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Convolutions of random functions

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1. Introduction

Let a probability space $[X, B, P]$ be given and denote the set of real numbers by R. Let L^p be the class of all random variables ξ with $E|\xi|^p < +\infty$, $p \geq 1$ and the norm $\|\xi\|_p = E^{1/p}|\xi|^p$. A random function is said to belong to L^p if $\xi(t) \in L^p$ for $t \in R$. We shall consider the topology in L^p given by this norm and deal with limits, continuity, etc., with respect to it. Then we talk about limits, continuity (L^p) or L^p-limits, L^p-continuity, etc.

A random function ξ is called a.s. non-decreasing if it is real and $\xi(t_1) \leq \xi(t_2)$ a.s. for any pair (t, t_2), $t_1 \leq t_2$. (Since we do not require that the random functions are separable, the sample functions need not be non-decreasing for a.s. all $x \in X$.) The L^p-limits $\xi(t+)$ and $\xi(t-)$ exist for such a random function (Theorem 2.1). If $\xi(t) = \frac{1}{2}[\xi(t-) + \xi(t+)]$ (L^p) we say that ξ is L^p-mean-continuous at that point and if such a relation holds for all t we say that ξ is L^p-mean-continuous. Let M^p be the class of L^p-mean-continuous a.s. non-negative, a.s. non-decreasing random functions and let $V^p = R(M^p)$ be the linear closure of M^p over R. We shall define a generalized convolution $\xi \circledast \eta \in V^p$ for $\xi \in V^{q_1}$, $\eta \in V^{q_2}$, $q_1 \geq 1$, $q_2 \geq 1$, $1/q_1 + 1/q_2 \geq 1/p$ and show that the commutative and associative laws hold for this convolution.

Let $M_0^p = \{\xi : \xi \in M^p, \ \xi(-\infty) = 0 \text{ a.s.}\}$ and let V_0^p be the linear closure of M_0^p. The L^p-FS-transform (F.S. read Fourier–Stieltjes) of $\xi \in V_0^p$ will be defined in section 5 as an RS-integral in respect to the L^p-norm and it will be shown that $\xi \circledast \eta$ has the L^p–FS-transform $\hat{\xi} \cdot \hat{\eta}$ when $\xi \in V_0^{q_1}$ and $\eta \in V_0^{q_2}$ have the L^{q_1}-FS-transform $\hat{\xi}$ and L^{q_2}-FS-transform $\hat{\eta}$ respectively ($1/q_1 + 1/q_2 \leq 1/p$, $q_1 \geq 1$, $q_2 \geq 1$, $p \geq 1$). In a forthcoming paper [2] we shall prove a generalized Bochner theorem which gives necessary and sufficient conditions for a random function to be the L^p-FS-transform of an a.s. non-decreasing random function belonging to V_0^p. Then it is also possible to define the convolution of random functions with the help of L^p-RS-transforms in such a way that the two definitions agree. We have also given limit theorems for convolution products of random functions [3].

In many cases the generalizations of theorems for functions on the real line to corresponding theorems for random functions are quite simple and we can refer to [1] for details in the proofs.

2. The linear space V^p

The following simple lemma will frequently be used.

Lemma 2.1. *If ξ and η are* a.s. *non-negative random variables belonging to L^p and if $\xi \geq \eta$* a.s. *then*

$$\| \xi - \eta \|_p \leqslant [\| \xi \|_p^p - \| \eta \|_p^p]^{1/p}.$$

Proof. $\xi^p = (\eta + \xi - \eta)^p \geqslant \eta^p + (\xi - \eta)^p$ a.s. Using then Minkowsky's inequality we get the desired result.

Theorem 2.1. *If ξ is an* a.s. *non-decreasing random function on $(-\infty, +\infty)$ then the limits $\xi(t-)$ and $\xi(t+)$ exist in thv L^p-norm for $t \in (-\infty, +\infty)$.*

Proof. We may assume that $\xi \geqslant 0$ a.s. Applying Lemma 2.1 we get for $t_1 \leqslant t_2$.

$$\| \xi(t_2) - \xi(t_1) \|_p \leqslant [\| \xi(t_2) \|_p^p - \| \xi(t_1) \|_p^p]^{1/p}. \tag{2.1}$$

But $\| \xi(t_2) \|_p \geqslant \| \xi(t_1) \|_p$ and thus the left-hand side of (2.1) tends to 0 as $t_1 \uparrow t_0$, $t_2 \uparrow t_0$ or $t_1 \downarrow t_0$, $t_2 \downarrow t_0$, $t_1 \leqslant t_2$. Hence the directed classes $\{\xi(t) : t < t_0\}$ and $\{\xi(t) : t > t_0\}$ are mutually convergent and thus the L^p-limits $\xi(t-)$ and $\xi(t+)$ exist.

Now consider the class M^p. We call the point c an L^p-discontinuity point of $\xi \in M^p$ if $\| \xi(c+) - \xi(c-) \|_p > 0$. Clearly $\xi(c-)$ and $\xi(c+)$ are L^p-limits of sequences $\{\xi(c - a_n)\}$ and $\{\xi(c + a_n)\}$ respectively where $a_n \downarrow 0$. Then $\{a_n\}$ may be chosen such that $\xi(c-)$ and $\xi(c+)$ are a.s. limits of these sequences ([4], p. 164). Hence $\xi(c-) \geqslant 0$, $\xi(c+) \geqslant 0$ a.s. Put $\alpha_c = \xi(c+) - \xi(c-)$ and let $\Lambda(\xi)$ be the set of numbers c for which $\alpha_c > 0$ a,s. By Lemma 2.1 we find that $c \in \Lambda(\xi)$ is a discontinuity point of the non-decreasing bounded function $\| \xi \|_p$ and hence $\Lambda(\xi)$ is a countable set. Let e be the mean-continuous unit distribution function ($e(t) = 0$ for $t < 0$, $= \frac{1}{2}$ for $t = 0$, $= 1$ for $t > 0$) and define e^c by $e^c(t) = e(t + c)$. It is easily seen that $\xi - \alpha_c e^c$ belongs to M^p and is L^p-continuous at $t = c$. By the help of induction we then get (cf. [1], p. 19) also observing that

$$\| \sum_{c \in \Lambda(\xi)} \alpha_c \|_p \leqslant \| \xi(+\infty) \|_p.$$

Theorem 2.2. *A random function $\xi \in M^p$ has the representation*

$$\xi = \xi_\infty + \sum_{c \in \Lambda(\xi)} \alpha_c e^c, \tag{2.2}$$

where ξ_∞ belongs to M^p and is L^p-continuous, $\alpha_c > 0$ a.s. *and $\sum \alpha_c$ is convergent in the L^p-norm.*

Corollary. *The representation 2.2 also holds for $\xi \in V^p$ and then ξ_∞ belongs to V^p and is L^p-continuous and $\sum |\alpha_c|$ is convergent in the L^p-norm.*

We say that ξ is uniformly L^p-continuous if there to any $\varepsilon > 0$ belongs a $\delta > 0$ such that $\| \xi(t + h) - \xi(t) \|_p < \varepsilon$ for $0 < h < \delta$ and all t.

Theorem 2.3. *If ξ belongs to V^p and is L^p-continuous then it is uniformly L^p-continuous.*

Proof. It is sufficient to deal with $\xi \in M^p$. Then if ξ is continuous we find by Minkowsky's inequality that $\| \xi \|_p$ is continuous and clearly $\| \xi \|_p$ is uniformly continuous since it is non-decreasing and bounded. By (2.1) we then find that ξ is uniformly L^p-continuous.

We say that ξ is of bounded variation in respect to the L^p-norm if

$$\sup_N \left\| \sum_{i=1}^{n} |\xi(t_i) - \xi(t_{i-1})| \right\| < +\infty$$

(N being any net fitted on any interval) and that ξ is of L^p-bounded variation if

$$\sup_N \sum_{i=1}^{n} \| \xi(t_i) - \xi(t_{i-1}) \|_p < +\infty.$$

It can be shown that V^p is the class of L^p-mean-continuous random functions of bounded variation in respect to the L^p-norm. However we omit the proof of this statement-

3. L^p-RS-integrals

Let $N: a = t_0 < t, < \ldots < t_n = b$ be a net fitted on a finite interval $[a, b]$. We call N' a refinement of N and write $N' > N$ if any subinterval of N' belongs to some subinterval of N. The set of nets on $[a, b]$ form a direction in respect to refinements ([4], p. 67). To random functions $\xi \in V^{q_1}$, $\eta \in V^{q_2}$ where $q_1 \geq 1$, $q_2 \geq 1$, $1/q_1 + 1/q_2 \leq 1/p$ we form the RS-sum (RS read Riemann–Stieltjes).

$$\sigma_l^N(\xi, \eta) = \sum_{i=1}^{n} \xi(t_{i-1}+)[\eta(t_i) - \eta(t_{i-1})]. \tag{3.1}$$

Definition. *A random variable is called the left L^p-RS-integral of ξ in respect to η on $[a, b]$ and is denoted by*

$$\sigma = \int_a^b \xi(t)\, d_l \eta(t)$$

if there to any $\varepsilon > 0$ belongs a net N_ε such that

$$\| \sigma_l^N(\xi, \eta) - \sigma \|_p < \varepsilon \quad \textit{for} \quad N > N_\varepsilon.$$

It is easily seen that σ is uniquely determined by this definition. Left L^p-RS integrals on infinite intervals are defined as L^p-limits of corresponding left integrals on finite intervals which tend nondecreasing to the infinite interval. Further we put

$$\int \xi(t)\, d_l \eta(t) = \xi(-\infty)\eta(-\infty) + \int_{-\infty}^{+\infty} \xi(t)\, d_l \eta(t).$$

Right integrals are defined in the same way.[1]

[1] Stochastic integrals as limits in probability of sums have been studied by K. Ito [5], [6].

Theorem 3.1. *If ξ and η satisfy the conditions given above, then ξ has left and right L^p-RS-integrals with respect to η. If furthermore η is of L^p-bounded variation, then the left L^p-RS-integral is equal to the corresponding right integral.*

Proof. Clearly it is sufficient to prove the theorem for $\xi \in M^{q_1}$, $\eta \in M^{q_2}$. Let $[a, b]$ be any finite interval. Since

$$0 \leqslant \sigma_l^N(\xi, \eta) \leqslant \sigma_l^{N'}(\xi, \eta) \leqslant \xi(+\infty)\eta(+\infty) \text{ a.s.}$$

for $N' > N$: we get by Hölder's inequality

$$\|\sigma_l^{N'}(\xi, \eta)\|_p \leqslant \|\sigma_l^N(\xi, \eta)\|_p \leqslant \|\xi(+\infty)\|_{q_1} \|\eta(+\infty)\|_{q_2}. \tag{3.2}$$

Applying Lemma 2.1 we further obtain

$$\|\sigma_l^{N'}(\xi, \eta) - \sigma_l^N(\xi, \eta)\|_p \leqslant \{\|\sigma_l^{N'}(\xi, \eta)\|_p^p - \|\sigma_l^N(\xi, \eta)\|_p\}^{1/p}. \tag{3.3}$$

When N and N' are infinitely refined and $N' > N$ the right-hand side of (3.3) tends to 0, according to (3.2). Hence the class $\sigma_1^N(\xi, \eta)$, directed in respect to refinements, is mutually convergent and thus convergent. The corresponding L^p-limit is the left L^p-integral on $[a, b]$. It belongs to L^p according to (3.3). The existence of the L^p-integral on any infinite interval then easily follows. The existence of right integrals is obtained in the same way.

Let now η be of L^{q_2}-bounded variation. By Hölder's inequality we get

$$\|\sigma_r^N(\xi, \eta) - \sigma_l^N(\xi, \eta)\|_p \leqslant \sum_{i=1}^{n} \|\xi(t_i -) - \xi(t_{i-1} +)\|_{q_1} \|\eta(t_i) - \eta(t_{i-1})\|_{q_2}. \tag{3.4}$$

We may choose the net N on $[a, b]$ such that

$$\|\xi(t_i -) - \xi(t_{i-1} +)\|_{q_1} < \varepsilon$$

for any $\varepsilon > 0$ and for all i (since $\|\xi\|_{q_2}$ is of bounded variation). Then

$$\|\sigma_r^N(\xi, \eta) - \sigma_1^N(\xi, \eta)\| \leqslant \varepsilon \sum_{i=1}^{n} \|\eta(t_i) - \eta(t_{i-1})\|_{q_2}.$$

Hence the left and right L^p-integrals on $[a, b]$ are equal (L^p).

Remark 1. When the left L^p-integral is equal (L^p) to the right L^p-integral it is also the L^p-limit of any RS-sum of the form 3.1 where $\xi(t_i +)$ is changed into $\xi(\tau_i)$, τ_i being any point on the open interval (t_{i-1}, t_i).

Remark 2. Since the left (right) L^p-integral can be given as the L^p-limit of a sequence of RS-sum it is also the a.s. limit of such a sequence.

A random variable ξ is called a.s. uniformly continuous in respect to a random variable $\xi_0 \in L^p$ if there to any positive number $\varepsilon > 0$ exists a positive number $h(\varepsilon) \geqslant 0$ such ihat

$$|\xi(t+h) - \xi(t)| \leqslant \varepsilon \xi_0 \text{ a.s. for } |h| \leqslant h(\varepsilon).$$

Theorem 3.2. *If $\xi \in V^{q_1}), \eta \in V^{q_2}), q_1 \geqslant 1, q_2 \geqslant 1, 1/q_1 + 1/q_2 \leqslant 1/p, p \geqslant 1$ and if ξ is uniformly continuous in respect to a random variable $\xi_0 \in L^{q_1}$, then the left and right L^p-integrals of ξ in respect to η are equal.*

Proof. For any $\varepsilon > 0$ we may choose the net N such that

$$|\sigma_r^N(\xi,\eta) - \sigma_l^N(\xi,\eta)| \leqslant \varepsilon \xi_0 \sum_{i=1}^n |\eta(t_i) - \eta(t_{i-1})| \quad \text{a.s.}$$

A sequence $\{\eta_n\}$ of random functions is said to converge L^p-completely to a random function η on an interval $[a, b]$ if $\|\eta_n - \eta\|_p$ tends to $o(n \to +\infty)$ at $t = a$, $t = \mathrm{b}$ and all other points on $[a, b]$ except at most a countable set. We shall state a generalized Helly's theorem as follows.

Theorem 3.3. *Let f be a continuous function and $\eta_n \in M^p$ for $n = 1, 2, \ldots$ If $\eta_n \to \eta$ L^p-completely on $[a, b]$, then*

$$\left\| \int_a^b f(t)\, d\eta_n(t) - \int_a^b f(t)\, d\eta(t) \right\|_p \to o(n \to +\infty).$$

Remark, If η is L^p-continuous and belnngs to M^p and G_n is a sequence of random functions tending to G at all finite points and at a and b, then

$$\int_a^b \eta(t)\, dG_n(t) \to \int_a^b \eta(t)\, dG(t) \quad (L^p).$$

The proof follows as in [1], section 2.7.

4. Convolutions

For $\xi \in V^{q_1}, \eta \in V^{q_2}$ where $1/q_1 + 1/q_2 \leqslant 1/p$, $q_1 \geqslant 1$, $q_2 \geqslant 1$. $p \geqslant 1$ we define left and right L^p-convolutions $\xi \underset{l}{*} \eta$ and $\xi \underset{r}{*} \eta$ by

$$\xi \underset{l}{*} \eta = \int \xi(\mathrm{t} - \tau)\, d_l \eta(\tau), \qquad \xi \underset{r}{*} \eta = \int \xi(t - \tau)\, d_r \eta(\tau) \quad (L^p).$$

If these convolutions are equal (L^p) we write

$$\xi \underset{l}{*} \eta = \xi * \eta = \xi \underset{r}{*} \eta.$$

Denote by $\xi(\cdot + c)$ that function which is equal to $\xi(t + c)$ at the point c (Hence $\xi(\cdot + c) = \xi * e^c$).

Theorem 4.1. *If ξ and η have the representations*

$$\xi = \xi_\infty + \sum_{c \in \Lambda(\xi)} \alpha_c e^c \quad (L^{q_1}),$$

$$\eta = \eta_\infty + \sum_{d \in \Lambda(\eta)} \beta_d e^d \quad (L^{q_2})$$

according to Theorem 2.2, then

$$\xi \underset{l}{*} \eta = \xi_\infty \underset{l}{*} \eta_\infty + \sum_{c \in \Lambda(\xi)} \alpha_c \eta(\cdot + c) + \sum_{d \in \Lambda(\eta)} \beta_d \xi(\cdot + d)$$

$$+ \sum_{c \in \Lambda(\xi)} \sum_{d \in \Lambda(\eta)} \alpha_c \beta_d e^{c+d} \quad (L^p), \tag{4.1}$$

where the series are absolutely and uniformly convergent in the L^p-norm.

The corresponding relation holds for right convolutions.
The proof follows immediately (cf. [1], p. 44).

Theorem 4.2. *The relations*

$$\xi \underset{l}{*} \eta = \eta \underset{l}{*} \xi, \; \xi \underset{r}{*} \eta = \eta \underset{r}{*} \xi$$

hold.

Proof. Clearly we may consider the case $\xi \in M^{q_1}$, $\tau \in M^{q_2}$ and by Theorem 4.1 we find that it is also sufficient to deal with that case when ξ is L^{q_1}-continuous and η is L^{q_2}-continuous. Further we may consider the convolution at the point $t = 0$. Then $\xi \underset{r}{*} \eta(0)$ can be approximated arbitrarily closely in the L^p-norm by a RS-sum

$$\xi(-t_{-n})\eta(t_{-n}) + \sum_{i=-n+1}^{n} \xi(-t_i)[\eta(t_i) - \eta(t_{i-1})].$$

By an Abelian transformation we can write this sum

$$\eta(t_n)\xi(-t_n) + \sum_{i=-n}^{n-1} \eta(t_i)[\xi(-t_i) - \xi(-t_{-i+1})],$$

and it approximates $\eta \underset{r}{*} \xi(0)$ arbitrarily closely in the L^p-norm for suitable choice of the net.

Lemma 4.1. *If ξ belongs to M^{q_1} and is L^{q_1}-continuous and η belongs to M^{q_2}, then $\xi \underset{l}{*} \eta$ is L^p-continuous to the right and $\xi \underset{r}{*} \eta$ is L^p-continuous to the left.*

Proof. Let $t \in (-\infty, +\infty)$ and let $[-a, a]$ be a finite interval and $N: -a = t_0 < t_1 < \dots < t_n = a$ some net fitted on $[-a, a]$ such that

$$0 \leqslant \left\| \int_{-a}^{a} \xi(t-\tau)\, d_l \eta(\tau) \right\|_p^p - \left\| \sum \xi(t - t_i)[\eta(t_i) - \eta(t_{i-1})] \right\|_p^p < \tfrac{1}{2} \varepsilon^p \tag{4.2}$$

for a given number $\varepsilon > 0$. We observe that the RS-sum is a.s. not larger than the integral since the RS-sums are a.s. non-decreasing in the direction of refinements of nets. Now ξ is L^p-continuous at t and hence we can determine $h > 0$ such that

$$\left\|\sum_{i=1}^{n} \xi(t-t_i)\left[\eta(t_i)-\eta(t_{i-1})\right]\right\|_p^p - \left\|\sum_{i=1}^{n} \xi(t-h-t_i)\left[\eta(t_i)-\eta(t_{i-1})\right]\right\|_p^p < \tfrac{1}{2}\varepsilon^p. \tag{4.3}$$

But the second sums is a RS-sum belonging to the right L^p-RS-integral and hence

$$\left\|\int_{-a}^{a} \xi(t-h-\tau)\, d_r\eta(\tau)\right\|_p \geqslant \left\|\sum_{i=1}^{n} \xi(t-h-t_i)\,\eta(t_i)-\eta(t_{i-1})\right\|_p. \tag{4.4}$$

Combining (4.2) – (4.4) we obtain

$$0 \leqslant \left\|\int_{-a}^{a} \xi(t-\tau)\, d_r\eta(\tau)\right\|_p^p - \left\|\int_{-a}^{a} \xi(t-\tau-h)\, d_r\eta(\tau)\right\|_p^p < \varepsilon^p,$$

Applying Lemma 2.1 we then get

$$\left\|\int_{-a}^{a} \xi(t-\tau)\, d_r\eta(\tau) - \int_{-a}^{a} \xi(t-h-\tau)\, d\eta_r(\tau)\right\|_p < \varepsilon.$$

Also observing that

$$\left\|\int_{-\infty}^{-a} + \int_{a}^{+\infty} \xi(t-\tau)\, d_r\eta(\tau)\right\|_p \leqslant \|\xi(+\infty)\|_{q_1} \left\|\int_{-\infty}^{-a} + \int_{a}^{+\infty} d_r\eta(\tau)\right\|_{q_2}$$

we find that

$$\|\xi \underset{r}{\ast} \eta(t) - \xi \underset{r}{\ast} \eta(t-h)\|_p \to 0 \quad \text{as} \quad h \downarrow 0.$$

The L^p-continuity of $\xi \underset{l}{\ast} \eta$ to the right follows in the same way.

Lemma 4.2. *Let* $\xi_i \in M^{q_i}$, $i=1,2,3$, *and let* ξ_1 *be* a.s. *uniformly continuous in respect to the random variable* $\alpha \in L^{q_1}$, *where* $q_i \geqslant 1$, $\sum_{i=1}^{3} 1/q_i \leqslant 1/p \leqslant 1$.

Then

$$(\xi_1 \ast \xi_2) \ast \xi_3 = \xi_1 \ast (\xi_2 \underset{l}{\ast} \xi_3) = \xi_1 \ast (\xi_2 \underset{r}{\ast} \xi_3) \quad (L^p).$$

Proof. The convolution $\xi_1 \ast \xi_2$ exists according to Theorem 3.2. Further to any $\varepsilon > 0$ we can find $(h)\varepsilon > 0$ such that

$$0 \leqslant \xi_1(t+h) - \xi_1(t) < \varepsilon\alpha \quad \text{a.s.}$$

for $0 < h < h(\varepsilon)$. Then

$$\xi_1 \ast \xi_2(t+h) - \xi_1 \ast \xi_2(t) = \int_{-\infty}^{+\infty} [\xi_1(t+h-\tau) - \xi_1(t-\tau)]\, d\xi_2(\tau) \leqslant \varepsilon\alpha\xi_2(+\infty) \quad \text{a.s.}$$

and thus $\xi_1 \ast \xi_2$ is a.s. uniformly continuous in respect to the random variable $\alpha\xi_2(+\infty)$. Hence $(\xi_1 \ast \xi_2) \ast \xi_3$ exists. In the same way we conclude that $\xi_1 \ast (\xi_2 \underset{l}{\ast} \xi_3)$ and $\xi_1 \ast (\xi_2 \underset{r}{\ast} \xi_3)$ exist.

Now choose the positive number a and a net N fitted on such that

$$(-a.a), \ -a=t_0<t_1< \ldots >t_n=a,$$

$$\|\xi_1(-a)-\xi_1(-\infty)\|_{q_1}\leqslant\varepsilon, \ \|\xi_1(+\infty)-\xi_1(a)\|_{q_1}\leqslant\varepsilon, \tag{4.1}$$

$$\xi_1(t_i)-\xi_1(t_{i-1})\leqslant\varepsilon\alpha \quad \text{a.s.} \tag{4.2}$$

Using the definition of the L^p-RS integrals and the fact that these are a.s. limits of sequences of RS-sums, we get the inequalities

$$\xi_2(+\infty)\,\xi_1(-\infty)+\sum_{i=1}^{n}\xi_2(t-t_i)\,[\xi_1(t_i)-\xi_1(t_{i-1})]\leqslant\xi_2 * \xi_1(t)$$

$$\leqslant\xi_2(+\infty)\,[\xi_1(-a)+\xi_1(+\infty)-\xi_1(a)]+\sum_{i=1}^{n}\xi_2(t-t_{i-1})\,[\xi_1(t_i)-\xi_2(t_{i-1})] \ \text{ a.s.} \tag{4.3}$$

Forming the left convolution by ξ_3 we then get from (4.3)

$$\xi_2(+\infty)\,\xi_1(-\infty)\,\xi_3(+\infty)+\sum_{i=1}^{n}\xi_2 \underset{l}{*} \xi_3(t-t_i)\,[\xi_1(t_i)-\xi_1(t_{i-1})]$$

$$\leqslant(\xi_2 * \xi_1) * \xi_3(t)\leqslant\xi_2(+\infty)\,[\xi_1(-a)+\xi_1(+\infty)-\xi_1(a)]\,\xi_3(+\infty)$$

$$+\sum_{i=1}^{n}(\xi_2 \underset{l}{*} \xi_3)(t-t_{i-1})[\xi_1(t_i)-\varepsilon_1(t_{i-1})] \quad \text{a.s.} \tag{4.4}$$

Now it is easily seen that this inequality also holds if we change $(\xi_2 * \xi_1) * \xi_3$ into $(\xi_2 \underset{l}{*} \xi_3) * \xi_1$. Hence we get regarding the inequalities (4.1) and (4.2)

$$\|(\xi_2 * \xi_1) * \xi_3(t)-(\xi_2 \underset{l}{*} \xi_3) * \xi_1(t)\|_p$$

$$\leqslant\|\xi_2(+\infty)\|_{q_2}\cdot\|\xi_3(+\infty)\|_{q_3}\{\|\xi_1(-a)-\xi_1(-\infty)\|_{q_1}+\|\xi_1(+\infty)-\xi_1(a)\|_{q_1}\}$$

$$+\varepsilon\left\|\sum_{i=1}^{n}[\xi_2(t-t_{i-1})-\xi_2(t-t_i)]\right\|_p$$

$$\leqslant\varepsilon\{2\|\xi_2(+\infty)\|_{q_2}\|\xi_3(+\infty)\|_{q_3}+\|\alpha\|_{q_1}\|\xi_2(+\infty)\|_{q_2}\|\xi_3(+\infty)\|_{q_3}.$$

Since ε is arbitrary we conclude

$$(\xi_2 * \xi_1) * \xi_3=(\xi_2 \underset{l}{*} \xi_3) * \xi_1 \quad (L^p).$$

In the same way we obtain the corresponding relation for the right convolution.

Theorem 4.3. *Let ξ and η satisfy the conditions in Theorem 4.1. Then $\xi \underset{l}{*} \eta$ and $\xi \underset{r}{*} \eta$ have the same L^p-discontinuity points and their jumps are equal (L^p) at given L^p-discontinuity points.*

Proof. Clearly it is sufficient to consider the case $\xi\in M^{q_1}$, $\eta\in M^{q_2}$ and, according to Theorem 4.1 it is also sufficient to deal with an L^{q_1}-continuous ξ and an L^{q_2}-

continuous η. Since $\xi \underset{l}{*} \eta$ and $\xi \underset{r}{*} \eta$ are L^p-continuous to the right and left respectively, they have representations (according to Theorem 2.3 and the remark on this theorem.)

$$\xi \underset{l}{*} \eta = \zeta_r + \sum_{c \in \Lambda(\xi \underset{l}{*} \eta)} \alpha_c e_r^c \quad (L^p) \tag{4.5}$$

and

$$\xi \underset{r}{*} \eta = \zeta_l + \sum_{d \in \Lambda(\xi \underset{r}{*} \eta)} \beta_d e_l^d \quad (L^p), \tag{4.6}$$

respectively, where ζ_l and ζ_r ard L^p-continuous,

$$e_r^c(t) = \begin{cases} 0 & \text{for} \quad t < c, \\ 1 & \text{for} \quad t \geqslant c, \end{cases} \qquad e_l^c = \begin{matrix} 0 & \text{for} \quad t \leqslant c, \\ 1 & \text{for} \quad t > c. \end{matrix}$$

Let G be a symmetrical continuous distribution function and let $G(\cdot/\sigma)$ denote that function which takes the value $G(t/\sigma)$ at $t(\sigma > 0)$. Applying Lemma 4.2 with $\xi_1 = G(\cdot/\sigma)$ we get

$$\zeta_r * G\left(\frac{\cdot}{\sigma}\right) + \sum_{c \in \Lambda(\xi \underset{l}{*} \eta)} \alpha_c G\left(\frac{\cdot + c}{\sigma}\right) = \xi \underset{l}{*} G\left(\frac{\cdot}{\sigma}\right) + \sum_{d \in \Lambda(\xi \underset{r}{*} \eta)} \beta_d G\left(\frac{\cdot + d}{\sigma}\right) \quad (L^p).$$

Letting $\sigma \to 0+$ and applying Helly's generalized (Theorem 3.3 and the remark on this theorem) we obtain

$$\zeta_r + \sum_{c \in \Lambda(\xi \underset{l}{*} \eta)} \alpha_c e^c = \zeta_l + \sum_{c \in \Lambda(\xi \underset{r}{*} \eta)} \beta_d e^d \quad (L^p)$$

and from this relation the proposition follows.

Now we define a generalized convolution $\zeta \circledast \eta$ by putting

$$\xi \circledast \eta = \tfrac{1}{2} [\xi \underset{l}{*} \eta + \xi \underset{r}{*} \eta].$$

The generalized convolution is a commutative operation according to Theorem 4.2 and by Theorem 4.3 $\xi \circledast \eta$ belongs to V^p if $\xi \in V^{q_1}$, $\eta \in V^{q_2}$ where $1/q_1 + 1/q_2 \leqslant 1/p$, $q_1 \geqslant 1, q_2 \geqslant 1, p \geqslant 1$.

Theorem 4.4. *The generalized convolution is an associative operation in the following sense. Let $\xi_i \in V^{q_i}$, $q_i \geqslant 1$ for $i = 1, 2, 3$, where $\sum_{i=1}^{3} 1/q_i \leqslant 1/p, p \geqslant 1$. Then*

$$(\xi_1 \circledast \xi_2) \circledast \xi_3 = \xi_1 \circledast (\xi_2 \circledast \xi_3) \quad (L^p).$$

Proof. Clearly it is sufficient to deal with the case $\xi_i \in M^{q_i}$. According to Lemma 4.2 this relation holds if furthermore ξ_1 is a.s. uniformly continuous in respect to a random variable $\alpha \in L^{q_1}$. Hence observing that $\xi_1 * G(\cdot/\sigma)$ is a.s. uniformly continuous in respect to $\xi_1(+\infty)$, we get

$$\left[\left(\xi_1 * G\left(\frac{\cdot}{\sigma}\right)\right) \circledast \xi_2\right] \circledast \xi_3 = \left(\xi_1 * G\left(\frac{\cdot}{\sigma}\right)\right) \circledast [\xi_2 \circledast \xi_3] = G\left(\frac{\cdot}{\sigma}\right) * [\xi_1 \circledast (\xi_2 \circledast \xi_3)] \quad (L^p). \tag{4.7}$$

Since $G(./\sigma)$ is uniformly continuous we also have

$$\left(\xi_1 * G\left(\frac{\cdot}{\sigma}\right)\right) \circledast \xi_2 = G\left(\frac{\cdot}{\sigma}\right) * (\xi_1 \circledast \xi_2) \quad (L^p) \tag{4.8}$$

and

$$\left[G\left(\frac{\cdot}{\sigma}\right) * (\xi_1 \circledast \xi_2)\right] \circledast \xi_3 = G\left(\frac{\cdot}{\sigma}\right) * [(\xi_1 \circledast \xi_2) \circledast \xi_3 \quad (L^p). \tag{4.9}$$

Combining (4.7)–(4.9) we obtain

$$G\left(\frac{\cdot}{\sigma}\right) * [(\xi_1 \circledast \xi_2) \circledast \xi_3] = G\left(\frac{\cdot}{\sigma}\right) * [\xi_1 \circledast (\xi_2 \circledast \xi_3)] \quad (L^p).$$

Letting $\sigma \downarrow 0$ we get

$$(\xi_1 \circledast \xi_2) \circledast \xi_3 = \xi_1 \circledast (\xi_2 \circledast \xi_3).$$

5. L^p-Fouriertransforms

If $\xi \in V_0^p$ then the L^p-RS-integral

$$\hat{\xi}(s) = \int_{-\infty}^{+\infty} \exp its \, d\xi(t)$$

exists. It is called the L^p-Fourierintegral of ξ at the point s.

Theorem 5.1. *Let* $\xi \in V_0^{q_1}$, $\eta \in V_0^{q_2}$, $\eta \in V_0^{q_2}$ where $1/q_1 + 1/q_2 \geqslant 1/p$, $q_1 \geqslant 1$, $q_2 \geqslant 1$, $p \geqslant 1$. Then

$$\widehat{\xi * \eta} = \hat{\xi} \cdot \hat{\eta} \quad (L^p).$$

Proof. It is sufficient to consider $\xi \in M^{q_1}$, $\eta \in M^{q_2}$. If G is a continuous distribution we have

$$G * (\xi \circledast \eta) = (G * \xi) * \eta \quad (L^p) \tag{5.1}$$

(c.f. Lemma 4.2). However then this relation also holds for any continuous function G of bounded variation since G is the difference between two continuous bounded and non-decreasing functions. Since $\xi * \eta(-\infty) = \eta(-\infty) = 0$ a.s. it then also follows that (5.1) remains true for any bounded continuous function which is of bounded variation on any finite interval. Thus particularly (5.1) holds for $G(-t) = \sin ts$, $G(t) = \cos ts$ and hence also for exp-its. Choosing $G(t) = \exp its$ we get successively

$$G * \xi(t) = \int_{-\infty}^{+\infty} \exp its\,(t-\tau)\, d\xi(\tau) = \hat{\xi}(s) \exp its \quad (L^{q_1}),$$

$$(G \ast \xi(t)) \ast \eta(t) = \hat{\xi}(s)\,\hat{\eta}(s) \exp its \quad (L^p),$$

$$G \ast (\xi \circledast \eta)(t) = \widehat{\xi \circledast \eta}(s) \exp its \quad (L^p),$$

and thus according to (5.1)

$$\widehat{\xi \circledast \eta} = \hat{\xi} \cdot \hat{\eta}.$$

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