0$. But

$$ab + ba = u^2 - a^2 - b^2 = u^2 - 2e = \varkappa u + \lambda e,$$

$$ab + ba = -(v^2 - a^2 - b^2) = -v^2 + 2e = \mu v + \nu e$$

for scalars $\varkappa, \gamma, \mu, \nu$. But then $\nu=\gamma$, and $\varkappa=\mu=0$ because of the linear independence and u^2, v^2 are (negative) multiples of e. Adjusting by scalars we get elements i, j such that

$$i^2=j^2=-e, \quad ij+ji=0$$

and A_0 is generated by $\{e, i, j\}$. But since A is also assumed alternative the elements e, i, j, $k=ij$ satisfy the defining relations for the four basis elements of the quaternion algebra. Hence A_0 is isomorphic to Q.

Now we have seen that the subalgebra generated by e and two elements x and y is isomorphic to R, C or Q. This subalgebra is of course a (pre-)hilbert algebra with identity. In [4], however, it was observed that the "usual" norms for R, C and Q are the only ones making them hilbert algebras with identity. Since these norms all satisfy (ii), the given norm, restricted to A_0, must also satisfy (ii). But this simply means that for any pair x, y

$$\|xy\|=\|x\|\cdot\|y\|.$$

The result now follows from Theorem 2.2 (or already from Hurwitz' original theorem [3]).

The corollary for complex scalars is a slight generalization of a result, implicit in the article [2] by H. F. Bohnenblust and S. Karlin as pointed out in [4, Theorem 1].

Corollary 3.2. *An alternative complex prehilbert algebra with identity is isomorphic to the complex numbers.*

Proof. If we restrict scalar multiplication to the real numbers and define a real inner product as $\langle x, y\rangle=\mathrm{Re}(x, y)$, the norm is unchanged and the algebra satisfies the assumptions of Theorem 3.1. Then it is isomorphic to R, C, Q or D, but since it is also of complex type it must be isomorphic to C (Lemma 1.1).

Remark. The assumption in Theorem 3.1 and [4, Theorem 2] can be replaced by: A is a normed algebra with identity e whose norm satisfies (iii) and is such that the unit sphere has a unique hyperplane of support at e (i.e. e is *regular* in the sense of E. Strzelecki [10]). In the reasoning the condition $(e, x)=0$ should then be replaced by $x\in H$, where $e+H$ is the hyperplane of support to the unit sphere at e.

4. A class of algebras

In this section we construct a certain class of algebras, illustrating the natural limitations of some of the results obtained.

Let Λ be a non-void set and $\{i_\lambda\}_{\lambda\in\Lambda}$ a set of symbols, indexed with Λ. Let A be the vector space generated by $\{i_\lambda\}_{\lambda\in\Lambda}$. We will define a multiplication on A, through

$$i_\lambda\cdot i_\mu=i_{\lambda\mu},$$

where $i_{\lambda\mu}$ is a doubly-indexed set in A. A distinguished element in Λ is called 0. $i_{\lambda\mu}$ shall have the properties

$$i_{\lambda 0} = i_{0\lambda} = i_\lambda \quad \text{for all } \lambda,$$

$$i_{\lambda\mu} = -i_{\mu\lambda} \quad \text{if } 0 \neq \lambda \neq \mu \neq 0,$$

and

$$i_{\lambda\lambda} = -i_0 \quad \text{if } \lambda \neq 0.$$

Then $i_0 = e$ is an identity for A. If we let

$$x = \alpha_0 e + \sum_{\lambda \neq 0} \alpha_\lambda i_\lambda, \quad y = \beta_0 e + \sum_{\lambda \neq 0} \beta_\lambda i_\lambda,$$

we have (all $\sum$'s in the sequel will be finite sums taken over all of Λ except 0)

$$xy = (\alpha_0\beta_0 - \sum \alpha_\lambda\beta_\lambda)\, e + \sum (\alpha_0\beta_\lambda + \beta_0\alpha_\lambda)\, i_\lambda + \tfrac{1}{2} \sum_{\lambda \neq \mu}\sum (\alpha_\lambda\beta_\mu - \alpha_\mu\beta_\lambda)\, i_{\lambda\mu}.$$

We also define the inner product

$$(x, y) = \alpha_0\beta_0 + \sum \alpha_\lambda\beta_\lambda$$

and the corresponding norm

$$\|x\|^2 = \alpha_0^2 + \sum \alpha_\lambda^2.$$

Proposition 4.1. *A is a quadratic (hence power associative) algebra with identity such that every non-zero element of A has a two-sided inverse (A is an "almost division algebra").*

Proof. Take $x = \alpha_0 e + \sum \alpha_\lambda i_\lambda$, $\alpha_\lambda \neq 0$ for some $\lambda \neq 0$. Then

$$(x - \alpha_0 e)^2 = -(\sum \alpha_\lambda^2)\, e$$

and the algebra generated by x and e is isomorphic to C. Define the conjugate $\bar{x}$ of x

$$\bar{x} = \alpha_0 e - \sum \alpha_\lambda i_\lambda.$$

Then

$$\bar{x} \cdot x = x \cdot \bar{x} = \|x\|^2 e$$

so that

$$\|x\|^{-2}\bar{x}$$

is two-sided inverse of x.

Hence we know the existence of quadratic, power associative algebras of any (finite or infinite) dimension. This, together with the result [1, Theorem 1] by Albert that alternative quadratic algebras over any field is of finite dimension 1, 2, 4 or 8, gives a fairly complete idea of what can be said about quadratic algebras in general.

Proposition 4.2. *If $i_{\lambda\mu} = 0$ for $0 \neq \lambda \neq \mu \neq 0$ A is a prehilbert algebra with identity.*

Proof. It only remains to prove $\|xy\| \leqslant \|x\| \cdot \|y\|$.

$$\|xy\|^2 = \|(\alpha_0\beta_0 - \sum \alpha_\lambda\beta_\lambda) e + \sum (\alpha_0\beta_\lambda + \beta_0\alpha_\lambda) i_\lambda\|^2$$

$$= \alpha_0^2\beta_0^2 + (\sum \alpha_\lambda\beta_\lambda)^2 - 2\alpha_0\beta_0 \sum \alpha_\lambda\beta_\lambda + \beta_0^2 \sum \alpha_\lambda^2 + \alpha_0^2 \sum \beta_\lambda^2 + 2\alpha_0\beta_0 \sum \alpha_\lambda\beta_\lambda$$

$$= (\alpha_0^2 + \sum \alpha_\lambda^2)(\beta_0^2 + \sum \beta_\lambda^2) - [\sum \alpha_\lambda^2 \sum \beta_\lambda^2 - (\sum \alpha_\lambda\beta_\lambda)^2]$$

$$\leqslant \|x\|^2 \cdot \|y\|^2 \quad \text{(using Schwartz' inequality).}$$

Thus we have constructed power associative prehilbert algebras with identity of any dimension. In particular, Theorem 3.1 would not be true if "alternative" was weakened to merely "power associative".

The A of Proposition 4.2 is also an example of an "almost division algebra" that is not a division algebra, i.e. such that the functions $x \to ax$ and $x \to xa$ map the algebra onto itself for every $a \neq 0$. For instance $x \to i_\lambda x$, $\lambda \neq 0$, maps A onto the subspace spanned by e and i_λ and thus A is not a division algebra if it has more than two dimensions.

Another special case of some interest is when the product of two basis elements is plus or minus a new basis element (e.g. R, C, Q and D are of this type). Then we have

$$i_\lambda \cdot i_\mu = i_{\lambda\mu} = s(\lambda, \mu)\, i_{f(\lambda, \mu)},$$

where f and s are functions

$$f\colon\ \Lambda \times \Lambda \to \Lambda,$$

$$s\colon\ \Lambda \times \Lambda \to \{-1, +1\}.$$

For $\lambda, \mu \neq 0$ f and s must satisfy

$$f(\lambda, \mu) = f(\mu, \lambda), \quad s(\lambda, \mu) = -s(\mu, \lambda) \ \text{ if } \ \lambda \neq \mu$$

and

$$f(\lambda, \lambda) = 0, \quad s(\lambda, \lambda) = -1.$$

An element $u = \sum \alpha_\lambda i_\lambda$ is called a *pure vector*.

Proposition 4.3. *If A has the properties mentioned above, and moreover f, restricted to the set where $0 \neq \lambda \neq \mu \neq 0$ and $s(\lambda, \mu) = +1$, is injective, then $\|uv\| = \|u\| \cdot \|v\|$ for all pure vectors u, v. It follows that multiplication is continuous in the normed topology and $\|xy\| \leqslant 2\|x\| \cdot \|y\|$.*

Proof. If $u = \sum \alpha_\lambda i_\lambda$, $v = \sum \beta_\lambda i_\lambda$ we get

$$uv = -(\sum \alpha_\lambda\beta_\lambda) e + \sum_{\lambda \neq \mu}\sum \alpha_\lambda\beta_\mu i_{\lambda\mu} = -(\sum \alpha_\lambda\beta_\lambda) e + \sum_{s(\lambda,\mu)=+1}\sum (\alpha_\lambda\beta_\mu - \alpha_\mu\beta_\lambda)\, i_{f(\lambda,\mu)}.$$

In this expression all the i-elements are different, and

$$\begin{aligned}\|uv\|^2 &= (\sum \alpha_\lambda\beta_\lambda)^2 + \sum_{s(\lambda,\mu)=+1}\sum (\alpha_\lambda\beta_\mu - \alpha_\mu\beta_\lambda)^2\\ &= (\sum \alpha_\lambda\beta_\lambda)^2 + \tfrac{1}{2}\sum\sum(\alpha_\lambda^2\beta_\mu^2 + \alpha_\mu^2\beta_\lambda^2 - 2\alpha_\lambda\beta_\lambda\alpha_\mu\beta_\mu)\\ &= (\sum \alpha_\lambda\beta_\lambda)^2 + (\sum\alpha_\lambda^2)(\sum\beta_\lambda^2) - (\sum\alpha_\lambda\beta_\lambda)^2 = \|u\|^2\cdot\|v\|^2.\end{aligned}$$

Now let $x=\alpha_0 e+u$, $y=\beta_0 e+v$. Then

$$\begin{aligned}\|xy\| &= \|\alpha_0\beta_0 e + \alpha_0 v + \beta_0 u + uv\|\\ &\leqslant |\alpha_0\beta_0| + |\alpha_0|\,\|v\| + |\beta_0|\,\|u\| + \|u\|\cdot\|v\|\\ &= (|\alpha_0| + \|v\|)(|\beta_0| + \|v\|) \leqslant 2\|x\|\cdot\|y\|\end{aligned}$$

(using the elementary inequality $\xi+\eta\leqslant\sqrt{2(\xi^2+\eta^2)}$).
This concludes the proof.

Remark 1. If A has finite dimension f cannot be injective in the sense mentioned in Proposition 4.3 unless $\dim A\leqslant 4$.

Remark 2. Multiplication is not necessarily continuous in the normed topology. For an example, let $\Lambda=\{\text{integers}\geqslant 0\}$ and define $i_{kl}=i_{k+l}$ if $0<k<l$, $i_{kl}=-i_{k+l}$ if $k>l>0$. With $x_n=\frac{1}{\sqrt{n}}(i_1+i_2+\dots+i_n)$ and $y_n=\frac{1}{\sqrt{n}}(i_{n+1}+\dots+i_{2n})$ we get

$$x_n y_n=\frac{1}{n}(i_{n+2}+2i_{n+3}+\dots+(n-1)i_{2n}+ni_{2n+1}+(n-1)i_{2n+2}+\dots+2i_{3n-1}+i_{3n}).$$

It is then easily verified that

$$\|x_n\|=\|y_n\|=1 \text{ but } \|x_n y_n\|=\left(\frac{2n^2+1}{3n}\right)^{\frac{1}{2}}\to\infty,$$

when $n\to\infty$. Hence multiplication is not (simultaneously) continuous.

Remark 3. The completion of A as a normed space is congruent to the Hilbert space $l^2(\Lambda)$. If multiplication is continuous on A (as in Propositions 4.2 and 4.3) it can be extended to the whole of $l^2(\Lambda)$, to produce examples of (not necessarily associative) Banach algebras with identity.

We finally give an example of an absolute-valued algebra that does not satisfy the assumptions of Theorem 2.2. Let A' be a vector space as above and define

$$i_\lambda i_\mu = s(\lambda,\mu)\, i_{f(\lambda,\mu)}.$$

Here s has values $+1$ and -1, $f(\lambda,\lambda)=0$, $s(\lambda,\lambda)=\text{constant}$, $f(\lambda,\mu)=f(\mu,\lambda)\neq 0$ and $s(\lambda,\mu)=-s(\mu,\lambda)$ when $\lambda\neq\mu$. Moreover f, restricted to λ,μ such that $\lambda\neq\mu$ and $s(\lambda,\mu)=+1$, shall be injective.

Proposition 4.4. *With the assumptions above A' is an absolute-valued algebra (without identity).*

Proof. Computation as in Proposition 4.3.

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