ARKIV FÖR MATEMATIK Band 5 nr 13

Communicated 11 September 1963 by ÅKE PLEIJEL and LARS GÅRDING

Compact linear mappings between interpolation spaces

By ARNE PERSSON

Introduction

Let L^p denote the space of all (equivalence classes of) functions f defined on some subset Ω of the ν-dimensional euclidean space R^ν and such that f is measurable and

$$\int_\Omega |f(x)|^p \, dx < \infty, \quad dx = dx_1 \ldots dx_\nu .$$

The well-known M. Riesz theorem states in particular that, if T is a linear operator which maps L^{p_j} continuously into L^{q_j} $(j=0,1)$, then T maps L^p continuously into L^q, where p and q are given by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \quad (0<\theta<1).$$

If in addition $T: L^{p_0} \to L^{q_0}$ is a compact mapping, it was proved by Krasnoselski [2] (cf. also Cotlar [1], where a similar, slightly weaker result is established) that the mapping $T: L^p \to L^q$ also is compact. J. L. Lions (personal communication) posed the problem if this theorem holds true if we replace L^{p_0}, L^{p_1} and L^{q_0}, L^{q_1} by more general interpolation pairs A_0, A_1 and E_0, E_1, respectively, of Banach spaces and L^p and L^q by interpolation spaces A_θ and E_θ of exponent θ with respect to these pairs. We shall prove here that this question can be answered in the affirmative as soon as the interpolation pair E_0, E_1 satisfies a certain approximation hypothesis, a special case of which was already considered by Lions [3] for other purposes. The approximation hypothesis is easily verified in almost all known concrete examples of interpolation pairs. We give the details of the verification in the case E_0, E_1 are L^p-spaces over an arbitrary locally compact space X with respect to a positive measure on X. Lions [3] has verified the condition in another important case.

Interpolation spaces

A pair E_0, E_1 of Banach spaces is called an interpolation pair, if E_0 and E_1 are continuously embedded in some separated topological linear space $\mathcal{E}$. One verifies easily that $E_0 \cap E_1$ and $E_0 + E_1$ are Banach spaces in the norms

$$x \to \max (\|x\|_{E_0}, \|x\|_{E_1})$$

and

$$x \to \inf_{x=x_0+x_1} (\|x_0\|_{E_0} + \|x_1\|_{E_1}),$$

respectively. Provided that A_0, A_1 and E_0, E_1 are interpolations pairs, A_θ and E_θ are called interpolations spaces of exponent θ $(0<\theta<1)$, with respect to A_0, A_1 and E_0, E_1 if we have the topological inclusions

$$A_0 \cap A_1 \subset A_\theta \subset A_0 + A_1; \; E_0 \cap E_1 \subset E_\theta \subset E_0 + E_1,$$

and if each linear mapping T from $\mathcal{A}$ into $\mathcal{E}$, which maps A_i continuously into E_i $(i=0, 1)$, maps A_θ continuously into E_θ in such a way that

$$M \leqslant M_0^{1-\theta} M_1^\theta,$$

where M denotes the norm of $T: A_\theta \to E_\theta$ and M_i the norm of $T: A_i \to E_i$ $(i=0,1)$. For given pairs A_0, A_1 and E_0, E_1 one can construct many interpolation spaces (see Lions-Peetre [4]). But there exist interpolation spaces $\underline{A}_\theta$, $\underline{E}_\theta$ and $\bar{A}_\theta$, $\bar{E}_\theta$, which for each fixed θ are in a certain sense minimal and maximal, respectively (loc. cit.). We shall make use of the following lemma, the proof of which can be found in Lions-Peetre [4].

Lemma. *Let A_0, A_1 be an interpolation pair and suppose that A and E are given Banach spaces with $A \subset \bar{A}_\theta$ $(0<\theta<1)$. Then, if $T: A_0 \to E$ is compact and $T: A_1 \to E$ is bounded, it follows that $T: A \to E$ is compact.*

The approximation hypothesis

If E and F are Banach spaces we denote by $L(E,F)$ the space of all linear bounded mappings T of E into F endowed with the norm

$$\|T\|_{L(E,F)} = \sup_{\|x\|_E \leqslant 1} \|Tx\|_F.$$

For interpolation pairs E_0, E_1 we shall consider the following condition:

(H) *To each compact set $K \subset E_0$ there exist a constant $C>0$ and a set $\mathfrak{P}$ of linear operators $P: \mathcal{E} \to \mathcal{E}$, which map E_i into $E_0 \cap E_1$ $(i=0, 1)$ and are such that*

$$\|P\|_{L(E_i,E_i)} \leqslant C \quad (i=0,1). \tag{1}$$

Furthermore, we suppose that to each $\varepsilon>0$ we can find a $P \in \mathfrak{P}$ so that

$$\|Px - x\|_{E_0} < \varepsilon \tag{2}$$

for all $x \in K$.

In practice it is often more convenient to verify one of the following stronger hypotheses:

(H_1) *There exist a constant $C>0$ and a set $\mathcal{D}$ of linear operators $P: \mathcal{E} \to \mathcal{E}$ with $P(E_i) \subset E_0 \cap E_1$ $(i=0,1)$, such that* (1) *is satisfied and such that to every $\varepsilon>0$ and every finite set $x_1, \ldots, x_N$ in E_0 we can find a $P \in \mathcal{D}$, so that*

$$\|Px_k - x_k\|_{E_0} < \varepsilon \quad (k=1, \ldots, N). \tag{3}$$

(H_2) *There exists a sequence $(P_n)_1^\infty$ of linear operators $P_n: \mathcal{E} \to \mathcal{E}$ with $P(E_i) \subset E_0 \cap E_1$ $(i=0,1)$ such that $P_n x \to x$ in E_i as $n \to \infty$ for each fixed $x \in E_i$ $(i=0,1)$.*

The latter condition is considered in Lions [3]. Using the Banach-Steinhaus theorem we see that (H_1) follows from (H_2), and it is easily verified that (H_1) implies (H). Thus (H_2) $\Rightarrow$ (H_1) $\Rightarrow$ (H). For later use we notice that in (H_1) it is clearly sufficient to consider elements $x_1, \ldots, x_N$ which belong to a dense subset of E_0.

We shall now prove the main result.

Theorem. *Let A_0, A_1 and E_0, E_1 be interpolation pairs and suppose that A_θ and E_θ are interpolation spaces of exponent θ $(0<\theta<1)$, with respect to these pairs. Suppose further that $A_\theta \subset \bar{A}_\theta$ and that E_0, E_1 satisfies* (H). *Then, if $T: A_0 \to E_0$ is compact and $T: A_1 \to E_1$ is bounded, it follows that $T: A_\theta \to E_\theta$ is compact.*

Proof. The image $K=T(B_0)$ in E_0 of the unit ball B_0 in A_0 is relatively compact in E_0. Hence, choosing P in accordance with (H), we find

$$\|PTa - Ta\|_{E_0} < \varepsilon$$

for all $a \in B_0$, that is $\qquad \|PT - T\|_{L(A_0, E_0)} < \varepsilon.$

In virtue of (1) we therefore obtain

$$\|PT-T\|_{L(A_\theta, E_\theta)} \leqslant \|PT-T\|_{L(A_0, E_0)}^{1-\theta} \|PT-T\|_{L(A_1, E_1)}^{\theta} \leqslant \varepsilon^{1-\theta}(C+1)^\theta \|T\|_{L(A_1, E_1)}^{\theta}.$$

This means that the mapping $T: A_\theta \to E_\theta$ can be approximated uniformly by operators of the form PT, where $P \in \mathcal{D}$, and hence the theorem will follow if we can prove that each mapping $PT: A_\theta \to E_\theta$, $P \in \mathcal{D}$, is compact.

By the closed graph theorem the mappings $P: E_i \to E_0 \cap E_1$ $(i=0,1)$, are continuous. Hence, the composition of a compact and a bounded operator being compact, $PT: A_0 \to E_\theta$ is compact and $PT: A_1 \to E_\theta$ is bounded, so that in view of the lemma $PT: A_\theta \to E_\theta$ is compact. This completes the proof.

An example

We shall verify the approximation hypothesis in an important concrete case. Let X be a locally compact space and μ a positive measure on X. Denote by L^p $(1 \leqslant p \leqslant \infty)$, the Banach space of all (equivalence classes of) measurable functions f on X with

$$\|f\|_{L^p} = \left(\int |f(x)|^p \, d\mu(x)\right)^{1/p} < \infty,$$

if $1 \leqslant p < \infty$, and $\qquad \|f\|_{L^\infty} = \operatorname*{ess\,sup}_{x \in X} |f(x)| < \infty,$

when $p=\infty$. The closed subspace of L^∞ which consists of all bounded functions vanishing at infinity is denoted by L_0^∞. When combined with the main theorem the following proposition generalizes the result of Krasnoselski [2] to arbitrary locally compact spaces. The method of proof is well known.

Proposition. *The interpolation pairs* L^{p_0}, L^{p_1} $(p_0<\infty)$, *and* L_0^∞, L^{p_1} *satisfy the approximation hypothesis* (H_1).

Proof. Let $f_1, \ldots, f_N$ be given functions in $L^{p_0}(L_0^\infty)$ and suppose $\varepsilon>0$ is a given number. Since the set $\mathcal{B}$ of all bounded measurable functions with compact supports is dense in $L^{p_0}(L_0^\infty)$, we may assume that $f_j \in \mathcal{B}$ $(j=1, \ldots, N)$. Let K be a compact set in X, outside which all f_j vanish, and choose $\eta>0$ such that

$$\eta \cdot \max (1, \mu(K))<\varepsilon.$$

It is easy to construct a finite partition (K_n) of K consisting of a set K_0 of measure zero and measurable sets K_1, K_2, ... with $\mu(K_n)>0$ and such that

$$\sup_{x, y \in K_n} |f_j(x)-f_j(y)|<\eta \quad (j=1, \ldots, N).$$

Let φ_n $(n=1,2, \ldots)$, denote the characteristic function of K_n and set

$$Pf=\sum_{n>0}\left(\mu(K_n)^{-1} \int f \varphi_n \, d\mu\right) \varphi_n$$

for each locally integrable function f. It is obvious that P maps L^{p_i} into $L^{p_0} \cap L^{p_1} (i=0,1)$, ($L_0^\infty$ into $L_0^\infty \cap L^{p_1}$). Moreover, for each $f \in L^p$, $p<\infty$, we have

$$\begin{aligned} \|Pf\|_{L^p}^p &=\sum_{n>0}\left(\mu(K_n)^{-1} \int f \varphi_n \, d\mu\right)^p \mu(K_n) \leqslant \sum_{n>0} \mu(K_n)^{1-p} \int_{K_n}|f|^p \, d\mu \cdot \mu(K_n)^{p/p'} \\ &\leqslant \sum_{n>0} \int_{K_n}|f|^p \, d\mu \leqslant\|f\|_{L^p}^p, \quad\left(\frac{1}{p}+\frac{1}{p'}=1\right). \end{aligned}$$

The case $p=\infty$ is easily treated directly, and hence we have proved (1) with $C=1$. In order to prove (3) we observe that

$$\begin{aligned} &\sum_{n>0}\left(\mu(K_n)^{-1} \int f_k \varphi_n \, d\mu\right) \varphi_n(x)-f_k(x) \\ &\quad=\sum_{n>0}\left[\mu(K_n)^{-1} \int(f_k(y)-f_k(x)) \varphi_n(y) \, d\mu(y)\right] \varphi_n(x). \end{aligned}$$

Since for each $x \in K_n$

$$\left|\mu(K_n)^{-1}\int(f_k(y)-f_k(x))\varphi_n(y)\,d\mu(y)\right|\leqslant\eta,$$

we conclude that, for all $1\leqslant p\leqslant\infty$,

$$\|Pf_k-f_k\|_{L^p}\leqslant\eta\left(\sum_{n>0}\int\varphi_n(x)\,d\mu\,(x)\right)^{1/p}=\eta(\mu(K))^{1/p}<\varepsilon\quad(k=1,\ldots,N).$$

Thus the proposition is proved.

Remark. Lions [3] showed that a very wide class of interpolation pairs satisfy condition (H_2).

University of Lund, Sweden.

REFERENCES

1. COTLAR, M., Continuity conditions for potential and Hilbert operators. Cursos y Seminarios de Matemática, Fasc. 2, Buenos Aires (1959). (Spanish.)
2. KRASNOSELSKI, M. A., On a theorem of M. Riesz. Dokl. Akad. Nauk *131*, 246–248 (1959).
3. LIONS, J. L., Sur les espaces d'interpolation; dualité. Math. Scand. *9*, 147–177 (1961).
4. LIONS, J. L. and PEETRE, J., Sur une classe d'espaces d'interpolation. To appear in Inst. Hautes Études Sci. Publ. Math.

Tryckt den 10 februari 1964

Uppsala 1964. Almqvist & Wiksells Boktryckeri AB