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Continuous functions and potential theory

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Introduction

Let F_0 be a compact set in R^m. If F_0 has α-capacity zero, $0 \leqslant \alpha < m$,[1] there exists, according to a well-known result by Evans [8], a positive measure μ concentrated on F_0 such that the potential of order α of μ is infinite everywhere on F_0. In Chapter I we consider a problem related to this result. We shall prove that a compact set F_0 has α-capacity zero if and only if every continuous function coincides everywhere on F_0 with a continuous α-potential of a measure with compact support. This is a consequence of the Theorems 1 and 2. However, these theorems contain much more than the above characterization of compact sets of α-capacity zero. In particular we consider the case when we have more general kernels than $r^{-\alpha}$.

Let h be a positive integer, $p \geqslant 1$, $0 \leqslant \alpha < m$, and let F_0 be a compact set with α-capacity zero. In Chapter II we use Theorem 1 to deduce conditions on h, p and α which guarantee that every function f_0 which is the restiction to F_0 of a continuous function can be extended to a function f having the following properties: f is defined and continuous everywhere in R^m and infinitely differentiable on the complement of F_0; all the partial derivatives of f of orders less than or equal to h and the function f itself belong to $L^p(R^m)$. The result is stated in Theorem 3. The conditions of the theorem imply in particular $\alpha < m-1$; i.e. all the compact sets considered in Theorem 3 have Hausdorff dimension less than $m-1$.

We formulate a converse of Theorem 3 in § 7 (Theorem 4) where we also consider a certain class of Beppo Levi functions, which, in the case $\alpha < m-1$, is more general than the class of functions considered in Theorem 3. The case $p=1$ is studied further in § 9.

As a by-product of our investigation we obtain a theorem on the existence of uniformly continuous harmonic functions with finite Dirichlet integrals in the unit sphere, which take given continuous values on a certain subset of the boundary of the unit sphere (Theorem 5).

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Chapter I. On the representation of continuous functions by potentials

1. *Notations and definitions*

R^m is the m-dimensional Euclidean space, $m \geqslant 1$, with points $x=(x^1,\ldots,x^m)$, $|x|$ the distance from x to the origin. By a closed cube in R^m we mean the set of points satisfying the inequalities $a_i \leqslant x^i \leqslant a_i + l$ where $a_i, i=1,\ldots,m$, are any numbers and $l>0$.

[1] $\alpha=0$ corresponds to the logarithmic capacity.

$S(x_0,r)$ denotes the closed sphere $|x-x_0|\leqslant r$. We write min $\{a,b\}$ for the smaller of the numbers a and b.

The complement of a set E we denote by $\complement E$. If E_1 and E_2 are two sets, then $E_1 \setminus E_2$ is the set of points belonging to E_1 but not to E_2. If $E_1 \supset E_2$, we write $E_1 - E_2$ instead of $E_1 \setminus E_2$.

By a *kernel* we mean a function K satisfying the following conditions:

(a) *K is defined in the interval $r>0$, is finite and continuous, non-negative and non-increasing and satisfies* $\lim_{r\to 0} K(r)=\infty$.

(b) $$\int_0^1 K(r)\, r^{m-1}dr < \infty.$$

(a) and (b) are for instance satisfied if $K(r)=r^{-\alpha}$, $0<\alpha<m$.

Let σ be a real measure on R^m, i.e. a completely additive real set function. The *support* of σ is denoted by S_σ. σ^+ is the positive and σ^- the negative part of $\sigma, \sigma = \sigma^+ - \sigma^-$, and $|\sigma| = \sigma^+ + \sigma^-$.

The potential of a measure σ belonging to a kernel K, the K-potential of σ, is denoted by u_K^σ,

$$u_K^\sigma(x) = \int K(|x-y|)\, d\sigma(y),$$ [1]

and the energy integral by $I_K(\sigma)$,

$$I_K(\sigma) = \iint K(|x-y|)\, d\sigma(x)\, d\sigma(y).$$

u_K^σ is well-defined at the point x provided $u_K^{\sigma^+}(x)$ and $u_K^{\sigma^-}(x)$ are not both infinite. If there will be no misunderstanding we write u^σ instead of u_K^σ. If σ is absolutely continuous and has a density g, $d\sigma = g\,dx$, we sometimes write u_K^g instead of u_K^σ.

The K-capacity of a bounded Borel set E, $C_K(E)$, is defined as

$$C_K(E) = \{\inf_{\nu\in\Gamma_E} I_K(\nu)\}^{-1},$$

where Γ_E is the class of positive measures ν with total mass 1 and $S_\nu \subset E$. The K-capacity is an inner measure, i.e.

$$C_K(E) = \sup_{F\subset E} C_K(F), \tag{1.1}$$

where F is a closed subset of E.

The α-potential and the logarithmic potential, i.e. the potentials in the cases $K(r)=r^{-\alpha}$, $0<\alpha<m$, and $K(r)=-\log r$,[2] we also denote by u_α^σ and u_0^σ respectively and analogously for the α-capacity and the logarithmic capacity.

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[1] The integration is to be extended over the whole space if no limits of integration are indicated.

[2] We shall also consider the case when $K(r) = -\log r$ in spite of the fact that $-\log r$ takes negative values too. We sometimes omit the special simple treatment which is needed in the case $K(r) = -\log r$.

Our kernel K satisfies the continuity principle (see for instance Ugaheri [18]): If μ is a positive measure with compact support and the restriction of u_K^μ to S_μ is continuous, then u_K^μ is continuous in the whole space.

Let F be a compact set with positive K-capacity. Then there exists a positive measure τ—which is not necessarily uniquely determined—with $\tau(R^m)=1$, $S_\tau \subset F$, such that

$$I_K(\tau) = \inf_{\nu \in \Gamma_F} I_K(\nu) = \{C_K(F)\}^{-1},$$

where Γ_F is the class of positive measures ν with $\nu(R^m)=1$, $S_\nu \subset F$. We call τ a *capacitary distribution* and u_K^τ a *capacitary potential* belonging to K and F. u_K^τ satisfies the following inequalities:

$u_K^\tau(x) \geqslant \{C_K(F)\}^{-1}$ *for every $x \in F$ except when x belongs to a set of K-capacity zero.* (1.2)

$$u_K^\tau(x) \leqslant \{C_K(F)\}^{-1} \quad \textit{for every } x \in S_\tau. \tag{1.3}$$

$u_K^\tau(x) \leqslant A \cdot \{C_K(F)\}^{-1}$ *everywhere, where A is a constant which only depends on the dimension m of the space R^m.* (1.4)

As to (1.2) and (1.3) we refer to Frostman [9, pp. 35 ff.] and Fuglede [11, p. 159]. (1.4) is a result by Ugaheri [18].

If F_0 is a compact set, we denote by $S(F_0)$ the class of functions which are restrictions to F_0 of real functions defined and continuous everywhere on R^m.

2. *Representation of continuous functions by potentials*

In this section we also make the following assumption on our kernel K:

(c) *If E is the union of a finite number of closed spheres and u_K^τ a capacitary potential belonging to K and E, we have*

$$u_K^\tau(x) \geqslant \{C_K(E)\}^{-1}, \quad \textit{for every } x \in E. \tag{2.1}$$

The condition (c) is for instance satisfied if $K(r) = r^{-\alpha}$, $0 < \alpha < m$, and, more generally if K satisfies

$$K(r) < MK(2r), \quad \text{for every } r > 0,$$

where M is a constant.[1]

We shall now prove the following theorem on the representation of continuous functions by potentials.

Theorem 1. *Suppose that K is a kernel satisfying* (c) *and that F_0 is a compact set with $C_K(F_0)=0$. Then there exists, for every function $f_0 \in S(F_0)$, an absolutely continuous measure σ with compact support having the following properties: The potential of σ, u_K^σ, is continuous in the whole space and equal to f_0 on F_0, i.e.*

$$u_K^\sigma(x) = f_0(x), \quad \textit{for every } x \in F_0;$$

[1] For a proof of this we refer to Kunugui [13] or Carleson [4, p. 16]. These authors have made further assumptions on the kernel K. However, it is easy to see that the result is true also for our kernels.

if $f_0(x)>0$ for every $x\in F_0$, then σ is a positive measure and if K is infinitely differentiable in the interval $r>0$ then u_K^σ is infinitely differentiable on $\complement F_0$.

For the proof we need the following simple lemma.

Lemma 1. *Let K be a kernel satisfying* (c) *and F_0 a compact set with $C_K(F_0)=0$. Suppose that a is given, $a>0$. Then there is a set E which is the union of a finite number of closed spheres, $E\supset F_0$, so that $C_K(E)<a$. If u_K^τ is a capacitary potential belonging to K and E, then u_K^τ is continuous everywhere and if A is the constant in* (1.4) *we have*

$$\{C_K(E)\}^{-1}\leqslant u_K^\tau(x)\leqslant A\{C_K(E)\}^{-1}\quad \textit{for every } x\in E \tag{2.2}$$

and thus in particular for every $x\in F$.

Proof of Lemma 1. We first observe that the existence of a set E follows immediately from the facts that F_0 is compact and $C_K(F_0)=0$. (2.2) is a consequence of (2.1) and (1.4). Using (1.3) we also find that the restriction of u_K^τ to S_τ is constant and thus u_K^τ is continuous everywhere according to the continuity principle.

Proof of Theorem 1. We first suppose that $f_0(x)>0$ for every $x\in F_0$. Let f be a continuous extension of f_0 to R^m and F a compact set such that every point of F_0 is an interior point of F and $f(x)>0$ for every $x\in F$.

We start by proving that for any $\varepsilon>0$ there exists a positive measure ν with $\nu(R^m)$ less than a given positive number so that S_ν is a subset of a given neighborhood of F_0 and $u^\nu=u_K^\nu$ is continuous and satisfies the following inequalities, if $M=3^mA+1$, m is the dimension of the space and A the constant in (1.4),

$$u^\nu(x)<f(x)\quad \textit{for every}\quad x\in F, \tag{2.3}$$

$$u^\nu(x)\geqslant f_0(x)-M\varepsilon\quad \textit{for every}\quad x\in F_0. \tag{2.4}$$

We introduce the sequence of nets $\mathcal{N}=\{\mathcal{N}_i\}$, where $\mathcal{N}_0$ consists of all closed cubes with corners having integer coordinates, and $\mathcal{N}_i$, $i>0$, consists of all closed cubes which we obtain by dividing the cubes in $\mathcal{N}_{i-1}$ into 2^m equal cubes by $(m-1)$-dimensional hyperplanes parallel to the coordinate planes. Let $\omega_1,\dots,\omega_r$ be a number of congruent cubes from $\mathcal{N}$, $\cup\,\omega_i\supset F$, so that the oscillation of f is less than ε in ω_i for $i=1,\dots,r$. We separate those cubes $\omega_i, i=1,\dots,r$, which are such that the maximum of f_0 on $F_0\cap\omega_i$ is larger than $M\varepsilon$.[1] In this way we get the cubes $\omega_1',\dots,\omega_s'$. Using Lemma 1 it is easy to realize that, for $i=1,\dots,s$, we can choose a positive measure with total mass less than a given positive number having the following properties: The support of the measure is a subset of a given neighborhood of $F_0\cap\omega_i'$; the K-potential of the measure is continuous everywhere and takes values between ε and $A\varepsilon$ on $F_0\cap\omega_i'$; it is less than or equal to $A\varepsilon$ on ω_i' and on those cubes $\omega_1,\dots,\omega_r$ which have non-void intersections with ω_i' and it is finally less than a given positive number elsewhere. In this way we get a positive measure associated with every cube ω_i', $i=1,\dots,s$. Let ν_1 be the sum of these measures. Due to the choice of the constant $M, M=3^mA+1$, and the fact that the oscillation of f is less than ε in every cube ω_i, $i=1,\dots,r$, it is easy to realize that we can make the above procedure so that

$$u^{\nu_1}(x)<f(x)\quad \textit{for every } x\in F, \tag{2.5}$$

$$u^{\nu_1}(x)\geqslant \min\{f_0(x)-M\varepsilon,\varepsilon\}\quad \textit{for every } x\in F_0. \tag{2.6}$$

[1] If $f_0(x)\leqslant M\varepsilon$ for every $x\in F_0$ there are no such cubes. In this case, however, (2.3) and (2.4) are satisfied with $\nu=0$.

Our procedure also guarantees that we can get $\nu_1(R^m)$ smaller than a given positive number and S_{ν_1} as a subset of a given neighborhood of F_0.

If (2.3) and (2.4) are not true for $\nu=\nu_1$ we consider $f-u^{\nu_1}$. As $f(x)-u^{\nu_1}(x)>0$ for every $x\in F$ and u^{ν_1} is continuous we can repeat the procedure leading to (2.5) and (2.6) but with f replaced by $f-u^{\nu_1}$ and f_0 by $f_0-u^{\nu_1}$. In this way we get a positive measure ν_2 having properties which are analogous to those of ν_1, and a continuous potential u^{ν_2} satisfying

$$u^{\nu_2}(x)<f(x)-u^{\nu_1}(x) \quad \textit{for every } x\in F, \tag{2.7}$$

$$u^{\nu_2}(x)\geq \min\{f_0(x)-u^{\nu_1}(x)-M\varepsilon, \varepsilon\} \quad \textit{for every } x\in F_0. \tag{2.8}$$

(2.6) and (2.8) give, for $x\in F_0$,

$$u^{\nu_1}(x)+u^{\nu_2}(x)\geq \min\{f_0(x)-M\varepsilon, 2\varepsilon\},$$

and so (2.7) and (2.8) yield

$$u^{\nu_1}(x)+u^{\nu_2}(x)<f(x) \quad \text{for every } x\in F,$$

$$u^{\nu_1}(x)+u^{\nu_2}(x)\geq \min\{f_0(x)-M\varepsilon, 2\varepsilon\} \quad \text{for every } x\in F_0.$$

If (2.3) and (2.4) are not true for $\nu=\nu_1+\nu_2$ we repeat our procedure anew. After n steps we have obtained n positive measures $\nu_1,\dots,\nu_n$ having continuous potentials $u^{\nu_1},\dots,u^{\nu_n}$ so that

$$\sum_{i=1}^{n} u^{\nu_i}(x)<f(x) \quad \textit{for every } x\in F, \tag{2.9}$$

$$\sum_{i=1}^{n} u^{\nu_i}(x)\geq \min\{f_0(x)-M\varepsilon, n\varepsilon\} \quad \textit{for every } x\in F_0, \tag{2.10}$$

and so that $\sum_1^n \nu_i(R^m)$ is less than a given positive number and $\bigcup_1^n S_{\nu_i}$ is a subset of a given neighborhood of F_0. As f_0 is bounded on F_0, there exists, according to (2.10), a smallest number n_0 so that (2.4) holds with $\nu=\nu_1+\dots+\nu_{n_0}$. From (2.9) we see that also (2.3) is true for this choice of ν.

As the second step of the proof we show that there even exists an absolutely continuous positive measure, ψdx, where ψ is infinitely differentiable and the total mass of ψdx is less than a given positive number and S_ψ is a subset of a given neighborhood of F_0, so that (2.3) and (2.4) are true with ν replaced by ψdx. In fact, let φ be an infinitely differentiable function with compact support, $\varphi\geq 0$, $\int\varphi(x)dx=1$, and let ψ be the convolution of φ and ν, $\psi=\varphi*\nu$. ψ is then infinitely differentiable and the measure ψdx has the same total mass as ν. By choosing r small enough and φ such that $S_\varphi\subset S(0,r)$, where $S(0,r)$ is the sphere with centre 0 and radius r, we can make S_ψ a subset of a given neighborhood of S_ν. As $u^\nu=K*\nu$ we have

$$u^\psi=K*\psi=K*\varphi*\nu=\varphi*u^\nu.$$

By choosing r small enough we can thus also make the difference $u^\psi-u^\nu$ less than a given number, uniformly on F. As we can prove an inequality (2.4) for every $\varepsilon>0$ we conclude that we also can prove the same inequality and the inequality (2.3) with ν replaced by a measure ψdx having the properties stated above.

Thus we have proved the existence of an absolutely continuous measure μ_1 having an infinitely differentiable density so that

$$u^{\mu_1}(x) < f(x) \quad \text{for every} \quad x \in F,$$

$$u^{\mu_1}(x) \geqslant f_0(x) - 2^{-1} \quad \text{for every} \quad x \in F_0.$$

By considering $f - u^{\mu_1}$ instead of f we get analogously a measure μ_2 so that

$$u^{\mu_2}(x) < f(x) - u^{\mu_1}(x) \quad \text{for every} \quad x \in F,$$

$$u^{\mu_2}(x) \geqslant f_0(x) - u^{\mu_1}(x) - 2^{-2} \quad \text{for every} \quad x \in F_0,$$

which gives

$$u^{\mu_1}(x) + u^{\mu_2}(x) < f(x) \quad \text{for every} \quad x \in F,$$

$$u^{\mu_1}(x) + u^{\mu_2}(x) \geqslant f_0(x) - 2^{-2} \quad \text{for every} \quad x \in F_0.$$

After n steps we have obtained n positive measures, $\mu_1, \ldots, \mu_n$, satisfying

$$\sum_{i=1}^{n} u^{\mu_i}(x) < f(x) \quad \textit{for every} \quad x \in F, \tag{2.11}$$

$$\sum_{i=1}^{n} u^{\mu_i}(x) \geqslant f_0(x) - 2^{-n} \quad \textit{for every} \quad x \in F_0. \tag{2.12}$$

We can furthermore suppose that μ_i is absolutely continuous and has an infinitely differentiable density for $i = 1, \ldots, n$, that $\sum_1^n \mu_i(R^m)$ is less than a given positive number which is independent of n, and that the maximal distance between F_0 and a point on the boundary of S_{μ_n} tends to zero as n tends to infinity. Under these assumptions it follows that $\{\sum_1^n \mu_i\}_{n=1}^{\infty}$ converges weakly to an absolutely continuous positive measure μ such that the density of μ is infinitely differentiable om $\complement F_0$. Using (2.12) we obtain, as the kernel K is non-negative,

$$u^{\mu}(x) \geqslant \sum_{1}^{n} u^{\mu_i}(x) \geqslant f_0(x) - 2^{-n}, \; n = 1, 2, \ldots, \quad \text{for every} \quad x \in F_0.$$

This gives $u^{\mu}(x) \geqslant f_0(x)$ for every $u \in F_0$. As $\{\sum_1^n \mu_i\}$ converges weakly to μ we have by (2.11):

$$f(x) \geqslant \varliminf_{n \to \infty} \sum_{1}^{n} u^{\mu_i}(x) \geqslant u^{\mu}(x) \quad \text{for every} \quad x \in F.$$

It follows that

$$u^{\mu}(x) = f_0(x) \quad \textit{for every} \quad x \in F_0, \tag{2.13}$$

$$u^{\mu}(x) \leqslant f(x) \quad \textit{for every} \quad x \in F. \tag{2.14}$$

From the construction we conclude at once that u^{μ} is continuous on $\complement F_0$. To prove that u^{μ} is continuous at a point x_0 belonging to F_0 we use (2.14). It is well known that

$$u^{\mu}(x_0) \leqslant \varliminf_{x \to x_0} u^{\mu}(x).$$

To prove that
$$u^\mu(x_0) \geqslant \overline{\lim_{x\to x_0}}\, u^\mu(x)$$
we observe that x_0 is an interior point of F and so (2.14) yields
$$\overline{\lim_{x\to x_0}}\, u^\mu(x) \leqslant \overline{\lim_{x\to x_0}}\, f(x) = f(x_0) = f_0(x_0) = u^\mu(x_0).$$
This proves that u^μ is continuous at x_0 and thus everywhere.

From the construction of u^μ we easily realize that u^μ is infinitely differentiable on $\complement F_0$ in the case when K is infinitely differentiable in the interval $r>0$. Because if x_1 is a point in $\complement F_0$ we can write u^μ:
$$u^\mu = \sum_1^{n_1} u^{\mu_i} + \Big(u^\mu - \sum_1^{n_1} u^{\mu_i}\Big) \tag{2.15}$$
and choose n_1 so large that the support of $\mu - \sum_1^{n_1}\mu_i$ does not contain x_1. The second term of the right member of (2.15) is then infinitely differentiable at x_1 due to our assumption on K, and the first term due to the fact that $\sum_1^{n_1}\mu_i$ is absolutely continuous and has an infinitely differentiable density.

By that our theorem is completely proved in the case when $f_0(x)>0$ for every $x\in F_0$. The general case follows immediately by writing f_0 as the difference between two functions which are strictly positive on F_0.

Remark 1. From the proof it is clear that the measure σ in Theorem 1 can be chosen with $\sigma^+(R^m)+\sigma^-(R^m)$ less than a given number so that S_σ is a subset of a given neighborhood of F_0. We can also make u^σ the difference between two continuous potentials generated by absolutely continuous positive measures having densities which are infinitely differentiable on $\complement F_0$.

Remark 2. The condition K non-negative is not necessary for the validity of Theorem 1. For instance, it is easy to realize that the theorem is true also for the kernel $-\log r$.

Remark 3. From Theorem 1 we can deduce the following result: *Suppose that g is a strictly positive function which is lower semi-continuous in R^m and that $C_K(F_0)=0$, F_0 compact. Then there exists a potential u_K^μ, continuous on $\complement F_0$, which is generated by a positive absolutely continuous measure μ such that S_μ is a subset of a given neighborhood of F_0 and*
$$u^\mu(x) = g(x) \quad \textit{for every} \quad x\in F_0.$$

In fact, there exists a sequence of continuous functions $\{g_i\}_1^\infty$, $0<g_{i-1}(x)<g_i(x)$ for every $x\in F_0$, $i=2,3,\ldots$, converging pointwise to g on F_0. Hence there is a positive measure μ_i having suitable properties and a continuous potential u^{μ_i} so that, if $g_0=0$,
$$u^{\mu_i}(x) = g_i(x) - g_{i-1}(x) \quad \text{for every} \quad x\in F_0,\ i=1,2,\ldots.$$
It is possible to arrange so that $\{\sum_1^n \mu_i\}_1^\infty$ converges weakly to a positive measure μ having the required properties and in fact also those stated for σ in Remark 1. This gives the required properties to u^μ including the equality $u^\mu(x)=g(x)$ for every $x\in F_0$. (Compare the calculations leading from (2.11) and (2.12) to (2.13).)

This result clearly contains the following result by Rudin [16] as a special case (the case $g(x)=\infty$ for every $x \in F_0$): Suppose that G is an open set containing the compact set F_0 and that $C_{m-2}(F_0)=0$, $m \geq 2$. Then there exists a positive absolutely continuous measure μ having compact support such that u^{μ}_{m-2} is infinite on F_0, continuous on $\complement F_0$ and harmonic in the complement of the closure of G.[1]

Remark 4. Let U be the open unit sphere in R^m. If F_0 is a closed subset of the boundary of U, $f_0 \in S(F_0)$ and $C_{m-2}(F_0)=0$, $m \geq 2$, it is not hard to prove, by modifications of the proof of Theorem 1, that there exists a measure σ such that S_σ is compact and does not contain any points from U and such that $u^{\sigma}_{m-2}(x)=f_0(x)$ for every $x \in F_0$. As before we can get u^{σ}_{m-2} as the difference between two continuous potentials generated by positive measures which means that the energy integral $I_{m-2}(|\sigma|)$ is finite. Therefore we can conclude: For any function $f_0 \in S(F_0)$ there exists a function u, $u(x)=f_0(x)$ for every $x \in F_0$, which is harmonic in U, continuous in the closure of U and has a finite Dirichlet integral,

$$\int_U \sum_{i=1}^{m} \left(\frac{\partial u}{\partial x^i}\right)^2 dx < \infty.$$

A converse of this proposition will be proved in § 8.

3. *Modulus of continuity of potentials*

We consider two kernels K and K_0 and suppose that

$$\lim_{r \to 0} K_0(r)\{K(r)\}^{-1} = \infty. \tag{3.1}$$

We shall show the existence of a modulus of continuity on the set where $u^{|\sigma|}_{K_0}$ is bounded by a given constant, a modulus of continuity which is common to all potentials u^{σ}_K with $\sigma^+(R^m)+\sigma^-(R^m)$ less than a given constant. In § 4 we shall then use this result to prove a converse of Theorem 1.

Lemma 2. *Suppose that the kernels K and K_0 satisfy* (3.1). *Then there exists a non-negative function t defined in the interval $r>0$, $\lim_{r\to 0} t(r)=0$, only depending on K and K_0 so that, if σ is a measure with compact support, $\sigma^+(R^m)+\sigma^-(R^m) \leq M_1$ and $u^{|\sigma|}_{K_0}(x_i) \leq M_2$, $i=1, 2$, then we have*

$$|u^{\sigma}_K(x_1)-u^{\sigma}_K(x_2)| \leq (M_1+M_2)\,t(|x_1-x_2|). \tag{3.2}$$

Proof. We consider only the case when σ is a positive measure, from which the general case is an immediate consequence. Let x_1 and x_2 be two points with $u^{\sigma}_{K_0}(x_i) \leq M_2$, $i=1,2$, and put $|x_1-x_2|=r_0$. Let S be the open sphere with centre $(x_1+x_2)\cdot 2^{-1}$ and radius $2^{-1}r_0+r_1$, where r_1 is a number which we shall choose later depending on r_0 so that r_1 tends to zero when r_0 tends to zero.

$$u^{\sigma}_K(x_1)-u^{\sigma}_K(x_2)=\int (K(|x_1-y|)-K(|x_2-y|))\,d\sigma(y)=\int_S+\int_{\complement S}=I+II.$$

[1] Rudin also treats the simple extension to arbitrary closed sets.

In order to estimate I we introduce the function t_1:

$$t_1(r) = \{K_0(r)\}^{-1} \cdot K(r).$$

t_1 satisfies $\lim_{r\to 0} t_1(r) = 0$ according to (3.1). If $\sigma(x;r)$ is the value of σ for the open sphere with centre x and radius r, we have

$$\begin{aligned} |I| &\leqslant \int_0^{r_0+r_1} K(r)\,d\sigma(x_1;r) + \int_0^{r_0+r_1} K(r)\,d\sigma(x_2;r) \\ &= \int_0^{r_0+r_1} t_1(r)\,K_0(r)\,d\sigma(x_1;r) + \int_0^{r_0+r_1} t_1(r)\,K_0(r)\,d\sigma(x_2;r) \\ &\leqslant \sup_{0\leqslant r\leqslant r_0+r_1} t_1(r)\,\{u^\sigma_{K_0}(x_1) + u^\sigma_{K_0}(x_2)\} \leqslant 2\,M_2 \sup_{0\leqslant r\leqslant r_0+r_1} t_1(r), \end{aligned}$$

i.e.

$$|I| \leqslant 2\,M_2 \sup_{0\leqslant r\leqslant r_0+r_1} t_1(r). \tag{3.3}$$

To be able to estimate II we have to examine the difference $K(|x_1-y|)-K(|x_2-y|)$ when $y\in \complement S$. We need a result of the following kind: There exists a non-negative function t_2 defined in the interval $r>0$, $\lim_{r\to 0} t_2(r)=0$, only depending on K, so that, for every $\eta>0$,

$$K(r)-K(r+\varrho) \leqslant t_2(\eta) \quad \textit{for} \quad r\geqslant t_2(\eta), \quad 0\leqslant\varrho\leqslant\eta. \tag{3.4}$$

We suppose for a moment that this has been proved and use it to estimate II. As $\big||x_1-y|-|x_2-y|\big| \leqslant |x_1-x_2| = r_0$ and $|x_i-y|\geqslant r_1$, $i=1,2$, when $y\in\complement S$, we have by using (3.4) with $\eta=r_0$ and choosing $r_1=t_2(r_0)$,

$$|II| \leqslant \int_{\complement S} |K(|x_1-y|)-K(|x_2-y|)|\,d\sigma(y) \leqslant M_1\cdot t_2(r_0). \tag{3.5}$$

We define t by

$$t(r_0) = 2\sup t_1(r) + t_2(r_0), \quad r_0>0,$$

where sup is taken for those r which satisfy $0\leqslant r\leqslant r_0+t_2(r_0)$. t satisfies the demands of the lemma; (3.3) and (3.5) give (3.2).

It remains to prove the existence of a function t_2, $\lim_{r\to 0} t_2(r)=0$, only depending on K, such that (3.4) is valid. We observe that K is uniformly continuous in the interval $a\leqslant r<\infty$ for every $a>0$. From this we conclude that for every $\varepsilon>0$ there exists a largest number $q(\varepsilon)$, which is possibly $+\infty$, so that

$$K(r)-K(r+\varrho) \leqslant \varepsilon \quad \textit{for} \quad r\geqslant\varepsilon,, \quad 0\leqslant\varrho\leqslant q(\varepsilon), \tag{3.6}$$

and it is not hard to realize that this implies the existence of a function t_2 with the desired properties so that (3.4) is true. With that the lemma is proved.

4. *A converse of Theorem 1*

We denote by $\mathcal{U}(K,F_0)$ the class of functions which are restrictions to F_0 of K-potentials u^σ_K of measures σ with compact supports. A function $f_0\in S(F_0)$ belongs to $\mathcal{U}(K,F_0)$ if there is a function $g\in\mathcal{U}(K,F_0)$ well-defined at every point of F_0, such that $f_0(x)=g(x)$ for every $x \in F_0$. We shall prove the following converse of Theorem 1:

Theorem 2. *Let K be a kernel and F_0 a compact set, $C_K(F_0)>0$. Then there exists a function f_0 belonging to $S(F_0)$ but not to $\mathfrak{U}(K, F_0)$.*

Remark 1. From the proof of Theorem 2 it will appear that also the following more general proposition is true: *We can find a function $f_0 \in S(F_0)$ such that there is no function from $\mathfrak{U}(K, F_0)$ coinciding with f_0 on F_0 except on a subset of F_0 of K-capacity zero.*

Remark 2. It will appear from the proof that the assumption K non-negative is not necessary for the validity of the theorem. For instance the theorem is true also for the kernel $-\log r$.

Remark 3. Theorem 1 and Theorem 2 give in particular the following characterization of compact sets of K-capacity zero: *Let K satisfy the condition* (c) *of* § 2. *A compact set F_0 has K-capacity zero if and only if every function from $S(F_0)$ is the restriction to F_0 of a continuous K-potential of a measure with compact support.*

Proof of Theorem 2. As $C_K(F_0)>0$ there exists a kernel K_0 such that

$$\lim_{r \to 0} K_0(r) \cdot \{K(r)\}^{-1} = \infty \tag{4.1}$$

and

$$C_{K_0}(F_0) > 0.^1 \tag{4.2}$$

According to (4.2) there is a positive measure ν_0 with total mass 1, $S_{\nu_0} \subset F_0$ and $I_{K_0}(\nu_0) < \infty$.

We first use Lemma 2 to deduce a property on F_0 of an arbitrary K-potential u_K^σ of a measure σ with compact support and after that we shall construct a continuous function f with restriction f_0 to F_0 such that f_0 does not have this property. Let σ be a measure with compact support and $M_1 = \sigma^+(R^m) + \sigma^-(R^m)$. As $u_{K_0}^{|\sigma|}$ is finite except on a set of K_0-capacity zero and ν_0, due to the fact that $I_{K_0}(\nu_0) < \infty$, does not concentrate any mass on such a set, we conclude that the set where $u_{K_0}^{|\sigma|}$ is finite has ν_0-measure 1, i.e. measure 1 with respect to ν_0. Consequently, for every $\varepsilon > 0$ there is a constant $M_2 = M_2(\varepsilon)$ so that $u_{K_0}^{|\sigma|}$ is less than M_2 on F_0 except on a subset of F_0 having ν_0-measure less than ε. According to (4.1) and Lemma 2 this means that there exists a function t defined in the interval $r>0$, $\lim_{r\to 0} t(r) = 0$, only depending on K and K_0 such that

$$|u_K^\sigma(x_1) - u_K^\sigma(x_2)| \leqslant (M_1 + M_2)\, t(|x_1 - x_2|), \tag{4.3}$$

for x_1 and x_2 belonging to F_0 except when x_1 and x_2 belong to a subset of F_0 having ν_0-measure less than ε.

We construct the function f in the proof of the following lemma:

Lemma 3. *Let K_0 be a kernel, F_0 a compact set with $C_{K_0}(F_0) > 0$, ν_0 a positive measure with $S_{\nu_0} \subset F_0$, $\nu_0(R^m) = 1$, $I_{K_0}(\nu_0) < \infty$ and t^* a non-decreasing function defined in the interval $r>0$ such that $t^*(r)>0$ if $r>0$, $\lim_{r\to 0} t^*(r) = 0$. Then there exists a function*

[1] Compare Carleson [3, p. 405].

f, defined and continuous everywhere, having the following property for all sufficiently small positive values of ε: For every Borel set E, $E \subset F_0$, with $\nu_0(E) < \varepsilon$, there are points x_1 and x_2 belonging to $F_0 - E$ with $|x_1 - x_2|$ arbitrarily small so that

$$|f(x_1) - f(x_2)| \geqslant M_3 t^*(|x_1 - x_2|), \quad M_3 \textit{ positive constant.}^1 \tag{4.4}$$

Using Lemma 3 we can easily finish the proof of Theorem 2. Let K_0, F_0 and ν_0 in the lemma be identical with the kernel K_0, the set F_0 and the measure ν_0 occurring in the proof of the theorem. We choose the function t^* in the lemma so that

$$\lim_{r \to 0} t^*(r) \{t(r)\}^{-1} = \infty. \tag{4.5}$$

If f is the function occurring in the lemma it is clear by (4.3), (4.4) and (4.5) that there is no measure σ with compact support such that u_K^σ and f coincide everywhere on F_0. This proves our theorem.

Proof of Lemma 3. We shall construct f as a sum, $f(x) = \sum_1^\infty f_i(x)$. To construct $\{f_i\}_1^\infty$ we use three sequences of positive numbers $\{a_i\}$, $\{b_i\}$ and $\{d_i\}$.

We first suppose that the intersection between F_0 and an arbitrary $(m-1)$-dimensional plane parallel to some coordinate plane has ν_0-measure zero.

To construct f_i, for a fixed i, we start by covering F_0 by means of congruent cubes from the sequence of sets $\mathcal{N} = \{\mathcal{N}_i\}$ which we used in the proof of Theorem 1, and we use cubes with diameters less than or equal to $2^{-1} \cdot a_i$.[2] After that we separate those cubes, say $\omega'_{i1}, \omega'_{i2}, \ldots$, which intersect F_0 in a set of positive ν_0-measure. For every cube ω'_{ij}, where j is a fixed number, $j = 1, 2, \ldots$, we consider the infinite strip which is determined by the points $(x^1, \ldots, x^m)$ where x^1 varies arbitrarily and $x^2, \ldots, x^m$ vary in the same intervals as the corresponding coordinates for an arbitrary point in ω'_{ij}. If there is no cube ω'_{is}, $s \neq j$, which is contained in this strip and has an $(m-1)$-dimensional edge plane in common with ω'_{ij}, we divide ω'_{ij} into two rectangles by an $(m-1)$-dimensional hyperplane perpendicular to the x^1-axis so that both rectangles get the property that they intersect F_0 in a set of positive ν_0-measure. (Compare (4.9) below.) The cubes $\omega'_{i1}, \omega'_{i2}, \ldots$ are in this way replaced by rectangles $\omega''_{i1}, \omega''_{i2}, \ldots$.

We now choose the number d_i satisfying

$$0 < d_i < \min_j \nu_0(F_0 \cap \omega''_{ij}). \tag{4.6}$$

If $\nu_0(F_0 \cap \omega''_{ij}) > 2d_i$ we divide, for $j = 1, 2, \ldots, \omega''_{ij}$ into two or more parts by $(m-1)$-dimensional hyperplanes perpendicular to the x^1-axis in such a way that $\omega''_{i1}, \omega''_{i2}, \ldots$ are replaced by rectangles $\omega_{i1}, \ldots, \omega_{in_i}$ with the following properties:

$$\nu_0(F_0 \setminus \bigcup_{j=1}^{n_i} \omega_{ij}) = 0. \tag{4.7}$$

$$d_i < \nu_0(F_0 \cap \omega_{ij}) \leqslant 2d_i, \quad j = 1, \ldots, n_i. \tag{4.8}$$

Every ω_{ij} has an $(m-1)$-dimensional edge plane in common with a rectangle ω_{is}, $s \neq j$, in the infinite strip which is determined by the points $(x^1, \ldots, x^m)$ where x^1 is arbitrary and $x^2, \ldots, x^m$ vary in the same intervals as the corresponding coordinates for an arbitrary point of ω_{ij}. (4.9)

[1] In general we cannot make (4.4) true for all x_1 and x_2 belonging to F_0. This is a consequence of a result by Besicovitch, [1, p. 183].

[2] We suppose for the moment that the number a_i is given.

As the intersection between F_0 and an $(m-1)$-dimensional hyperplane parallel to some coordinate plane has ν_0-measure zero, we can form a subset Δ_{ij} of ω_{ij} containing the boundary of ω_{ij} such that the distance from the boundary of ω_{ij} to $\omega_{ij}-\Delta_{ij}$ is larger than zero and, due to (4.8),

$$\nu_0(F_0 \cap (\omega_{ij}-\Delta_{ij}))>d_i, \quad j=1,\dots,n_i.$$

We can also make the choice of Δ_{ij} such that, if η is a given number, $\eta>0$, and we define Δ by

$$\Delta=\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{n_i}(\Delta_{ij}\cap F_0), \tag{4.10}$$

then we have $\nu_0(\Delta)<\eta$.

We now choose f_i. For every rectangle ω_{ij} there is an infinite strip of the kind described in (4.9). In each such strip we choose f_i identically equal to $t^*(a_i)$ in every second set $\omega_{ij}-\Delta_{ij}$ and identically equal to zero in every second, counted from sets $\omega_{ij}-\Delta_{ij}$ with points having smaller x^1-coordinates to sets with points having larger x^1-coordinates. In the remaining points of R^m we define f_i such that f_i becomes continuous and less than or equal to $t^*(a_i)$ everywhere and zero on $\complement\bigcup_{j=1}^{n_i}\omega_{ij}$.

We now fix two points x_i and x_i' from $\bigcup_{=1}^{n_i}(\omega_{ij}\cap F_0)\setminus\Delta$ belonging to two different rectangles ω_{ij} bordering on each other in an infinite strip of the kind described in (4.9). If we introduce the notation $\delta_{js}^{(i)}$ for the distance between $\omega_{ij}-\Delta_{ij}$ and $\omega_{is}-\Delta_{is}$ and define b_i by

$$b_i=\min_{j\neq s}\delta_{js}^{(i)}, \tag{4.11}$$

then, clearly, we have

$$|f_i(x_i)-f_i(x_i')|=t^*(a_i) \quad \text{and} \quad b_i\leqslant|x_i-x_i'|\leqslant a_i.$$

As the function t^* is non-decreasing, we obtain

$$|f_i(x_i)-f_i(x_i')|\geqslant t^*(|x_i-x_i'|). \tag{4.12}$$

For $j>i$ we have $\qquad |f_j(x_i)-f_j(x_i')|\leqslant 2\max_y|f_j(y)|\leqslant 2t^*(a_j),$

which gives $\qquad |f_j(x_i)-f_j(x_i')|\leqslant t^*(|x_i-x_i'|)\cdot 2t^*(a_j)\cdot\{t^*(b_i)\}^{-1}, \quad j>i. \qquad (4.13)$

If we furthermore assume that

$$a_i<b_{i-1}, \quad i=2,3,\dots, \tag{4.14}$$

we get $\qquad f_j(x_i)-f_j(x_i')=0, \quad j<i. \qquad (4.15)$

Putting $f(x)=\sum_1^\infty f_i(x)$ we have by (4.12), (4.13) and (4.15)

$$|f(x_i)-f(x_i')|\geqslant t^*(|x_i-x_i'|)\Big[1-2\{t^*(b_i)\}^{-1}\cdot\sum_{j>i}t^*(a_j)\Big].$$

If the expression in square brackets is larger than or equal to 2^{-1}, i.e. if

$$\sum_{j>i}t^*(a_j)\leqslant\tfrac14 t^*(b_i), \tag{4.16}$$

then
$$|f(x_i)-f(x_i')|\geqslant \tfrac{1}{2}t^*(|x_i-x_i'|),\ i=1,2,\ldots. \tag{4.17}$$

It is easy to realize that it is possible to choose the sequence $\{a_i\}$ such that (4.14) and (4.16) are satisfied and this shows that it is possible to perform our construction. (4.16) guarantees that $\sum f$ converges uniformly and therefore f is continuous.

Suppose now that E is a Borel set, $E\subset F_0$, $\nu_0(E)<\varepsilon$. To finish the proof of the lemma it is, according to (4.17), enough to prove that, for all sufficiently small values of ε and for all i, there exist points x_i and x_i' from $\bigcup_{j=1}^{n_i}(\omega_{ij}\cap F_0)\setminus(E\cup\Delta)$ belonging to two different rectangles ω_{ij} bordering on each other in an infinite strip of the kind described in (4.9). Suppose, on the contrary, that this is not the case. Thus, for some i, at least one third of the rectangles $\omega_{i1},\ldots,\omega_{in_i}$ do not contain any points from $F_0\setminus(E\cup\Delta)$. Using (4.8) and the fact that we have chosen Δ so that $\nu_0(\Delta)<\eta$ we get

$$\varepsilon+\eta>\nu_0(E\cup\Delta)>d_i\cdot\frac{n_i}{3}, \tag{4.18}$$

where n_i is the number of rectangles ω_{ij} for a fixed i. But (4.7) and (4.8) give

$$1=\nu_0(F_0)\leqslant 2d_i n_i,$$

which, combined with (4.18), gives $\varepsilon+\eta>6^{-1}$. As η can be chosen arbitrarily small we get a contradiction if ε is small enough.

The lemma is thus proved in the case when the intersection between F_0 and an arbitrary $(m-1)$-dimensional hyperplane parallel to some coordinate plane has ν_0-measure zero. If this condition is not satisfied we choose an $(m-1)$-dimensional hyperplane P parallel to some coordinate plane such that $\nu_0(P\cap F_0)>0$ and carry through the above construction of f—with F_0 replaced by $P\cap F_0$—in the $(m-1)$-dimensional hyperplane P. The fact that we may have $\nu_0(P\cap F_0)<1$ does not change the idea of the construction. After having constructed f in P we extend f to a continuous function in R^m. The extended function clearly satisfies the conditions of the lemma which thus is proved.

Chapter II. An extension problem for continuous functions

5. *Statement of the problem*

We first introduce some more notations. We denote the sequence $(s_1,\ldots,s_h)$ of indices between 1 and m by s and its length h by $|s|$ and we put

$$D^s=\frac{\partial^h}{\partial x^{s_1}\ldots\partial x^{s_h}}.$$

For any number $p\geqslant 1$ we denote by L^p the class of all Lebesgue measurable functions f in R^m such that $\int|f(x)|^p dx<\infty$; we denote by L^p_{loc} the class of all measurable functions f in R^m such that $\int_F|f(x)|^p dx<\infty$ for every compact set F. We use the notation

$$\|f\|_{L^p(E)}=\left\{\int_E|f(x)|^p dx\right\}^{1/p},$$

and we write $\|f\|_{L^p}$ instead of $\|f\|_{L^p(R^m)}$.

We shall use a class of distributions in R^m (in the sense of Schwartz [17]) consisting of functions from L^1_{loc} and, in order to avoid confusion with the usual pointwise derivation, we always state explicitly when it concerns derivation in the distribution sense. δ denotes the Dirac measure and we write D^s instead of $D^s\delta$.

We shall deal with the following extension problem: Let F_0 be a compact set having m-dimensional Lebesgue measure zero. We are interested in finding conditions on F_0 which guarantee that every function $f_0 \in S(F_0)$ has a continuous extension f to R^m such that all the derivatives of f—in a certain sense—of orders less than or equal h belong to L^p, where h and p are given numbers, $p \geq 1$. It turns out that a relevant condition is that $C_\alpha(F_0)=0$ for a certain α, $0 \leq \alpha < m$, irrespective of the regularity of the set F_0 in other respects. The results are formulated in the Theorems 3, 4 and 6 as conditions on the connections between h, p, α and the dimension m of the space.

In the case when we are searching for a positive solution of our extension problem it is natural to require that the extended function f is to be infinitely differentiable on $\complement F_0$ (Theorem 3). When we try to find a converse (Theorem 4) of Theorem 3 we shall use a certain class of Beppo Levi functions of order h, $BL_h(L^p_{\text{loc}})$,[1] $p \geq 1$. This is the class of distributions T in R^m such that all the derivatives (in the distribution sense) of order h are functions belonging to L^p_{loc}, i.e. $D^sT \in L^p_{\text{loc}}$ for all s with $|s|=h$.

We start our investigation by discussing some properties of the class $BL_h(L^p_{\text{loc}})$, $p \geq 1$, and the class of functions which are infinitely differentiable on $\complement F_0$, where $C_\alpha(F_0)=0$ for some $\alpha < m-1$:

1°. Following Deny–Lions [7, p. 314] we say that a function g, defined in R^m, has the property (AC) in R^m if g is absolutely continuous on almost every line[2] with a given direction if this direction coincides with the direction of some coordinate axis.

Using the property (AC) we get the following characterization of the distributions in $BL_h(L^p_{\text{loc}})$:

$BL_h(L^p_{\text{loc}})$ consists of those distributions T in R^m which have the following properties: Every derivative D^sT (in the distribution sense) with $0 \leq |s| < h$[3] is a function which—properly defined on a set of Lebesgue measure zero—gives a function g_s having the property (AC); all the derivatives of the first order of g_s in the usual pointwise sense are in L^p_{loc} and they also constitute the derivatives of g_s in the distribution sense.

This characterization is given by Deny and Lions [7, p. 315] for $h=1$. The general case follows easily from the case $h=1$ if we use a theorem by Kryloff [17, part II, p. 37] to conclude that if all the derivatives of the first order of a distribution are functions in L^p_{loc} then the distribution itself is a function in L^p_{loc}.[4]

2°. F_0 is a compact set with $C_\alpha(F_0)=0$ for some α satisfying $\alpha < m-1$. Suppose that the function g is defined and infinitely differentiable on $\complement F_0$ and that all the partial derivatives of g of order h belong to L^p_{loc}, where $p \geq 1$. $C_\alpha(F_0)=0$ and $\alpha < m-1$ mean that almost every line with a given direction does not intersect F_0. Otherwise

[1] The discussion of this chapter concerning Beppo Levi functions in the case $h=1$, should be compared to the discussions in Deny [5] and Deny-Lions [7]. The definition of $BL_h(L^p_{\text{loc}})$ is found in the work of Deny-Lions. Compare also Nikodym [14].

[2] A function is absolutely continuous on a line if the restriction of the function to an arbitrary compact interval of the line is absolutely continuous.

[3] D^sT with $|s|=0$ denotes T.

[4] Observe that by means of Kryloff's theorem it is possible to enunciate more than we have done in our characterization of the distributions in $BL_h(L^p_{\text{loc}})$.

there would exist a direction such that the orthogonal projection F_0' of F_0 on an $(m-1)$-dimensional normal plane to this direction would satisfy $C_\beta(F_0')>0$ for every $\beta<m-1$, contrary to the fact that $C_\alpha(F_0)=0$.[1] This shows that g and all the partial derivatives of all orders of g have the property (AC). As all the partial derivatives of order h of g are in L^p_{loc} we can use the result by Deny–Lions [7, p. 315] and Kryloff's theorem as in 1° to conclude that g defines a distribution belonging to $BL_h(L^p_{\text{loc}})$. In particular we obtain that g and all the partial derivatives of g of orders less than h are in L^p_{loc} too.

3°. As we in the extension problem only are interested in continuous functions we introduce two classes $\mathcal{A}_h(L^p_{\text{loc}})$ and $\mathcal{A}^*_h(L^p_{\text{loc}}, F_0)$, $p\geq 1$, of functions in the following ways: *$\mathcal{A}_h(L^p_{\text{loc}})$ is the class of functions f defined and continuous everywhere which—considered as distributions—belong to $BL_h(L^p_{\text{loc}})$. $\mathcal{A}^*_h(L^p_{\text{loc}}, F_0)$ is the class of functions f defined and continuous everywhere in R^m, infinitely differentiable on $\complement F_0$, and such that all the partial derivatives of f of orders less than or equal to h belong to L^p_{loc}.*

From the discussion in 2° we conclude that $\mathcal{A}^*_h(L^p_{\text{loc}}, F_0)$, with $C_\alpha(F_0)=0$ for some $\alpha<m-1$, is a subclass of $\mathcal{A}_h(L^p_{\text{loc}})$.

4°. We shall need the following fact: Let $f_0\in S(F_0)$. If there exists a function $f\in\mathcal{A}^*_h(L^p_{\text{loc}}, F_0)$ (or $\mathcal{A}_h(L^p_{\text{loc}})$) coinciding pointwise with f_0 on F_0, then there exists a function $f^*\in\mathcal{A}^*_h(L^p_{\text{loc}}, F_0)$ (or $\mathcal{A}_h(L^p_{\text{loc}})$) which is zero outside a compact set and coincides pointwise with f_0 on F_0.

In fact, as the function f^* we only have to choose $f\varphi$, where φ is identically 1 on F_0, infinitely differentiable and has a compact support.

6. *An extension theorem*

We need the following lemma:

Lemma 4. *Let $0<\alpha<\beta<m$. Let μ be a positive measure with $\mu(R^m)<\infty$ and suppose that u^μ_α is bounded. Then we have*

$$u^\mu_\beta\in L^p \text{ if } 2\leq p=\frac{m-\alpha}{\beta-\alpha} \tag{6.1}$$

and

$$u^\mu_\beta\in L^p_{\text{loc}} \text{ if } 1\leq p<\frac{m-\alpha}{\beta-\alpha}<2. \tag{6.2}$$

More exactly, for $p>1$ we have, if $a=\mu(R^m)$,

$$\|u^\mu_\beta\|_{L^p}\leq M_1(a)\cdot\{\sup_{x\in R^m} u^\mu_\alpha(x)\}^{(p-1)/p}, \text{ if } 2\leq p=\frac{m-\alpha}{\beta-\alpha} \tag{6.3}$$

and for every sphere S with radius r, we have

$$\|u^\mu_\beta\|_{L^p(S)}\leq M_2(a,r)\cdot\{\sup_{x\in R^m} u^\mu_\alpha(x)\}^{(p-1)/p}, \text{ if } 1<p<\frac{m-\alpha}{\beta-\alpha}<2. \tag{6.4}$$

$M_1(a)$ is a constant depending further on m, p and α and $M_2(a,r)$ a constant depending further on m, p, α and β.

[1] Compare for instance Frostman [9, p. 91] and Brelot [2, p. 330].

For a proof of the lemma, based essentially on Hölders inequality, we refer to du Plessis [15], Deny [6] and Fuglede [10]. A proof based on function theory is given in Carleson [4, p. 61 and p. 80] for some special values of α and β. The method is, however, applicable in the general case, too.

Theorem 3. *Let F_0 be a compact set, $p \geq 1$ and h a positive integer. Suppose that either*

$$C_\alpha(F_0) = 0 \quad \textit{for} \quad \alpha = m - ph \quad \textit{where} \quad m - ph \geq 0 \quad \textit{and} \quad p \geq 2 \tag{6.5}$$

or

$$C_\alpha(F_0) = 0 \quad \textit{for some} \quad \alpha \quad \textit{satisfying} \quad \alpha < m - ph \quad \textit{where} \quad m - ph > 0 \quad \textit{and} \quad 1 \leq p < 2. \tag{6.6}$$

Then every function belonging to $S(F_0)$ can be extended to a function which is defined and continuous everywhere in R^m and infinitely differentiable on $\complement F_0$ and such that all the partial derivatives of orders less than or equal to h of the extended function and the extended function itself are in L^p.

Proof. We first treat the case $\alpha > 0$ and assume that either (6.5) or (6.6) is valid.

Suppose that f_0 is a function from $S(F_0)$. According to § 5, 4° it is enough to prove that there is a function from $A_h^*(L_{\text{loc}}^p, F_0)$ coinciding pointwise with f_0 on F_0.

As $C_\alpha(F_0) = 0$ there exists an absolutely continuous measure σ, constructed as in Theorem 1, with compact support such that u_α^σ is continuous everywhere, infinitely differentiable on $\complement F_0$, $u_\alpha^{|\sigma|}$ is bounded and $u_\alpha^\sigma(x) = f_0(x)$ for every $x \in F_0$. We shall prove that $u_\alpha^\sigma \in A_h^*(L_{\text{loc}}^p, F_0)$. An easy consequence of the properties of σ is that

$$D^s u_\alpha^\sigma(x) = \int D_x^s \frac{1}{|x-y|^\alpha}\, d\sigma(y) \quad \textit{for every} \quad x \in \complement F_0, \tag{6.7}$$

for all sequences s with $|s| \leq h$ if $\alpha + h < m$. The index x in D_x^s denotes derivation with respect to x.

(6.7) yields, if $|s| = h, \alpha + h < m$,

$$|D^s u_\alpha^\sigma(x)| \leq M u_{\alpha+h}^{|\sigma|}(x) \quad \textit{for every} \quad x \in \complement F_0, \tag{6.8}$$

where M is a constant only depending on h, α and m. As $u_\alpha^{|\sigma|}$ is bounded we can, by means of (6.8) and Lemma 4 used with $\beta = \alpha + h$, conclude that all the derivatives of order h of u_α^σ belong to L_{loc}^p; we use (6.1) if (6.5) is valid and (6.2) if (6.6) is valid. From the discussion in § 5, 2° we finally conclude that $u_\alpha^\sigma \in A_h^*(L_{\text{loc}}^p, F_0)$, and so the theorem is proved if $\alpha > 0$.

The case $\alpha = 0$ is treated analogously to the case $\alpha > 0$, by using the following lemma instead of Lemma 4:

Lemma 5. *Let $0 < \beta < m$. Let μ be a positive measure with compact support. If either $I_0(\mu)$ is finite and $2 \leq p = m/\beta$ or $1 \leq p < m/\beta$, then $u_\beta^\mu \in L_{\text{loc}}^p$.*

The proof of Lemma 5 is analogous to that of Lemma 4 (see Fuglede [10]).

7. *A converse of Theorem 3*

Theorem 3 gives a positive solution of our extension problem with the extended function in the class $A_h^*(L_{\text{loc}}^p, F_0)$ if F_0 satisfies (6.5) or (6.6). The following theorem gives a converse of Theorem 3 as $A_h^*(L_{\text{loc}}^p, F_0)$ is a subclass of $A_h(L_{\text{loc}}^p)$ if $C_\alpha(F_0) = 0$ for some $\alpha < m - 1$:

Theorem 4. *Let F_0 be a compact set. A sufficient condition for the existence of a function in $S(F_0)$, which cannot be extended to a function in $A_h(L^p_{\mathrm{loc}})$, is that either*

$$C_\alpha(F_0)>0 \text{ for some } \alpha \text{ satisfying } \alpha>m-ph, \text{ where } m-ph\geqslant 0 \text{ and } p>2 \tag{7.1}$$

or

$$C_\alpha(F_0)>0 \text{ for } \alpha=m-ph \text{ where } m-ph>0 \text{ and } 1\leqslant p\leqslant 2. \tag{7.2}$$

In order not to encumber the proof with details we shall not treat the case when $m-ph=0$, $1\leqslant p\leqslant 2$. It is, however, no difficulty to modify the proof which we shall give to obtain: *The theorem is also true in the case when*

$$C_0(F_0)>0, \quad m-ph=0, \quad 1\leqslant p\leqslant 2. \tag{7.3}$$

An inspection of the Theorems 3 and 4 shows that they give a complete solution of our extension problem when $p=2$ but not when $p\neq 2$.

Theorem 4 is proved in § 8. For the proof we need integral representations of those functions in $BL_h(L^p_{\mathrm{loc}})$ which are zero outside a compact set. To deduce these we use the following formulas (Schwartz [17, part I, p. 47]): If Δ_h is the Laplace operator iterated h times, $h\geqslant 1$, then we have, in the distribution sense, if δ is the Dirac measure and M_1 and M_2 are certain constants only depending on h and m,

$$M_1\cdot\Delta_h|x|^{2h-m}=\delta \text{ if } m-2h>0 \text{ or } m-2h<0,\ m \text{ odd}, \tag{7.4}$$

$$M_2\cdot\Delta_h(|x|^{2h-m}\log|x|)=\delta \text{ if } m-2h\leqslant 0,\ m \text{ even}. \tag{7.5}$$

We can write

$$\Delta_h=\sum a_s D^s D^s, \tag{7.6}$$

where a_s are constants and the sum is extended over a number of multiindices s with $|s|=h$. If $f\in BL_h(L^p_{\mathrm{loc}})$ and f has compact support we obtain, by means of (7.6), in the case when (7.4) is valid,

$$f=\delta\ast f=M_1\Delta_h|x|^{2h-m}\ast f=M_1\sum a_sD^s\ast D^s|x|^{2h-m}\ast f=M_1\sum a_sD^s|x|^{2h-m}\ast D^sf.$$

The distributions $D^s|x|^{2h-m}$ and D^sf are functions and D^sf has compact support. The convolution $D^s|x|^{2h-m}\ast D^sf$ can consequently be written as an integral and as $D^sf\in L^p$, we get:

If $f\in BL_h(L^p_{\mathrm{loc}})$, f has compact support and the function $k_{s,h}$ is defined by $k_{s,h}(x)=D^s|x|^{2h-m}$, then there are functions g_s with compact supports, $g_s\in L^p$, and constants b_s so that

$$f(x)=\sum b_s\int k_{s,h}(x-y)g_s(y)\,dy \text{ a.e., if } m-2h>0 \text{ or } m-2h<0,\ m \text{ odd}. \tag{7.7}$$

Analogously we get by (7.5)*—with other values on the constants b_s—if the function $k^*_{s,h}$ is defined by $k^*_{s,h}(x)=D^s(|x|^{2h-m}\log|x|)$,*

$$f(x)=\sum b_s\int k^*_{s,h}(x-y)\,g_s(y)\,dy \text{ a.e., if } m-2h\leqslant 0,\ m \text{ even}. \tag{7.8}$$

The sums in (7.7) *and* (7.8) *are extended over a number of multiindices s with $|s|=h$.*

In the proof of Theorem 4 we also need the following lemma:

Lemma 6.[1] *Let K be a kernel. Suppose that g_i, $i=1, 2$, are functions having the property that for every $\varepsilon>0$ there exists a Borel set E with $C_K(E)<\varepsilon$ such that the restrictions of g_1 and g_2 to $\complement E$ both are continuous. If we, furthermore, assume that $g_1(x)=g_2(x)$ a.e. and that the set of points x where $g_1(x)\neq g_2(x)$ is a Borel set, then $g_1(x)=g_2(x)$ except on a set of K-capacity zero.*

Proof. Let E_1 be the set of points x where $g_1(x)\neq g_2(x)$ and suppose that $C_K(E_1)=a>0$. Choose ε, $0<\varepsilon<a$, and let E be a Borel set with $C_K(E)<\varepsilon$ such that the restrictions of g_1 and g_2 to $\complement E$ both are continuous. We further choose a compact subset F of $E_1 \setminus E$ with $C_K(F)>a-\varepsilon$. Let F_n, for $n=1,2,\ldots$, $F_n\supset F$, be the union of finitely many closed spheres having radii $\leqslant r_n$, where $r_n\to 0$ when $n\to\infty$, and centres belonging to F and let F_n^* consist of the set of points situated at a distance $\leqslant r_n$ from F_n.

If μ_n is a capacitary distribution belonging to the kernel K and F_n, then

$$u_K^{\mu_n}(x)\leqslant A\cdot\{C_K(F_n)\}^{-1}, \tag{7.9}$$

where A is the constant in (1.4). Let φ_n, $\int\varphi_n dx=1$, be a non-negative function which is infinitely differentiable and let S_{φ_n} be a subset of $S(0, r_n)$. We define ψ_n by $\psi_n=\varphi_n * \mu_n$. $\psi_n dx$ is a positive measure with total mass 1 and $S_{\psi_n}\subset F_n^*$. Since E_1 has Lebesgue measure zero there exists for every $\eta>0$ a closed set $H_n(\eta)$, $H_n(\eta)\subset F_n^*\setminus E_1$, such that the restriction of the measure $\psi_n dx$ to $H_n(\eta)$ has total mass $\geqslant 1-\eta$. Hence

$$I_K(\psi_n)\geqslant(1-\eta)^2\,\{C_K(H_n(\eta))\}^{-1}\geqslant(1-\eta)^2\,\{C_K(F_n^*\setminus E_1\}^{-1}.\quad \eta\to 0 \text{ gives}$$

$$I_K(\psi_n)\geqslant\{C_K(F_n^*\setminus E_1)\}^{-1}. \tag{7.10}$$

On the other hand we have by (7.9)

$$u_K^{\psi_n}(x)=K*\varphi_n*\mu_n(x)=\varphi_n*u_K^{\mu_n}(x)\leqslant A\cdot\{C_K(F_n)\}^{-1},$$

which gives $I_K(\psi_n)\leqslant A\cdot\{C_K(F_n)\}^{-1}$. This inequality and (7.10) give

$$C_K(F_n^*\setminus E_1)\geqslant A^{-1}C_K(F_n)\geqslant A^{-1}C_K(F)>A^{-1}\cdot(a-\varepsilon).$$

By choosing ε so small that $A^{-1}(a-\varepsilon)>\varepsilon$ we get $C_K(F_n^*\setminus E_1)>C_K(E)$ as $C_K(E)<\varepsilon$, i.e. the set $(F_n^*\setminus E_1)\setminus E$ is non-empty.

Let $x_n\in(F_n^*\setminus E_1)\setminus E$. According to our construction we have

$$g_1(x_n)=g_2(x_n),\quad n=1,2,\ldots, \text{ and} \tag{7.11}$$

There exists a point $y_n\in F$ such that $|x_n-y_n|\leqslant 2r_n$, $n=1,2,\ldots$. (7.12)

We now prove the existence of a point x_0 from F where g_1 and g_2 coincide which gives a contradiction to the fact that $F\subset E_1$. We suppose that the sequence of points $\{x_n\}$ converges to a point x_0. (If this is not the case, we choose a convergent subsequence.) From (7.12) we conclude that $\{y_n\}$ converges to x_0 too. Hence $x_0\in F$. We now use the estimate

$$|g_1(x_0)-g_2(x_0)|\leqslant|g_1(x_0)-g_1(x_n)|+|g_1(x_n)-g_2(x_n)|+|g_2(x_n)-g_2(x_0)|.$$

[1] A special case of this lemma is given in Deny and Lions [7, p. 353].

The second term of the right member is zero according to (7.11). As x_n and x_0 belong to $\complement E$ and the restrictions of g_1 and g_2 to $\complement E$ are continuous, we conclude that the two remaining terms of the right member are arbitrarily small when n is large. This proves that $g_1(x_0)=g_2(x_0)$ which gives a contradiction to the fact that $x_0 \in F$.

8. *Proof of Theorem 4 and a theorem on harmonic functions*

The idea of the proof of Theorem 4 is as follows: We start by observing that, due to § 5,4°, we only have to prove the existence of a function $f_0 \in S(F_0)$ which cannot be extended to a function in $A_h(L^p_{\text{loc}})$ with compact support if (7.1) or (7.2) is satisfied. But every function in $A_h(L^p_{\text{loc}})$ is—considered as a distribution—an element in $BL_h(L^p_{\text{loc}})$. This means that every function in $A_h(L^p_{\text{loc}})$ with compact support can be written on the form (7.7) or (7.8). Following the method of the proof of Theorem 2 we shall deduce moduli of continuity of the right members of (7.7) and (7.8) having properties analogous to those of the modulus of continuity of the potentials in the proof of Theorem 2. This will also show that the right members of (7.7) and (7.8) have such properties that we can use Lemma 6 to conclude that every function in $A_h(L^p_{\text{loc}})$ with compact support has a representation (7.7) or (7.8) valid not only a.e. but everywhere except on a set of K_0-capacity zero, where the kernel K_0 is defined below. (Compare (8.1) and (8.17).) The choice of K_0 is such that $C_{K_0}(F_0)>0$ and this will enable us to infer, from Lemma 3, the existence of a function $f_0 \in S(F_0)$ which cannot be extended to a function in $A_h(L^p_{\text{loc}})$ if (7.1) or (7.2) is valid.

The case $1 \leqslant p \leqslant 2$. We start by treating the case $1 \leqslant p \leqslant 2$ and we suppose consequently that (7.2) is true for the given set F_0 and given values on h and p, $1 \leqslant p \leqslant 2$. Since $C_\alpha(F_0)>0$ for $\alpha = m-ph$ there exists a kernel K_0 satisfying

$$\lim_{r\to 0} K_0(r)\, r^{m-ph} = \infty,\ C_{K_0}(F_0) > 0,\ \lim_{r\to\infty} K_0(r) > 0. \tag{8.1}$$

As $C_{K_0}(F_0)>0$ we can choose a measure ν_0 with

$$\nu_0 \geqslant 0, \quad \nu_0(R^m) = 1, \quad S_{\nu_0} \subset F_0, \quad I_{K_0}(\nu_0) < \infty. \tag{8.2}$$

We formulate the information which we need about the right members of (7.7) and (7.8) in the following lemma:

Lemma 7. *Let the compact set F_0, the kernel K_0, and the measure ν_0 satisfy* (8.1) *and* (8.2) *and suppose that p and h are given numbers, h a positive integer, with $m-ph>0$, $1 \leqslant p \leqslant 2$. Using the notations of* (7.7) *and* (7.8) *we define the function v_h by*

$$v_h(x) = \sum b_s \int k_{s,h}(x-y)\, g_s(y)\, dy \quad \textit{if} \quad m-2h>0 \quad \textit{or} \quad m-2h<0,\ m \quad \textit{odd}, \tag{8.3}$$

and by
$$v_h(x) = \sum b_s \int k^*_{s,h}(x-y)\, g_s(y)\, dy \ \textit{if}\ m-2h \leqslant 0,\ m\ \textit{even}, \tag{8.4}$$

at those points where the right members are well defined. The sums are extended over the same multiindices s as in (7.7) *and* (7.8), *i.e. over a number of s with $|s| = h$; b_s are constants and g_s functions in L^p with compact supports. Then there exists a non-negative function t_h defined in the interval $r>0$, $\lim_{r\to 0} t_h(r)=0$, only depending on m, h and K_0 such that the following assertion is true:*

For every $\varepsilon>0$ there exists a Borel set E_1 with $C_{K_0}(E_1)<\varepsilon$ and a constant M only depending on ε, m, h, K_0, b_s and g_s so that

$$|v_h(x_1)-v_h(x_2)| \leqslant M t_h(|x_1-x_2|) \quad \textit{for all} \quad x_1,x_2 \in \complement E_1. \tag{8.5}$$

Furthermore, there exists a Borel set E_2 with $\nu_0(E_2)<\varepsilon$ such that (8.5) *is valid with E_1 replaced by E_2.*

We can easily finish the proof of Theorem 4 in the case $1\leqslant p\leqslant 2$ if for a moment we suppose that Lemma 7 has been proved. As we have mentioned above it is enough to prove the existence of a function $f_0 \in S(F_0)$ which cannot be extended to a function in $\mathcal{A}_h(L^p_{\text{loc}})$ with compact support. To prove this we first study those functions in $\mathcal{A}_h(L^p_{\text{loc}})$ which have compact supports. Let f be such a function. Then f has a representation (7.7) if $m-2h>0$ or $m-2h<0$, m odd, and a representation (7.8) if $m-2h\leqslant 0$, m even. This gives an integral representation of f valid a.e. But since f is continuous we can use (8.5) of Lemma 7 and Lemma 6 to conclude that this integral representation of f is valid everywhere except on a set of K_0-capacity zero.[1] By means of the last sentence of Lemma 7 we infer, as the set where the integral representation of f is not true has K_0-capacity zero and accordingly also ν_0-measure zero: For every $\varepsilon>0$ there exists a Borel set E, $E\subset F_0$, $\nu_0(E)<\varepsilon$, such that, if t_h is the function occurring in Lemma 7, then

$$|f(x_1)-f(x_2)| \leqslant M t_h(|x_1-x_2|) \quad \textit{for all} \quad x_1,x_2 \in F_0-E. \tag{8.6}$$

Using Lemma 3 and (8.6) the proof is now completed in the same way as we finished the proof of Theorem 2 by means of Lemma 3.

It remains to prove Lemma 7.

Proof of Lemma 7. We first deal with the case $p=2$. According to our assumptions we have $m-2h>0$ which means that the case when v_h is defined by (8.4) does not occur. The kernel K_0 has when $p=2$ the properties

$$\lim_{r\to 0} K_0(r) r^{m-2h} = \infty,\ C_{K_0}(F_0)>0,\ \lim_{r\to\infty} K_0(r)>0. \tag{8.7}$$

Let q be a function defined in the interval $r>0$ satisfying

$$\begin{aligned}&q \textit{ is non-increasing, non-negative and continuous, } q(r)\leqslant A_1 q(2r) \textit{ for every } r>0,\\ &\quad A_1 \textit{ constant, } \lim_{r\to 0} q(r)=\infty.\end{aligned} \tag{8.8}$$

We define the function g by

$$g(y)=\sum |g_s(y)|,$$

where the sum is extended over those functions g_s which occur in (8.3). Finally we introduce the function w_h:

$$w_h(x)=\int \frac{q(|x-y|)\,g(y)}{|x-y|^{m-h}}\,dy.$$

[1] Due to the degree of arbitrariness in the choice of K_0 in (8.1) it is, in fact, easy to realize, that the integral representation of f holds true except on a set of α-capacity zero with $\alpha=m-ph$.

The reason for introducing the function w_h is the following: Since $k_{s,h}(x) = D^s|x|^{2h-m}$, $|s| = h$, we have

$$|k_{s,h}(x)| \leqslant M \cdot |x|^{h-m}.^{1} \tag{8.9}$$

This and (8.3) show that v_h is majorized by a potential generated by the kernel r^{h-m}. But w_h is a potential generated by the kernel $r^{h-m}\cdot q(r)$. Since $q(r)$ tends to infinity as r tends to zero we may proceed as in the proof of Lemma 2 to show an inequality of the type of (8.5) for v_h, valid for points x_1 and x_2 from a set of points where w_h is majorized by a certain constant. We start by performing this and then we shall finish the proof of Lemma 7 in the case $p=2$ by showing that w_h is bounded except on a set having K_0-capacity and ν_0-measure less than a prescribed positive number, if q is properly chosen.

We accordingly suppose that $w_h(x_i) \leqslant a$, $i=1, 2$, where a is a given positive number. As in the proof of Lemma 2 we put $|x_1 - x_2| = r_0$ and introduce the open sphere S with centre $2^{-1}(x_1+x_2)$ and radius $2^{-1}r_0 + r_1, r_1 > r_0$, and write

$$v_h(x_1) - v_h(x_2) = \int_S + \int_{\complement S} = I + II.$$

Using (8.9) we obtain $\quad |I| \leqslant a \cdot M \{q(r_0+r_1)\}^{-1}.$

If $y \in \complement S$ we obtain, by using the mean value theorem, if $|s| = h$,

$$|k_{s,h}(x_1-y) - k_{s,h}(x_2-y)| \leqslant M \frac{|x_1-x_2|}{(r_1-r_0)^{m-h+1}}.$$

As in the proof of Lemma 2 we now realize that we can choose r_1 depending on r_0 and find a function t_h, defined in the interval $r>0$, $t_h(r) \to 0$ when $r \to 0$, only depending on m, h and q, such that

$$|v_h(x_1) - v_h(x_2)| \leqslant (a+1) M t_h(|x_1 - x_2|) \quad \textit{if} \quad w_h(x_i) \leqslant a,\ i = 1,2. \tag{8.10}$$

The lemma follows, in the case $p=2$, by means of (8.10), if we prove that it is possible to choose q only depending on K_0 such that, for every $\varepsilon > 0$, there exists a constant a with $w_h(x) \leqslant a$ except on a set of K_0-capacity and ν_0-measure less than ε. But this is a consequence of the assertion that we can choose q only depending of K_0 such that, if G_a denotes the set where $w_h(x) > a > 0$, then

$$C_{K_0}(G_a) < M \cdot a^{-2} \|g\|^2_{L^2}. \tag{8.11}$$

To prove this last assertion we consider an arbitrary positive measure μ with $\mu(R^m) = 1$, $S_\mu \subset G_a$. We have, by Schwarz's inequality,

$$a^2 < \left(\int w_h(x)\,d\mu(x)\right)^2 = \left\{\int g(y) \left(\int \frac{q(|x-y|)}{|x-y|^{m-h}}\,d\mu(x)\right) dy\right\}^2$$

$$\leqslant \|g\|^2_{L^2} \cdot \iint d\mu(x)\,d\mu(z) \int \frac{q(|x-y|)\,q(|z-y|)}{|x-y|^{m-h}\,|z-y|^{m-h}}\,dy.$$

[1] In the proof of this lemma M denotes a constant—not necessarily the same each time it occurs—only depending on the parameters shown for the constant M in Lemma 7.

We define H by $$H(x,z)=\int\frac{q(|x-y|)\,q(|z-y|)}{|x-y|^{m-h}\,|z-y|^{m-h}}\,dy$$

and shall prove that we can choose q, satisfying (8.8) and only depending on K_0 such that

$$H(x,z)\leqslant MK_0(|x-z|). \tag{8.12}$$

In fact, if for a moment we suppose that this has been proved we obtain from our estimate of a^2,

$$a^2<\|g\|_{L^2}^2\cdot M\cdot I_{K_0}(\mu).$$

Hence $$a^2<M\cdot\|g\|_{L^2}^2\cdot\{C_{K_0}(G_a)\}^{-1}$$

which yields (8.11).

We now prove the assertion leading to (8.12). If we put $z-x=\xi$ we obtain, by a substitution in the integral defining $H(x,z)$,

$$H(x,z)=\int\frac{q(|y|)\,q(|y-\xi|)}{|y|^{m-h}\,|y-\xi|^{m-h}}\,dy.$$

We denote this expression by $H_1(\xi)$ and estimate $H_1(\xi)$ by the following division of the integral, where D is the set of points y satisfying

$$|y|>\frac{|\xi|}{2},\ |y-\xi|>\frac{|\xi|}{2}\ \text{and}\ |y|<2|\xi|,$$

$$H_1(\xi)=\int_{|y|\leqslant\frac{|\xi|}{2}}+\int_{|y-\xi|\leqslant\frac{|\xi|}{2}}+\int_{|y|\geqslant 2|\xi|}+\int_D=I+II+III+IV.$$

By using $q(r)\leqslant A_1\,q(2r)$ we obtain

$$I\leqslant 2^{m-h}A_1\,q(|\xi|)\,|\xi|^{h-m}\int_{|y|\leqslant\frac{|\xi|}{2}}|y|^{h-m}\,q(|y|)\,dy$$
$$\leqslant MA_1q(|\xi|)\,|\xi|^{h-m}\int_0^{|\xi|}r^{h-1}\,q(r)\,dr.$$

For II we obtain the same estimate as for I.

$$III\leqslant 2^{m-h}\{q(|\xi|)\}^2\cdot\int_{|y|\geqslant 2|\xi|}\frac{dy}{|y|^{2m-2h}}\leqslant M\,|\xi|^{2h-m}\{q(|\xi|)\}^2,$$

as $m-2h>0$.

$$IV\leqslant A_1^2\cdot 2^{2m-2h}\{q(|\xi|)\}^2\cdot|\xi|^{2h-2m}\int_D dy\leqslant M\cdot A_1^2\cdot|\xi|^{2h-m}\{q(|\xi|)\}^2.$$

These estimates show that (8.12) is true if we can choose q satisfying (8.8) so that

$$q(|\xi|)\cdot|\xi|^{-h}\int_0^{|\xi|}r^{h-1}q(r)\,dr+\{q(|\xi|)\}^2\leqslant MK_0(|\xi|)\cdot|\xi|^{m-2h},\ |\xi|\neq 0.$$

But this is a consequence of the facts that $m-2h>0$, $\lim_{r\to 0} K_0(r)\, r^{m-2h}=\infty$, $\lim_{r\to\infty} K_0(r)>0$ and, for instance, the fact that

$$|\xi|^{-h}\int_0^{|\xi|} r^{h-1}q(r)\,dr \leqslant M_2 q(|\xi|), \tag{8.13}$$

if q satisfies (8.8) with A_1 sufficiently close to 1. M_2 is a constant depending on A_1 and h. (8.13) is proved by dividing the integration interval $(0,|\xi|)$ by the points $2^{-n}|\xi|$, $n=0,1,2,\ldots$

Hence Lemma 7 is proved in the case $p=2$.

We now prove Lemma 7 when $1\leqslant p<2$. In this case it may occur that $m-2h\leqslant 0$. Consequently we also have to consider the case when the function v_h is defined by the formula (8.4).

We let q be a function satisfying (8.8) and introduce the functions g and w_h in the same way as in the case $p=2$. (8.10) is, of course, still valid, when $m-2h>0$ or $m-2h<0$, m odd, i.e. when v_h is defined by the formula (8.3), but we can also deduce the same formula when $m-2h\leqslant 0$, m even, i.e. when v_h is defined by the formula (8.4). In fact, when $m-2h\leqslant 0$, m even, and $m-h>0$ it is easy to prove that the estimate (8.9) holds also for $k^*_{s,h}$ i.e. that

$$|k^*_{s,h}(x)|\leqslant M|x|^{h-m},\ |s|=h. \tag{8.14}$$

Analogously to the proof of (8.10) when v_h is defined by (8.3) it is possible to show, by means of (8.14), the existence of a function t_h, defined in the interval $r>0$, $\lim_{r\to 0} t_h(r)=0$, only depending on m, h and q such that (8.10) is true also when v_h is defined by (8.4).

An inspection of the proof when $p=2$ now shows that the lemma follows in the case $1\leqslant p<2$ if we prove that we can choose q only depending on K_0 such that the following substitute of (8.11) is valid,

$$C_{K_0}(G_a)<M_3 a^{-p}\|g\|^p_{L^p},\ a>0, \tag{8.15}$$

where G_a denotes the set of points x where $w_h(x)>a$ and M_3 is a constant depending on the same parameters as the constant M in (8.5) and on the constant A in (1.4).

To prove (8.15) we consider, as in the proof of (8.11), a positive measure $\mu,\mu(R^m)=1$, $S_\mu\subset G_a$. By using Hölder's inequality twice we have in the case $1<p<2$,[1] if $p'=p(p-1)^{-1}$:

$$\begin{aligned}
a^{p'} &< \left(\int w_h(x)\,d\mu(x)\right)^{p'} \leqslant \|g\|^{p'}_{L^p}\int\left\{\int\frac{q(|x-y|)}{|x-y|^{m-h}}\,d\mu(x)\right\}^{p'}dy\\
&= \|g\|^{p'}_{L^p}\int\left\{\int |x-y|^{\frac{(ph-m)(p'-2)}{p'}}\cdot|x-y|^{\frac{ph-2m}{p'}}\cdot q(|x-y|)\,d\mu(x)\right\}^{p'}dy\\
&\leqslant \|g\|^{p'}_{L^p}\int\left\{\int |x-y|^{ph-m}\,d\mu(x)\right\}^{p'-2}\cdot\left\{\int |x-y|^{\frac{ph}{2}-m}q(|x-y|)^{\frac{p'}{2}}\,d\mu(x)\right\}^2dy\\
&\leqslant \|g\|^{p'}_{L^p}\{\sup_{y\in R^m} u^\mu_{m-ph}(y)\}^{p'-2}\int\left\{\int |x-y|^{\frac{ph}{2}-m}\cdot q(|x-y|)^{\frac{p'}{2}}\,d\mu(x)\right\}^2dy.
\end{aligned}$$

[1] As to the latter use of Hölder's inequality, compare [15, p. 130]. The case $p=1$ requires a simple special treatment which we omit.

Defining q_1 by $q_1(r)=\{q(r)\}^{\frac{p'}{2}}$ we get

$$\int\left\{\int |x-y|^{\frac{ph}{2}-m}\cdot q(|x-y|)^{\frac{p'}{2}}\,d\mu(x)\right\}^2 dy$$

$$=\iint d\mu(x)\,d\mu(z)\int \frac{q_1(|x-y|)\cdot q_1(|z-y|)}{|x-y|^{m-\frac{ph}{2}}\,|z-y|^{m-\frac{ph}{2}}}\,dy.$$

If we replace h by $ph/2$ in the proof of (8.12) we realize that we can choose q satisfying (8.8) such that the last integral above is majorized by $M\cdot I_{K_0}(\mu)$ where K_0 satisfies (8.1). Hence

$$a^{p'}< M\,\|g\|_{L^p}^{p'}\cdot \sup_{y\in R^m}\{u^{\mu}_{m-ph}(y)\}^{p'-2}\cdot I_{K_0}(\mu). \tag{8.16}$$

Now let F be an arbitrary closed subset of G_a. For μ we choose a capacitary distribution belonging to the kernel K_0 and the set F. This gives

$$\sup_{y\in R^m} u^{\mu}_{m-ph}(y)\leqslant M \sup_{y\in R^m} u^{\mu}_{K_0}(y)\leqslant M\cdot A\cdot\{C_{K_0}(F)\}^{-1},$$

where A is the constant in (1.4). Consequently (8.16) yields

$$a^{p'}< MA^{p'-2}\,\|g\|_{L^p}^{p'}\,\{C_{K_0}(F)\}^{-p'+1},$$

i.e.

$$C_{K_0}(F)< M_4\,a^{-p}\,\|g\|_{L^p}^{p}.$$

Since F is an arbitrary closed subset of G_a we conclude that (8.15) holds true.

By that Lemma 7 is proved and accordingly also Theorem 4 in the case $1\leqslant p\leqslant 2$.

The case $p>2$. We now prove Theorem 4 in the case $p>2$ and we consequently assume that (7.1) is valid. According to (7.1) we can choose ε_0 such that

$$C_{\alpha_0}(F_0)>0,\ \alpha_0=m-ph+\varepsilon_0,\ \alpha_0<m,\ \varepsilon_0>0. \tag{8.17}$$

Furthermore, we choose a positive measure ν_0 having the properties

$$\nu_0\geqslant 0,\ \nu_0(R^m)=1,\ S_{\nu_0}\subset F_0,\ I_{\alpha_0}(\nu_0)<\infty. \tag{8.18}$$

The proof is now analogous to the proof of the theorem in the case $1\leqslant p\leqslant 2$ if we replace the kernel K_0 by $r^{-\alpha_0}$ and use the following lemma instead of Lemma 7.[1]

Lemma 8. *Let the compact set F_0, the number α_0 and the measure ν_0 satisfy* (8.17) *and* (8.18). *Suppose that p and h, h a positive integer, are given numbers with $m-ph\geqslant 0$, $p>2$. If we define the function v_h by the same formula* (8.3) *as in Lemma* 7, *then exactly the same conclusions as in Lemma* 7 *are true, word by word, concerning v_h if we replace the kernel K_0 occurring in Lemma* 7 *by $r^{-\alpha_0}$.*

[1] Observe that the case when the function v_h is defined by (8.4) does not occur when $p>2$ since $m-2h>0$ if $p>2$, due to our assumption (7.1).

The proof of Lemma 8 is analogous to the proof of Lemma 7 for the case $p=2$. The differences are that the function w_h which is used in the proof of Lemma 7 now is defined by

$$w_h(x)=\int \frac{g(y)}{|x-y|^{\gamma}}\,dy,\ \gamma=m-h+\varepsilon_0',\ \varepsilon_0'>0,$$

where g is defined in the same way as in the proof of Lemma 7, and that we, instead of (8.11), use the fact that if ε_0' is chosen small enough and G_a as usual denotes the set of points x where $w_h(x)>a>0$, then

$$C_{\alpha_0}(G_a)<M\,a^{-p}\,\|g\|_{L^p}^p.$$

This fact is proved by using formula (6.4) in Lemma 4 and as the result can be extracted from du Plessis [15, § 5] and Fuglede [10, § 4] we omit the details.

By that Theorem 4 is completely proved.

We now use Theorem 4 to prove the following theorem on harmonic functions.

Theorem 5. *Let U be the open unit sphere in R^m, $m\geqslant 2$, and F_0 a closed subset of the boundary of U. Then $C_{m-2}(F_0)=0$ if and only if every function $f_0\in S(F_0)$ can be extended to a function u_0 which is harmonic in U, continuous in the closure of U, has a finite Dirichlet integral*

$$\int_U \sum_{i=1}^{m}\left(\frac{\partial u_0}{\partial x^i}\right)^2 dx<\infty, \tag{8.19}$$

and satisfies $u_0(x)=f_0(x)$ for every $x\in F_0$.

Proof. One half of the theorem has been proved in § 2, Remark 4. To prove the other half we assume that $C_{m-2}(F_0)>0$ and assert that there exists a function $f_0\in S(F_0)$ which cannot be extended to a function u_0 having the properties stated in the theorem.

Let u be a function which is continuous in the closure of U, harmonic in U and such that (8.19) holds with u_0 replaced by u. We define the function u^* by

$$u^*(x)=\begin{cases} u(x) \text{ if } |x|\leqslant 1,\\ u\left(\dfrac{x}{|x|^2}\right) \text{ if } |x|>1, \text{ where } \dfrac{x}{|x|^2}=\left(\dfrac{x^1}{|x|^2},\ \dots,\ \dfrac{x^m}{|x|^2}\right).\end{cases}$$

Our assertion clearly follows from Theorem 4 if we prove that $u^*\in A_1(L^2_{\mathrm{loc}})$. We first observe that all the partial derivatives of the first order of u^* are in L^2_{loc}. Furthermore, it is easy to realize that u^* has the property (AC). In fact, if a and b are given finite numbers, we have on almost every line with the direction x^i

$$\int_a^b \left(\frac{\partial u^*}{\partial x^i}\right)^2 dx^i<\infty,$$

which proves that u^* is absolutely continuous on almost every line with the direction x^i.

Since u^* has the property (AC) and all the partial derivatives of the first order of u^* are in L^2_{loc}, we conclude (compare Deny–Lions [7, p. 315]) that $u^* \in \mathcal{A}_1(L^2_{\text{loc}})$, which proves our theorem.

9. *The case $p=1$*

Theorem 3 gives, for $p=1$, a result in one direction of our investigation of the extension problem in the case when $C_\alpha(F_0)=0$ for some $\alpha<m-h$, and Theorem 4 a result in the other direction when $C_\alpha(F)>0$ for $\alpha=m-h$. We now give some additional results when $C_\alpha(F_0)=0$ for $\alpha=m-h$ but $C_\alpha(F_0)>0$ for every $\alpha<m-h$.

We start by the following example[1]:

Example 1. Let F_0 be the set of points $x=(x^1, \ldots, x^m)$ with $x^1=2^{-n}$ for $n=0, 1, 2, \ldots$ or $x^1=0$ and $0 \leqslant x^i \leqslant 1$, $i=2, \ldots, m$. We choose a sequence $\{a_i\}_0^\infty$ such that $\sum_0^\infty a_i$ is convergent and $\sum_0^\infty |a_i| = \infty$ and define a function $f_0 \in S(F_0)$ by putting, for $n=0, 1, \ldots,$

$$f_0(x)=\sum_0^n a_i \text{ if } x \in F_0,\ x^1=2^{-n}$$

and

$$f_0(x)=\sum_0^\infty a_i \text{ if } x \in F_0,\ x^1=0.$$

If f is an extension of f_0 to R^m which is continuous everywhere and infinitely differentiable on $\complement F_0$, we obtain, for $0 \leqslant x^i \leqslant 1$, $i=2, \ldots, m$,

$$\int_0^1 \left|\frac{\partial f(x^1, \ldots, x^m)}{\partial x^1}\right| dx^1 = \sum_0^\infty \int_{2^{-n-1}}^{2^{-n}} \left|\frac{\partial f(x^1, \ldots, x^m)}{\partial x^1}\right| dx^1 \geqslant \sum_0^\infty |a_{n+1}|$$

and hence

$$\int_0^1 \cdots \int_0^1 \left|\frac{\partial f(x^1, \ldots, x^m)}{\partial x^1}\right| dx^1 \ldots dx^m = \infty.$$

The given set F_0 is thus an example of a compact set which has α-capacity zero for $\alpha=m-1$, if $m \geqslant 2$, is enumerable if $m=1$, and has the following property: There exists a function $f_0 \in S(F_0)$ that cannot be extended to a function which is continuous everywhere, infinitely differentiable on $\complement F_0$ and such that all the partial derivatives of the first order of the extended function are in L^1_{loc}.

We use the above example to prove the following theorem:

Theorem 6. *Let h be a positive integer, $\alpha=m-h$ and $\alpha \geqslant 0$. Then there exists a compact set F_0 which satisfies $C_\alpha(F_0)=0$ if $\alpha>0$ and is enumerable if $\alpha=0$, and a function $f_0 \in S(F_0)$ that cannot be extended to a function f which is continuous everywhere in R^m, infinitely differentiable on $\complement F_0$ and such that all the partial derivatives of order h of f belong to L^1_{loc}.*

Proof. For $h=1$ the theorem is a consequence of Example 1. Hence we assume that $h>1$.

[1] The idea to use an example of this kind has been proposed to me by Dr. G. Aronsson.

Put $m^* = m-(h-1)$ and let R^{m^*} be the m^*-dimensional subspace of R^m which is defined by the points $(x^1, \ldots, x^{m^*}, 0, \ldots, 0)$ where $x^1, \ldots, x^{m^*}$ varies arbitrarily. According to Example 1 we can then choose a compact set F_0, $F_0 \subset R^{m^*}$, which satisfies $C_\alpha(F_0)=0$ if $\alpha>0$ and is enumerable if $\alpha=0$, and a function $f_0 \in S(F_0)$ which cannot be extended to a function g, which is defined and continuous in R^{m^*}, infinitely differentiable on $R^{m^*}-F_0$ and such that

$$\int_a^b \cdots \int_a^b \left| \frac{\partial g(x^1, \ldots, x^{m^*})}{\partial x^1} \right| dx^1 \ldots dx^{m^*} < \infty$$

for all finite values of a and b.

Suppose that we have chosen F_0 and f_0 as indicated and that there exists an extension f of f_0 having the properties stated for the function f in the theorem. We shall prove that this gives a contradiction by studying the derivative

$$\frac{\partial^h f}{\partial x^1 \partial x^{m^*+1} \partial x^{m^*+2} \ldots \partial x^m}. \tag{9.1}$$

For every choice of $x^1, \ldots, x^{m-1}$ such that the line through the point $(x^1, \ldots, x^{m-1}, 0)$ parallel to the x^m-axis does not intersect F_0, we get,

$$\int_0^{t_0} \frac{\partial^h f(x^1, \ldots, x^{m-1}, t)}{\partial x^1 \partial x^{m^*+1} \ldots \partial x^m} dt = \frac{\partial^{h-1} f(x^1, \ldots, x^{m-1}, t_0)}{\partial x^1 \partial x^{m^*+1} \ldots \partial x^{m-1}} - \frac{\partial^{h-1} f(x^1, \ldots, x^{m-1}, 0)}{\partial x^1 \partial x^{m^*+1} \ldots \partial x^{m-1}}. \tag{9.2}$$

From the discussion in § 5, 2° we conclude that for almost all t_0

$$\int_a^b \cdots \int_a^b \left| \frac{\partial^{h-1} f(x^1, \ldots, x^{m-1}, t_0)}{\partial x^1 \partial x^{m^*+1} \ldots \partial x^{m-1}} \right| dx^1 \ldots dx^{m-1} < \infty \tag{9.3}$$

for all finite values of a and b. (9.2) is valid for almost every line parallel to the x^m-axis as $C_\alpha(F_0)=0$, $\alpha=m-h$ and $h>1$. Hence we obtain from (9.2) and the assumption on f that (9.3) is true with $t_0=0$ for all finite values of a and b. If $h=2$ this gives a contradiction to the choices of F_0 and f_0. If $h>2$ we repeat the above procedure with (9.1) replaced by the second term of the right member of (9.2). An induction argument finally shows that

$$\int_a^b \cdots \int_a^b \left| \frac{\partial f(x^1, \ldots, x^{m^*}, 0, \ldots, 0)}{\partial x^1} \right| dx^1 \ldots dx^{m^*} < \infty$$

for all finite values of a and b which once more gives a contradiction to the choices of F_0 and f_0. This proves the theorem.

The type of sets F_0 used in Example 1 and in the proof of Theorem 6 has a rather complicated structure. As a comparison we consider a case when F_0 has a simple structure and still satisfies $C_\alpha(F_0)=0$, $\alpha=m-h$.

Example 2.[1] *Consider the case $m=2$ and let F_0 be the boundary of the open unit circle U. Then the following assertions are true:*

[1] This example is due to Professor L. Carleson. The example is related to results by Gagliardo [12].

1°. *Every function $f_0 \in S(F_0)$ can be extended to a function f which is defined and continuous everywhere in R^2, infinitely differentiable on $\complement F_0$, and such that the partial derivatives of the first order of f are in L^1.*

2°. *Let G be a non-decreasing postitve function defined in the interval $r \geqslant 0$ satisfying*

$$\lim_{r\to\infty} \frac{G(r)}{r} = \infty. \tag{9.4}$$

Then there exists a function $f_0 \in S(F_0)$ that cannot be extended to a function f which is continuous in the closure of U, infinitely differentiable in U and satisfies

$$\iint_U G(|\operatorname{grad} f|)\, dx^1\, dx^2 < \infty. \tag{9.5}$$

Proof of 1°. Let f_0 be a given function from $S(F_0)$. We introduce polar coordinates (r, θ) and consider f_0 as a function of θ. We write f_0 on the following form, where φ_n, for $n = 1, 2, \ldots$, is an infinitely differentiable function of one variable with period 2π and $\max_{0\leqslant\theta\leqslant 2\pi} |\varphi_n(\theta)| \leqslant M \cdot 2^{-n}$ for a certain constant M,

$$f_0(\theta) = \sum_1^\infty \varphi_n(\theta).$$

Put $\max_{0\leqslant\theta\leqslant 2\pi} |\varphi_n'(\theta)| = a_n$, $n = 1, 2, \ldots$, and choose $\{\varepsilon_n\}$, $1 > \varepsilon_n > 0$, $\lim_{n\to\infty} \varepsilon_n = 0$, such that $\sum_1^\infty \varepsilon_n a_n < \infty$.

We start by extending f_0 to U. Let $\{q_n\}$ be a sequence of functions defined in the interval $r \geqslant 0$ with q_n non-decreasing, infinitely differentiable and such that $q_n(r) = 0$ if $r \leqslant 1 - \varepsilon_n$ and $q_n(r) = 1$ if $r \geqslant 1$. As our extension of f_0 to U we choose the function f defined by

$$f(x^1, x^2) = \sum_1^\infty q_n(r)\varphi_n(\theta),\ x^1 = r\cos\theta,\ x^2 = r\sin\theta.$$

f is clearly infinitely differentiable in U and for a certain constant M_1 we have

$$\iint_U \left(\left|\frac{\partial f}{\partial x^1}\right| + \left|\frac{\partial f}{\partial x^2}\right|\right) dx^1\, dx^2 \leqslant \sum_{n=1}^\infty M_1 \int_0^1 \int_0^{2\pi} (|q_n'(r)\varphi_n(\theta)| + |q_n(r)\varphi_n'(\theta)|) r\, d\theta\, dr$$

$$\leqslant M_1 \sum_1^\infty (2\pi M \cdot 2^{-n} + 2\pi \cdot a_n \varepsilon_n) < \infty,$$

i.e. the partial derivatives of the first order of f are absolutely integrable over U. Analogously we can, of course, extend f_0 to the exterior of U with the extended function equal to zero outside a compact set, which proves 1°.

Proof of 2°. Suppose that f_0 is a function from $S(F_0)$ which can be extended to a function f having the properties stated for the function f in 2°. Let $f(r, \theta)$

be the value of f at the point with polar coordinates (r, θ). If we again consider f_0 as a function of θ we have $f(1,\theta)=f_0(\theta)$. We get, if $0<\varrho<1$ and $\delta>0$,

$$|f_0(\theta_0+\delta)-f_0(\theta_0)|\leqslant \int_\varrho^1\left|\frac{\partial f(r,\theta_0+\delta)}{\partial r}\right|dr+\int_{\theta_0}^{\theta_0+\delta}\left|\frac{\partial f(\varrho,\theta)}{\partial\theta}\right|d\theta+\int_\varrho^1\left|\frac{\partial f(r,\theta_0)}{\partial r}\right|dr.$$

Integration over θ_0 yields

$$\int_0^{2\pi}|f_0(\theta_0+\delta)-f_0(\theta_0)|\,d\theta_0\leqslant M_1\int_0^{2\pi}\int_\varrho^1|\operatorname{grad} f(r,\theta_0)|\,dr\,d\theta_0$$

$$+M_1\int_0^{2\pi}d\theta_0\int_{\theta_0}^{\theta_0+\delta}|\operatorname{grad} f(\varrho,\theta)|\,d\theta=I+II, \tag{9.6}$$

where M_1 is a constant. We divide the domain of integration of I into two parts, one of which consists of those points where $|\operatorname{grad} f(r,\theta_0)|>(1-\varrho)^{-1/2}$. Using (9.4) and (9.5) we obtain, with a constant M_2 only depending on f and a number $\varepsilon(\varrho)$ only depending on ϱ, $\varepsilon(\varrho)\to 0$, $\varrho\to 1$,

$$I<M_2(\varepsilon(\varrho)+\sqrt{1-\varrho}). \tag{9.7}$$

To estimate II we observe that

$$II=M_1\delta\int_0^{2\pi}|\operatorname{grad} f(\varrho,\theta)|\,d\theta,$$

and, by (9.5), this quantity is less than $M_3\delta(1-\varrho)^{-1}$ for a certain value $\varrho=\varrho_n$ where $2^{-n-1}\leqslant 1-\varrho_n\leqslant 2^{-n}$, $n=1,2,\ldots$. Using this and (9.7) it is easy to realize that we can choose ϱ depending on δ with $\varrho\to 1$ when $\delta\to 0$, so that we obtain a modulus of continuity $t(\delta)$, $t(\delta)\to 0$ when $\delta\to 0$, for the left member of (9.6), a modulus of continuity which is independent of f_0:

$$\int_0^{2\pi}|f_0(\theta+\delta)-f_0(\theta)|\,d\theta\leqslant Mt(\delta), \quad \textit{for every} \quad \delta>0. \tag{9.8}$$

M is a constant depending on f_0 and f. Thus (9.8) holds true for all the functions from $S(F_0)$ which can be extended to functions having the properties stated for the function f in 2°. It is, however, not hard to realize that there exists a function $f_0\in S(F_0)$ which does not satisfy (9.8).

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