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A problem of Newman on the eigenvalues of operators of convolution type

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In [1] Newman has studied the problem of uniqueness of the class of equations

$$\lambda F(x) - \int_E F(t)\,K(x-t)\,dt = G(x) \quad \text{for} \quad x \in E$$

under the restriction that K has compact support. However, the result in [1] is true also without that restriction:

Theorem 1. *Let G be a locally compact abelian group with Haar measure dt and let $K(x) \in L^1(G)$. Then for any measurable set E*

$$\lambda F(x) - \int_E F(t)\,K(x-t)\,dt = 0 \quad \text{for} \quad x \in E \quad \text{and} \quad F \in L^\infty \Rightarrow F \equiv 0$$

if $\lambda \notin H_k = \overline{CH\{\hat{K}(\xi) \mid \xi \in \hat{G}\}}$ = the closed convex hull of the values assumed by the Fourier transform $\hat{K}$ of K.

An equivalent theorem is obtained by looking at the class of operators on L^∞

$$\mathbf{K}_E F = \begin{cases} F \ast K & \text{for} \quad x \in E, \\ 0 & \text{for} \quad x \notin E, \end{cases}$$

where the kernel $K \in L^1$. The theorem then states that for any measurable set E, $\mathbf{K}_E$ has all its eigenvalues inside H_K. Thus H_K is a bound, uniform in E, for the eigenvalues of $\mathbf{K}_E$. The question of the "best" uniform bound has not been settled. The eigenvalue problem when $G = \mathbf{R}$ or $\mathbf{Z}$ has been solved in the cases $E = (-\infty, \infty)$, $(-\infty, 0)$ and $(0, \infty)$ (see e.g. Krein [2]). Together these eigenvalues form the set

$$A_K = \{\hat{K}(\xi) \mid \xi \in \hat{G}\} \cup \{\lambda \mid \operatorname{ind}(\lambda - \hat{K}) = (2\pi)^{-1} \int_{-\infty}^{\infty} d_\xi \arg(\lambda - \hat{K}(\xi)) \neq 0\},$$

i.e. the set of points on or "inside" the curve described by the Fourier transform $\hat{K}$. Consequently, if M_K is the best uniform bound for the eigenvalues then

$A_K \subseteq M_K \subseteq H_K$. In [1] there is an example where $G = \mathbf{Z}$ and $\mathbf{K}_E$ for suitable E has eigenvalues outside A_K. In that example M_K is strictly between A_K and H_K.

For the proof of Theorem 1 the following expression for the integral equation in question will be useful

$$|F(x)|^2 = \overline{F(x)}\,(F * K)(x) \quad \text{for all} \quad x \in G. \tag{1}$$

Proof. It is sufficient to study the case $\lambda = 1$. Following [1] we observe that $1 \notin H_K$ implies the existence of a complex number α such that

$$\operatorname{Re}(\alpha(1 - \hat{K}(\xi))) \geqslant 1 \quad \text{for all} \quad \xi \in \hat{G}. \tag{2}$$

Let x_0 denote an arbitrary point of G and let V be a compact symmetric subset of G such that

$$\int_{\complement(V)} |K(x)|\, dx < \tfrac{1}{2}|\alpha|^{-1}.$$

For abbreviation we define $f(x) = \chi_n(x) \cdot F(x)$ where χ_n is the characteristic function of $x_0 + V^n = \{x_0 + \sum_{i=1}^n x_i \mid x_i \in V\}$. With these notations Parseval's theorem gives us

$$|\alpha|^{-1} \int_{x_0+V^n} |F(x)|^2 dx = |\alpha|^{-1} \int_G |f(x)|^2 dx = |\alpha|^{-1} \int_{\hat{G}} |\hat{f}(\xi)|^2 d\xi.$$

From the inequality (2) and from Parseval's theorem once again it follows that

$$|\alpha|^{-1} \int_{\hat{G}} |\hat{f}(\xi)|^2 d\xi \leqslant \left| \int_{\hat{G}} |\hat{f}(\xi)|^2 (1 - \hat{K}(\xi))\, d\xi \right|$$
$$= \left| \int_G |f(x)|^2 - \overline{f(x)}\,(f * K)(x)\, dx \right| = \left| \int_{x_0+V^n} |F(x)|^2 - \overline{F(x)} \int_{x_0+V^n} F(t)\, K(x-t)\, dt\, dx \right|.$$

Now by the relation (1) this equals

$$\left| \int_{x_0+V^n} \overline{F(x)} \int_{x_0+\complement(V^n)} F(t)\, K(x-t)\, dt\, dx \right|.$$

After a change of variables ($y = x - t$; $x = x$) we can use Fubini's theorem to get

$$\left| \iint_{\substack{x \in x_0+V^n \\ y-x \in -x_0+\complement(V^n)}} \overline{F(x)}\, F(x-y)\, K(y)\, dy\, dx \right| = \left| \iint_{\substack{x \in x_0+E_y \\ y \in V^n+\complement(V^n)}} \overline{F(x)}\, F(x-y)\, K(y)\, dx\, dy \right|$$
$$\leqslant \int_G |K(y)| \int_{x_0+E_y} |F(x)\, F(x-y)|\, dx\, dy,$$

where $E_y = V^n \cap \{y + \complement(V^n)\}$. Thus we have arrived at the inequality

$$\int_{x_0+V^n} |F(x)|^2 dx \leqslant |\alpha| \int_G |K(y)| \int_{x_0+E_y} |F(x)\, F(x-y)|\, dx\, dy. \tag{3}$$

Define
$$\varphi_{x_0}(n)=\int_{x_0+V^n}|F(x)|^2dx$$
and
$$\psi(n)=\sup_{x_0\in G}\varphi_{x_0}(n).$$
Thus (3) yields
$$\varphi_{x_0}(n)\leqslant|\alpha|\int_V|K(y)|\,dy\int_{x_0+E_y}|F(x)\,F(x-y)|\,dx\,dy$$
$$+|\alpha|\int_{\complement(V)}|K(y)|\int_{x_0+E_y}|F(x)\,F(x-y)|\,dx\,dy=I_1+I_2.$$
Now $E_y\subseteq V^n$ so
$$I_2\leqslant|\alpha|\cdot\int_{\complement(V)}|K(y)|\,\psi(n)\,dy<\tfrac{1}{2}\cdot\psi(n).$$
Schwarz's inequality gives us
$$\int_{x_0+E_y}|F(x)\,F(x-y)|\,dx\leqslant\left\{\int_{x_0+E_y}|F(x)|^2dx\int_{x_0-y+E_y}|F(x)|^2dx\right\}^{\frac{1}{2}}. \tag{4}$$
But for $\quad y\in VE_y=V^n\cap\{y+\complement(V^n)\}\subseteq V^n\setminus V^{n-1}\subseteq V^{n+1}\setminus V^{n-1}$

and $\quad -y+E_y=\complement(V^n)\cap\{-y+V^n\}\subseteq V^{n+1}\setminus V^n\subseteq V^{n+1}\setminus V^{n-1}$

so the right member of (4) is less than $\varphi_{x_0}(n+1)-\varphi_{x_0}(n-1)$. Therefore
$$\varphi_{x_0}(n)<|\alpha|\cdot\int_V|K(y)|\,dy\,\{\varphi_{x_0}(n+1)-\varphi_{x_0}(n-1)\}+\tfrac{1}{2}\psi(n)$$
$$\leqslant\|K\|_1\{\varphi_{x_0}(n+1)-\varphi_{x_0}(n-1)\}+\tfrac{1}{2}\psi(n).$$
But as $\varphi_{x_0}(n)$ and $\psi(n)$ are increasing this yields
$$(\|K\|_1+1)\,\varphi_{x_0}(n-1)<\|K\|_1\,\varphi_{x_0}(n+1)+\tfrac{1}{2}\psi(n+1).$$
Varying x_0 we get
$$\psi(n-1)<\frac{\|K\|_1+\frac{1}{2}}{\|K\|_1+1}\cdot\psi(n+1)=\mu\cdot\psi(n+1)\quad(\mu<1).$$
So
$$\psi(2)<\mu^{n-1}\psi(2n). \tag{5}$$
We need the following simple lemma of Newman [1].

Lemma. *Let V be a compact subset of G. Then there exist constants c and d such that*
$$m(V^n)\leqslant c\cdot n^d.$$

Applying it in this situation we get $\psi(2n) \leqslant C \cdot M \cdot (2n)^d$ and thus from (5)

$$\psi(2) < C \cdot M \cdot \mu^{n-1} \cdot (2n)^d \to 0 \quad (n \to \infty).$$

From the definition of ψ we get that $F(x) \equiv 0$ and the theorem is proved.

Remark. The proof is valid with a minor modification also in the case $F \in L^p(G)$, $2 \leqslant p < \infty$. The only place where we used $|F(x)| \leqslant M$ was to prove that $\psi(2n) \leqslant C \cdot M \cdot (2n)^d$. We can write $|F|^2 \in L^{p/2}$, $|F|^2 = F_1 + F_2$ where $F_1 \in L^1$ and $F_2 \in L^\infty$.
Then $\psi(2n) \leqslant C \cdot (2n)^d \cdot \|F_2\|_\infty + \|F_1\|_1$ and the result follows just as above.

The only remaining case of interest is $F \in L^p$, $1 \leqslant p < 2$. It is reduced to the above by the following

Lemma. *If* $|F(x)|^2 = F(x) \cdot (F * K)(x)$ *where* $K \in L^1(G)$ *and* $F \in L^p(G)$, $1 \leqslant p < 2$ *then* $F \in L^2(G)$.

Proof. Suppose $p = 1$. Other cases are treated similarly. It follows that $|F| \leqslant |F| * |K|$. Now K can be written $K = K_1 + K_2$ where $K_1 \in L^1$, $K_2 \in L^1 \cap L^2$ and $\|K_1\|_1 < \varepsilon < \frac{1}{2}$, $\|K_2\|_2 = M < \infty$. Then

$$|F| \leqslant |F| * |K_1| + |F| * |K_2|. \tag{6}$$

Using this estimate of $|F|$ in the first term of the right member we get a new estimate $|F| < g_1 + h_1$ where $g_1 = |F| * |K_1| * |K_1|$

and
$$h_1 = |F| * |K_2| * |K_1| + |F| * |K_2|.$$

We repeat this procedure on the term $|F| * |K_1|$ in (6) to get successively new estimates $|F| \leqslant g_i + h_i$ where

$$g_{i+1} = g_i * |K_1| \quad \text{and} \quad h_{i+1} = h_i * |K_1| + |F| * |K_2|.$$

Therefore $\|g_{i+1}\|_1 < \varepsilon \|g_i\|$ and $\|h_{i+1}\|_2 < \varepsilon \|h_i\|_2 + M \|F\|_1$. It follows that $\|g_i\|_1 < \varepsilon^{i+1} \cdot \|F\|_1$ and $\|h_i\|_2 < 2M \cdot \|F\|_1$ and we get

$$\|F - h_i'\|_1 < \varepsilon^{i+1} \cdot \|F\|_1 \to 0 \quad \text{as} \quad i \to \infty \quad \text{where} \quad \|h_i'\|_2 \leqslant \|h_i\| < 2M \|F_1\|.$$

We can choose a subsequence $\{h_{i_k}\}$ such that $h_{j_k}(x) \to |F(x)|$ a.e. The conclusion that $F \in L^2$ now follows from Fatou's lemma:

$$\int_G |F|^2 dx \leqslant \liminf \int_G |h_{i_k}|^2 dx \leqslant (2M \|F_1\|)^2.$$

Thus we have proved the more general form of Theorem 1.

Theorem 2. *If* $K \in L^1(G)$ *then the operator* $\mathbf{K}_E$ *defined in any one of the spaces* $L^p(G)$, $1 \leqslant p \leqslant \infty$, *by*

$$\mathbf{K}_E F = K * F \quad \text{for} \quad x \in E, \ \mathbf{K}_E F = 0 \quad \text{for} \quad x \notin E$$

has no eigenvalue outside the set H_K.

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