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Some remarks about the limit point and limit circle theory

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ABSTRACT

Let L be a formally selfadjoint differential operator and p a real-valued function, both on $a \leqslant x < \infty$. The deficiency indices are the numbers of solutions of $Lu = \lambda pu$ for $\operatorname{Im} \lambda > 0$ and for $\operatorname{Im} \lambda < 0$ which have a certain regularity at $x = \infty$. (A) If $p(x) \geqslant 0$ this regularity means that the integral of $p(x)|u|^2$ converges at infinity. (B) If p changes its sign for arbitrarily large values of x but L has a positive definite Dirichlet integral it is natural to relate the regularity to this integral. Weyl's classical study of the deficiency indices is reviewed for (A) with the help of elementary theory of quadratic forms. Individual bounds are found for the deficiency indices also when L is of odd order. It is then indicated how the method carries over to (B).

0. Introduction

If q is a continuous real-valued function on $a \leqslant x < \infty$ and λ a non-real parameter, then

$$-u'' + qu = \lambda u \tag{0.1}$$

has at least one solution which is square-integrable on $a \leqslant x < \infty$. This result was deduced by H. Weyl in his fundamental treatise [3] on spectral properties of ordinary differential equations. If the equation is replaced by

$$-u'' + qu = \lambda pu, \tag{0.2}$$

where p and q are real-valued functions, Weyl's method can be applied even if p takes both positive and negative values but has a definite sign for sufficiently large values of x. If the last assumption is abandoned, but q is non-negative, it seems natural to relate the behaviour at infinity of the solutions to the Dirichlet integral

$$\int_a^\infty |u'|^2 + q|u|^2.$$

As a base for such considerations we shall present Weyl's method in its dependence on quadratic boundary forms. For (0.1) Weyl proved that given any solution ψ one can find another one, v, satisfying a boundary condition at $x = a$ (for instance $v(a) = 0$) such that $\psi - v$ becomes square-integrable over $a \leqslant x < \infty$. This result can be generalized to any formally selfadjoint equation of even order

(Everitt [1]). But for equations of odd order a set of solutions satisfying a boundary condition at $x=a$ is generally too small for the purpose. However, according to the elementary theory of quadratic forms one can instead use a linear set of solutions determined by a quadratic boundary *inequality* at $x=a$. In this way Weyl's result can be generalized to formally selfadjoint equations of arbitrary order.

The numbers of square-integrable solutions of $Lu=\lambda u$ on $0\leqslant x<\infty$ for $\operatorname{Im}\lambda>0$ and for $\operatorname{Im}\lambda<0$ are the deficiency numbers of the selfadjoint operator L. A complete hilbert space treatment of *real* selfadjoint differential operators of arbitrary (even) order was first given by I. M. Glazman [2]. In a footnote Glazman observed that the order (even or odd) of a differential operator is a lower bound for the sum of its deficiency numbers but gave no individual estimates.

After the presentation of Weyl's method we indicate how it carries over to $Lu=\lambda pu$ when the integral

$$\int_a^\infty p|u|^2$$

is indefinite in an essential way but the differential operator has a positive definite Dirichlet integral.

1. The differential operator

Let the linear differential operator

$$L=\sum_{j=0}^{M}A_j(x)\,D^j,\quad D=\frac{1}{\sqrt{-1}}\frac{d}{dx}$$

be defined on an open interval I containing $a\leqslant x<\infty$. The coefficients $A_j(x)$ are assumed continuous and so regular that the adjoint

$$L^*=\sum_{j=0}^{M}D^j\,\overline{A_j(x)}$$

can be formed. Let L be formally selfadjoint i.e. let L and L^* coincide. Then Green's formula reads

$$\int_\alpha^\beta \bar{v}\,Lu-u\,\overline{Lv}=i\overset{\beta}{\underset{\alpha}{[}}k(u,v)]. \tag{1.1}$$

Here the "boundary form" $k(u,v)$ is linear in u and hermitean. Thus k is a quadratic form with coefficients depending on x. By computation

$$k(u,v)=\sum_{j=0}^{m-1}(\overline{D^jv}\,B_ju+D^ju\,\overline{B_jv})-A_{2m+1}\overline{D^mv}\,D^mu \tag{1.2}$$

when $M=2m$ or $M=2m+1$, where m is an integer. In the case $M=2m$ it is understood that $A_{2m+1}=0$ in (1.2). The B_ju $(j=0,1,\ldots,(m-1))$, are linear in u. The expression for $k(u,u)$ can be written

$$k(u,u)=\sum_{j=0}^{m-1}\tfrac{1}{2}|D^j u+B_j u|^2-\sum_{j=0}^{m-1}\tfrac{1}{2}|D^j u-B_j u|^2-A_{2m+1}|D^m u|^2 \tag{1.3}$$

with $A_{2m+1}=0$ if $M=2m$.

On a finite dimensional linear set of functions the quadratic form k has a certain signature $[\pi,\nu]$ or $[\pi_x,\nu_x]$ for every x. This means that on the linear set $k(u,u)$ can be written as a sum of π squares (of absolute values) minus ν squares of altogether $\pi+\nu$ linearly independent linear forms. From (1.3) we conclude that

$$\pi\leqslant m^+,\quad \nu\leqslant m^-, \tag{1.4}$$

where

$$m^+=m^-=m \quad \text{if} \quad M=2m, \tag{1.5}$$

$$m^+=m+1,\quad m^-=m \quad \text{if} \quad M=2m+1 \quad \text{and} \quad A_{2m+1}(x)<0, \tag{1.6}$$

$$m^+=m,\quad m^-=m+1 \quad \text{if} \quad M=2m+1 \quad \text{and} \quad A_{2m+1}(x)>0. \tag{1.7}$$

Observe that in all cases $m^+ + m^- = M$.

2. Solution space

We shall study solutions of the differential equation

$$Lu=\lambda pu \tag{2.1}$$

on $a\leqslant x<\infty$. Here p denotes a real-valued function which for instance is continuous. It should not vanish identically. In our presentation of Weyl's method (§§ 2–5) we suppose

$$p(x)\geqslant 0 \quad \text{for} \quad a\leqslant x<\infty. \tag{2.2}$$

In (2.1) λ is a non-real parameter. To fix the ideas we assume in general that

$$\operatorname{Im}\lambda>0. \tag{2.3}$$

The highest order coefficient of a formally selfadjoint differential operator is always real. We assume $A_M(x)\neq 0$ for all x in $a\leqslant x<\infty$. Because of this the equation (2.1) has M linearly independent solutions on $a\leqslant x<\infty$ which form the *solution space*

$$l=\{u\,|\,Lu=\lambda pu\}$$

of dimension M. A solution cannot vanish on an interval without being 0 everywhere.

3. Signature of the boundary form

If $u\in l$ i.e. if u satisfies $Lu=\lambda pu$ and if we put $v=u$ in the Green's formula (1.1), we obtain

$$k_\beta(u,u)-k_\alpha(u,u)=\frac{\lambda-\bar\lambda}{i}\int_\alpha^\beta p\,|u|^2. \tag{3.1}$$

Let us assume that the interval $\alpha \leqslant x \leqslant \beta$ is so large that p does not vanish identically on this interval. Because of (2.3) the factor $(\lambda - \bar{\lambda})/i$ is positive. Thus the right-hand side of (3.1) is positive definite.

The signatures of k_α and k_β on l are $[\pi_\alpha, \nu_\alpha]$, $[\pi_\beta, \nu_\beta]$. The left-hand side of (3.1) can be written as a sum of $\pi_\beta + \nu_\alpha$ squares minus $\pi_\alpha + \nu_\beta$ squares. We shall first prove that $M \leqslant \pi_\beta + \nu_\alpha$. Assume $\pi_\beta + \nu_\alpha < M$. If we equate to 0 the $\pi_\beta + \nu_\alpha$ squares in $k_\beta(u,u) - k_\alpha(u,u)$, we obtain a linear subspace of l which because of $\pi_\beta + \nu_\alpha < M$ has a positive dimension. But this is contradicted by the fact that (3.1) implies $u=0$ for every u belonging to the subspace. Thus $M \leqslant \pi_\beta + \nu_\alpha$.

But according to (1.4), our last inequality can be completed to $M \leqslant \pi_\beta + \nu_\alpha \leqslant m^+ + m^- = M$. Hence, $\pi_\beta = m^+$ and $\nu_\alpha = m^-$. This is true for arbitrary values of α and β in the interval I, where L is defined. This shows *that the signature* $[\pi, \nu]$ *of* k *on* l *is independent of* x *and that*

$$\pi = m^+, \quad \nu = m^-,$$

where m^+ *and* m^- *are defined in* (1.5)–(1.7).

4. Weyl's method

Let $a \leqslant x \leqslant b$ be a finite interval. The signature of k_a on l is $[m^+, m^-]$. According to the theory of quadratic forms there exist linear subspaces of l of dimension m^+, but of no higher dimension, on which k_a is positive definite. Let l_a^+ be such a subspace

$$\dim l_a^+ = m^+, \quad k_a(u,u) \geqslant 0 \quad \text{on } l_a^+, \quad \text{equality only if } u=0. \tag{4.1}$$

Similarly, since the signature of k_b on l is $[m^+, m^-]$ we can chose a linear subspace l_b^- of l such that

$$\dim l_b^- = m^-, \quad k_b(u,u) \leqslant 0 \quad \text{on } l_b^-, \quad \text{equality only if } u=0. \tag{4.2}$$

For $\alpha = a$, $\beta = b$ the formula (3.1) takes the form

$$k_b(u,u) - k_a(u,u) = \frac{\lambda - \bar{\lambda}}{i} \int_a^b p\,|u|^2. \tag{4.3}$$

If $u \in l_b^- \cap l_a^+$ the left-hand side of (4.3) is non-positive according to (4.1) and (4.2). Since the right-hand side of (4.3) is non-negative this gives $u=0$. Thus $l_b^- \cap l_a^+ = \{0\}$. But $\dim l_b^- + \dim l_a^+ = m^- + m^+ = M$ which is the dimension of l. It so follows that the solution space equals the direct sum

$$l = l_b^- \dotplus l_a^+. \tag{4.4}$$

Let $\psi \in l$. On account of (4.4) we write $\psi = u + v$, where $u \in l_b^-$ and $v \in l_a^+$. Insert $u = \psi - v$ in (4.3). Since $k_b(u,u) \leqslant 0$ it follows that

$$k_a(\psi - v, \psi - v) + \frac{\lambda - \bar{\lambda}}{i} \int_a^b p\,|\psi - v|^2 \leqslant 0. \tag{4.5}$$

Let $\varphi_1, \varphi_2, \ldots, \varphi_{m^+}$ be a base in l_a^+ and write v as a linear combination

$$v = \sum_{j=1}^{m^+} t_j \varphi_j.$$

For all v in l_a^+ we have $k_a(v, v) \geqslant 0$ with equality only if $v = 0$. Thus, when $\operatorname{Im} \lambda > 0$, the part of the left-hand side of (4.5) which is quadratic in $T = t_1, t_2, \ldots, t_{m^+}$ is positive definite on l_a^+.

Geometrically (4.5) tells us that T belongs to the interior or the boundary of a m^+-dimensional unitary ellipsoid E_b. Our reasoning shows that the ellipsoid defined by (4.5) is not empty since there exists a function v for which (4.5) is satisfied. If $b' > b$ the definition by (4.5) of the ellipsoids E_b shows that $E_{b'} \subset E_b$. Thus there is at least one point T which belongs to all the compact sets E_b. Let $v = t_1 \varphi_1 + t_2 \varphi_2 + \ldots + t_{m^+} \varphi_{m^+}$ be the function in l_a^+ which corresponds to such a T. For this v the inequality (4.5) is satisfied for all values of b. Consequently

$$k_a(\psi - v, \psi - v) + \frac{\lambda - \bar{\lambda}}{i} \int_a^\infty p\,|\psi - v|^2 \leqslant 0.$$

It follows that
$$\int_a^\infty p\,|\psi - v|^2 < \infty,$$

i.e. that $\psi - v \in \mathfrak{L}^2(a, \infty; p)$. We have proved that *every solution* ψ *of* $Lu = \lambda p u$ *can be "compensated" by a solution* v *in* l_a^+ *so that*

$$\psi^* = \psi - v \in \mathfrak{L}^2(a, \infty; p).$$

Let $\psi_1, \psi_2, \ldots, \psi_{m^-}$ be a completion of $\varphi_1, \varphi_2, \ldots, \varphi_{m^+}$ to a base of l. Then the compensated functions

$$\psi_j^* = \psi_j - v_j \in \mathfrak{L}^2(a, \infty; p)$$

$(j = 1, 2, \ldots, m^-)$, also form a base together with $\varphi_1, \varphi_2, \ldots, \varphi_{m^+}$.

Thus, $Lu = \lambda p u$ *has at least* m^- *linearly independent solutions in* $\mathfrak{L}^2(a, \infty; p)$. *Under the condition* $\operatorname{Im} \lambda > 0$ *we have* $m^- = m$ *if the order of* L *is* $M = 2m$, *or if* $M = 2m + 1$ *and* $A_{2m+1}(x) < 0$, *while* $m^- = m + 1$ *if* $M = 2m + 1$ *and* $A_{2m+1}(x) > 0$. *If* $\operatorname{Im} \lambda < 0$ *the inequalities for* $A_{2m+1}(x)$ *should be reversed.*

5. Compensating functions and boundary conditions

The signature of k_a on l is $[m^+, m^-]$, where $m^+ + m^- = \dim l$. According to the theory of quadratic forms there exist linear subspaces of l of dimension $\min(m^+, m^-)$, but of no higher dimension, such that $k_a(u, v) = 0$ for every u and v belonging to the subspace. Let l_a^0 be such a *maximal nullspace* with respect to k_a. Thus $k_a(u, u) = 0$ if $u \in l_a^0$. If $u \in l_b^- \cap l_a^0$ (compare (4.2)) the relation (4.3) proves that $u = 0$.

The dimension of l_b^- is m^-. If M is even, or if $M = 2m + 1$ and $A_{2m+1}(x) > 0$, then $m^+ \leqslant m^-$ and $\dim l_a^0 = m^+$ (provided $\operatorname{Im} \lambda > 0$). In these cases

$$l = l_b^- \dotplus l_a^0. \tag{5.1}$$

But if (5.1) holds true we can (as Weyl actually did) use l_a^0 instead of l_a^+ in the reasoning of section 4 (after formula (4.4)). Thus if $M=2m$, or if $M=2m+1$ and $A_{2m+1}(x)>0$, we can compensate any solution ψ of $Lu=\lambda pu$ by means of a solution v in l_a^0 so that $\psi-v$ becomes square-integrable with respect to p.

If $M=2m+1$ and $A_{2m+1}(x)<0$ the relation (5.1) is not valid and a set l_a^0 is generally not sufficient to compensate every solution ψ to square-integrability over $a\leqslant x<\infty$.

A boundary condition for L on $a\leqslant x\leqslant b$ can be defined as a maximal null-space of the quadratic form

$$k_b(u,v)-k_a(u,v).$$

If M is even, any two subspaces of l which are maximal nullspaces l_a^0 and l_b^0 with respect to k_a and k_b determine a boundary condition. This is not true when M is odd but it seems anyhow fit to call a maximal nullspace with respect to k_a *a boundary condition at* $x=a$ both when M is even and odd. Under this agreement the result of the present section can be formulated as follows.

If $M=2m$ *or if* $M=2m+1$ *and* $A_{2m+1}(x)>0$, *we can use solutions which satisfy a boundary condition at* $x=a$ *to compensate any solution of* $Lu=\lambda pu$ *to square-integrability on* $a\leqslant x<\infty$ *with respect to* p. *If* $M=2m+1$ *and* $A_{2m+1}(x)<0$ *this is not generally possible. These statements are true when* $\operatorname{Im}\lambda>0$. *The changes when* $\operatorname{Im}\lambda<0$ *are obvious.*

Remark I. One might want wider possibilities to find spaces of restricted dimension which are sufficient to compensate any solution to square-integrability. For this we can note that the maximal dimension of subspaces of l on which k_a is positive (not necessarily definite) is also m^+. A space $l_a^{+,0}$ of this type is sufficient for the compensation.

Remark II. For $M=2$ the ellipsoids E_b are one-dimensional i.e. of the form

$$|t-c(b)|\leqslant\varrho(b).$$

They are circles in the complex t-plane and coincide with the classical Weyl circles. If $b'>b$ the circle $|t-c(b')|\leqslant\varrho(b')$ is contained in the circle $|t-c(b)|\leqslant\varrho(b)$. If for $b\to\infty$ the circles shrink to a single point we are in Weyl's *limit point case*. If they shrink to a limit circle we are in the *limit circle case* and all solutions are square-integrable. If $M>2$ the classification is similar but more elaborate.

6. Weyl's method for operators with positive Dirichlet integrals

A real, formally selfadjoint linear differential operator can be written

$$L=\sum_{j=0}^{m}D^j a_j(x)\,D^j$$

with real-valued functions $a_j(x)$. As in § 1, let L be given and sufficiently regular on an open interval I containing $a\leqslant x<\infty$. By partial integrations

$$\int_\alpha^\beta \bar v\,Lu=i\,\overset{\beta}{\underset{\alpha}{[B(u,v)]}}+\int_\alpha^\beta D(u,v), \tag{6.1}$$

where
$$D(u,v)=\sum_{j=0}^{m}a_j(x)\,D^ju\,\overline{D^jv} \tag{6.2}$$

and
$$B(u,v)=\sum_{j=0}^{m-1}\overline{D^jv}\,B_ju.$$

The expressions B_ju are linear in u. The Dirichlet form (6.2) is evidently hermitean. If for instance $a_j(x)\geqslant 0$ $(j=0,1,\ldots,m)$, the integral of $D(u,u)$ from α to β is non-negative and increases when the interval $\alpha\leqslant x\leqslant\beta$ increases. We assume that this is so and also that if the interval $\alpha\leqslant x\leqslant\beta$ is sufficiently large, then

$$\int_\alpha^\beta D(u,u)=0$$

holds true for a "regular" u only if $u(x)=0$ in a subinterval of $\alpha\leqslant x\leqslant\beta$. If these conditions are fulfilled we say that L has a positive definite Dirichlet integral.

Interchanging u and v in (6.1) and taking the complex conjugate we obtain

$$\int_\alpha^\beta u\overline{Lv}=-i\,\overline{[B(v,u)]}_\alpha^\beta+\int_\alpha^\beta D(u,v). \tag{6.3}$$

Let $Lu=\lambda pu$ and take $v=u$. Then (6.1) and (6.3) give

$$\lambda\int_\alpha^\beta p\,|u|^2=i\,[B(u,u)]_\alpha^\beta+\int_\alpha^\beta D(u,u), \tag{6.4}$$

$$\bar\lambda\int_\alpha^\beta p\,|u|^2=-i\,\overline{[B(u,u)]}_\alpha^\beta+\int_\alpha^\beta D(u,u). \tag{6.5}$$

From (6.4), (6.5) it follows that

$$h_\beta(u,u)-h_\alpha(u,u)=\frac{\lambda-\bar\lambda}{i}\int_\alpha^\beta D(u,u), \tag{6.6}$$

where
$$h(u,v)=\sum_{j=0}^{m-1}(B_ju\,\overline{\lambda D^jv}+\lambda D^ju\,\overline{B_jv}). \tag{6.7}$$

For the signature $[\pi,\nu]$ of the evidently quadratic form h, considered on any finite dimensional linear set, it follows from (6.7) (see § 1) that

$$\pi\leqslant m,\quad \nu\leqslant m.$$

If $\operatorname{Im}\lambda>0$, the equality (6.6) shows that $h_\beta(u,u)-h_\alpha(u,u)$ is positive definite on the solution space
$$l=\{u\,|\,Lu=\lambda pu\}.$$

A reasoning similar to the one performed in § 3 proves that the signature of h on l is independent of x and always equals $[m,m]$.

Repeating, but with (6.6) instead of (4.3), what was done in §§ 4 and 5 we arrive at the result *that every solution ψ of $Lu = \lambda pu$ on $a \leqslant x < \infty$ can be compensated by a solution v belonging to a maximal nullspace for h_a, so that $u = \psi - v$ gives a finite value to*

$$\int_a^\infty D(u, u).$$

That v belongs to a maximal nullspace for h_a can be considered as a kind of boundary condition for v at $x = a$. But in general the boundary conditions at $x = a$ determined by h_a do not coincide with the boundary conditions of § 5 which are related to k_a. Dirichlet's boundary condition at $a(D^j u(a) = 0, j = 0, 1, \ldots, (m-1))$ is a boundary condition at a, both with respect to k_a and h_a. Observe that a maximal nullspace in l with respect to h_a has dimension m. *If $\mathcal{H}$ denotes a hilbert space with norm*

$$|u| = \left(\int_a^\infty D(u, u)\right)^{\frac{1}{2}}$$

it follows that $Lu = \lambda pu$ has at least m solutions in $\mathcal{H}$. This is true if L is real formally selfadjoint of order $2m$ and has a positive definite Dirichlet integral. The result is essentially independent of p.

7. Conclusions

Clearly questions related to Weyl's result have their similarities in a theory about differential operators with positive Dirichlet integrals. Some of these questions are easily settled, others give rise to certain difficulties. It is the intention to discuss some of them in a forthcoming paper from the Department.

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