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On an unsolved question concerning the Diophantine equation $Ax^3 + By^3 = C$

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§ 1.

The Diophantine equation

$$x^3 + Dy^3 = 1 \tag{1}$$

was solved completely by B. Delaunay [1][1] who showed that it has at most one solution in integers x and y when $y \neq 0$; if x, y is an integral solution, then

$$\eta = x + y\sqrt[3]{D} \tag{2}$$

is the fundamental unit of the ring $\boldsymbol{R}(1, \sqrt[3]{D}, (\sqrt[3]{D})^2)$.

T. Nagell [2], [3], [4], and [5] proved the same theorem independently of Delaunay and, moreover, a stronger form of the latter part of the theorem.

Nagell [4] and [5] proved that η is the fundamental unit of the field $\boldsymbol{K}(\sqrt[3]{D})$, except when $D = 19$, 20, and 28, in which cases η is the square of the fundamental unit. These values of D correspond to the solutions $x = -8$, $y = 3$; $x = -19$, $y = 7$; and $x = -3$, $y = 1$.

To solve (1), one has thus to determine the fundamental unit of $\boldsymbol{K}(\sqrt[3]{D})$, and to examine whether it has the form (2) or not.

Nagell [4] generalized these results and showed that the Diophantine equation

$$Ax^3 + By^3 = C, \tag{3}$$

where $C = 1$, or $C = 3$, where A and B are > 1 when $C = 1$ and where AB is not divisible by 3 when $C = 3$, has at most one solution in integers x and y.

He also established the following result: Put $A = ac^2$ and $B = bd^2$, where a, b, c, and d are positive integers, relatively prime in pairs, and possessing no square factors. Then, if x, y is a solution, one has

$$\eta = \frac{1}{C}(x\sqrt[3]{A} + y\sqrt[3]{B})^3 = \xi^{2^r}, \tag{4}$$

[1] Figures in [ ] refer to the Bibliography at the end of this paper.

where ξ is the fundamental unit of the field $\boldsymbol{K}(\sqrt[3]{ac^2b^2d})$, $0 < \xi < 1$, and where r is an integer $\geqq 0$.[1]

This theorem may also be expressed in the following way: If x, y is a solution, then

$$\eta = \frac{1}{C}(x\sqrt[3]{A} + y\sqrt[3]{B})^3 = \zeta^{2^s}, \tag{5}$$

where ζ is the fundamental unit of the ring $\boldsymbol{R}(1, \sqrt[3]{ac^2b^2d}, \sqrt[3]{a^2cbd^2})$, $0<\zeta<1$, and where s is an integer $\geqq 0$.

The relation between ζ and ξ is $\zeta = \xi$, or $\zeta = \xi^2$. Hence we have $r = s$, or $r = s + 1$ (Cp. [4], p. 267).

Although an upper limit of the integers r and s could generally not be determined, NAGELL succeeded in constructing an algorithm to decide if (3) is solvable or not. In the former case, this algorithm gives a method to determine the solution of the equation (Cp. [4], p. 257 and p. 263). This method, a sort of *descente finie*, is, however, too cumbersome to be practical. It would thus be of value to solve the question of the upper limit of r and s.

NAGELL [4] has treated this question and proved that $s = 0$ when $C = 1$, and $r = 0$ when $C = 3$, if A is even and B is divisible by a prime factor of the form $8t - 1$, or $8t + 5$, and if A and B are both divisible by a prime factor of the form $8t - 1$, or $8t + 5$. He further proved in [4] that there is an infinite number of fields $\boldsymbol{K}$ in which $s = 0$ and $s = 1$ when $C = 1$, and that there is an infinite number of fields $\boldsymbol{K}$ in which $r = 0$ and $r = 1$ when $C = 3$.

NAGELL [6] and [7] has proved that $s \leqq 1$ when $C = 1$ if A and B contain at most three distinct prime factors each.

Finally, NAGELL [7] has proved that $s \leqq 1$ when $C = 1$ if A and B contain no prime factors of the form $3t + 1$.

The purpose of the present paper is to show that the method employed by NAGELL in [6] may be extended and used in a few more cases in order to find an upper limit of r and s.

μ and λ denote the largest number of distinct prime factors of A and B respectively. By ν, we denote the largest of the numbers μ and λ. The following results are obtained in this paper:

$r \leqq 1$ *when* $C = 3$ *if* $\nu \leqq 2$;

$r \leqq 1$ *when* $C = 3$ *if* A *and* B *are odd and* $\nu \leqq 4$;

$r \leqq 1$ *when* $C = 3$ *if* A *or* B *is divisible by* 4 *and* $\nu \leqq 4$;

$s \leqq 1$ *when* $C = 1$ *if* A *and* B *are odd and* $\nu \leqq 6$;

$s \leqq 1$ *when* $C = 1$ *if* A *or* B *is divisible by* 4 *and* $\nu \leqq 6$.

*

[1] There is one exception from this theorem, viz. the equation

$$2x^3 + y^3 = 3,$$

which has the two solutions $x = y = 1$ and $x = 4$, $y = -5$. This exception is not taken into consideration in the following.

In order to prove the theorems mentioned above, we start from some of the results in [4].

Let us first consider the case $C = 1$. If the number s of (5) is > 1, η may be written as the biquadrate of a unit. It is proved in [4] that it is a necessary condition for η being a biquadrate of $\boldsymbol{R}$ that the following equation has a solution in integers f, g, N_1, and N_2:

$$1 = f^2 N_1^6 - 27\, g^2 N_2^6, \tag{6}$$

where $fg = A$ or $fg = B$. This condition is not sufficient. A necessary and sufficient condition consists in the following system having a solution in integers X, Y, and Z:

$$\begin{cases} abZ^2 + 2XY = NM, \\ dY^2 + 2acXZ = dM^2, \\ cX^2 + 2bdYZ = -\tfrac{1}{2}cN^2, \end{cases} \tag{7}$$

with $N = 2N_1N_2$ and $(M, N) = 1$.

If $C = 3$ and the number r of (4) is > 1, η may be written as a biquadrate of a unit. It is a necessary condition for η being a biquadrate of $\boldsymbol{K}$ that the following equation has a solution in integers f, g, N_1, and N_2:

$$1 = f^2 N_1^6 - 3\, g^2 N_2^6, \tag{8}$$

where $fg = A$ or $fg = B$. This condition is not sufficient. A necessary and sufficient condition consists in system (7) having a solution in integers X, Y, and Z.

§ 2.

We begin by proving the following proposition:

Theorem 1. *If the equation*

$$p^n x^3 + q^m y^3 = 3,$$

where p and q are distinct primes $\neq 3$ and where m and n only take the value 1 or 2, has a solution in integers x and y, then

$$\eta = \tfrac{1}{3}\left(x\sqrt[3]{p^n} + y\sqrt[3]{q^m}\right)^3$$

is the fundamental unit of the field $\boldsymbol{K}(\sqrt[3]{p^n q^{3-m}})$, or the square of this unit.

We have to consider the equation

$$1 = f^2 N_1^6 - 3\, g^2 N_2^6, \tag{9}$$

where $fg = p^n$ (or $fg = q^m$), and we distinguish three cases:

1) p is odd, $f = p^n$ and $g = 1$.

(9) takes the form

$$1 = p^{2n} N_1^6 - 3 N_2^6. \tag{10}$$

If N_1 is even, (10) is impossible since the congruence

$$3(N_2^3)^2 + 1 \equiv 0 \pmod 8$$

is impossible. Hence N_1 is odd and N_2 even, and we get from (10)

$$p^n N_1^3 \pm 1 = 3 \cdot 2^\alpha N_3^6, \quad p^n N_1^3 \mp 1 = 2^\beta N_4^6,$$

which gives

$$\pm 2 = 3 \cdot 2^\alpha N_3^6 - 2^\beta N_4^6. \tag{11}$$

We have either $\alpha = 1$, $\beta = 5$, or $\alpha = 5$, $\beta = 1$. Hence we get from (11) the two equations

$$\pm 1 = 3 N_3^6 - 16 N_4^6, \tag{12}$$

$$\pm 1 = 48 N_3^6 - N_4^6. \tag{13}$$

From (12) we get the congruences

$$3(N_3^3)^2 \pm 1 \equiv 0 \pmod 8,$$

so that this equation is impossible. The upper sign of (13) is impossible modulo 3. (13) may be written

$$(4 N_3^3 + 1)^3 - (4 N_3^3 - 1)^3 = 2 N_4^6,$$

but, as is well known, the Diophantine equation

$$u^3 + v^3 = 2 w^3$$

has the only solution $u^3 = v^3 = w^3$ when $w \neq 0$. Hence the impossibility of the equation.

2) p is odd or even, $f = 1$ and $g = p^n$.

(9) takes the form

$$1 = N_1^6 - 3 p^{2n} N_2^6, \tag{14}$$

or

$$(p^n N_2^3 + 1)^3 - (p^n N_2^3 - 1)^3 = 2 N_1^6,$$

and we can see that (14) is impossible.

3) $p = 2$, $f = 2^n$ and $g = 1$.

(9) takes the form

$$1 = 2^{2n} N_1^6 - 3 N_2^6.$$

If $n = 2$, we get

$$1 = 16 N_1^6 - 3 N_2^6;$$

but this equation is impossible since (12) is impossible. If $n = 1$, we get

$$1 = 4 (N_1^2)^3 - 3 (N_2^2)^3.$$

The Diophantine equation $4x^3 - 3y^3 = 1$ has the only solution $x = y = 1$. This gives $|N_1| = |N_2| = 1$ and (Cp. [4], p. 263) $a = 2$, $b = c = d = 1$, $N = 2$, $M = 1$, or $a = 2$, $b = 41$, $c = d = 1$, $N = 2$, $M = -1$.

The former solution corresponds to the equation

$$2x^3 + y^3 = 3,$$

which we do not take into consideration. The latter solution corresponds to the equation

$$2x^3 + 41y^3 = 3,$$

which has the solution $x = -52$, $y = 19$. However, the number

$$\tfrac{1}{3}(-52\sqrt[3]{2} + 19\sqrt[3]{41})^3 = (329 + 22\sqrt[3]{164} - 30\sqrt[3]{3362})^2$$

is not a biquadrate of the field $\boldsymbol{K}(\sqrt[3]{164})$. If it were, system (7) would have a solution in integers X, Y, and Z. In this case the system may be written

$$\begin{cases} 41Z^2 + XY = -1, \\ Y^2 + 4XZ = 1, \\ X^2 + 82YZ = -2, \end{cases} \tag{15}$$

which gives

$$82Z^2 + 2XY - X^2 - 82YZ = 0,$$

$$41Z^2 + XY + Y^2 + 4XZ = 0.$$

From the third equation of (15) we get $Z \neq 0$. If we put $\frac{X}{Z} = u$, $\frac{Y}{Z} = v$, and eliminate u, we get

$$v^4 + 30v^3 + 246v^2 + 328v + 123 = 0. \tag{16}$$

If (15) had a solution in integers, then v would be a rational number. However, (16) has no rational solution.

§ 3.

We shall prove the following proposition:

Theorem 2. *Let a, b, c, and d denote positive integers, relatively prime in pairs, and possessing no square factors. If the equation*

$$Ax^3 + By^3 = 3,$$

where AB *is not divisible by* 3 *and where* $A = ac^2$ *and* $B = bd^2$ *have at most two distinct prime factors each, has a solution in integers* x *and* y, *then*

$$\eta = \tfrac{1}{3}(x\sqrt[3]{A} + y\sqrt[3]{B})^3$$

is the fundamental unit of the field $\mathbf{K}(\sqrt[3]{ac^2b^2d})$, *or the square of this unit.*

We have to consider the equation

$$1 = f^2 N_1^6 - 3g^2 N_2^6, \tag{17}$$

where $fg = A$ (or $fg = B$), and we distinguish three cases:

1) fg is even and $f = 2h$.

h cannot be even, for then

$$3(gN_2^3)^2 + 1 \equiv 0 \pmod 8,$$

which is impossible. Hence h is odd, and for the same reason N_1 is odd. It is easily seen that g is odd. From (17) we get

$$1 = 4h^2 N_1^6 - 3g^2 N_2^6. \tag{18}$$

Since $fg = 2hg$ is supposed to contain at most two distinct prime factors, and since $(h, g) = 1$ according to (18), we have either $h = 1$, or $g = 1$.

If we put $h = 1$, the equation (18) may be written

$$(gN_2^3 + 1)^3 - (gN_2^3 - 1)^3 = (2N_1^2)^3.$$

However, the Diophantine equation

$$x^3 + y^3 = z^3$$

has integral solutions only when $xyz = 0$. Thus we get the only solution $|N_1| = |N_2| = 1$, $f = 2$, and $g = 1$. As is shown in § 2, this solution corresponds to the equations

$$2x^3 + y^3 = 3 \quad \text{and} \quad 2x^3 + 41 \cdot y^3 = 3.$$

We do not take the former equation into consideration. The latter equation satisfies the conditions of the theorem.

If we put $g = 1$, we get from (18)

$$1 = 4h^2 N_1^6 - 3N_2^6,$$

which gives

$$2hN_1^3 \pm 1 = 3N_3^6, \quad 2hN_1^3 \mp 1 = N_4^6,$$

and

$$\pm 2 = 3N_3^6 - N_4^6,$$

where the lower sign is impossible modulo 3. The equation may be written

$$(N_4^2)^3 = 3(N_3^3)^2 - 2.$$

However, as was shown by T. NAGELL [5], the equation

$$x^3 = 3y^2 - 2$$

has the only integral solutions $x = 1$, $y = \pm 1$. We thus get $|N_4| = |N_3| = |N_2| = |N_1| = 1$ and $h = 1$, and again the above-mentioned equations.

2) fg is even and $g = 2h$.

Let us first suppose that h is odd. (17) may be written

$$1 = f^2 N_1^6 - 3 \cdot 4 h^2 N_2^6. \tag{19}$$

It is immediately evident that N_1 and f are odd integers. Further we have $f \neq 1$, because if $f = 1$, (19) could be written

$$(2hN_2^3 + 1)^3 - (2hN_2^3 - 1)^3 = 2(N_1^2)^3;$$

but this equation is impossible. Since $fg = 2fh$ is supposed to contain at most two distinct prime factors, and since $(f, h) = 1$ according to (19), we have $h = 1$. Hence (19) may be written

$$1 = f^2 N_1^6 - 3 \cdot 4 N_2^6,$$

which gives

$$fN_1^3 \pm 1 = 3 \cdot 2 N_3^6, \quad fN_1^3 \mp 1 = 2 N_4^6,$$

and

$$\pm 1 = 3 N_3^6 - N_4^6,$$

where the upper sign is impossible modulo 3. The equation may be written

$$(N_3^3 + 1)^3 - (N_3^3 - 1)^3 = 2(N_4^2)^3.$$

Hence the impossibility of (19) when h is odd.

Let us now suppose that h is even. Then $g = 4h_1$, where h_1 is odd, since A and B possess no cubic factors. (17) may be written

$$1 = f^2 N_1^6 - 3 \cdot 2^4 h_1^2 N_2^6. \tag{20}$$

As before, it is clear that f is odd and $\neq 1$. By (20) we have $(f, h_1) = 1$, and thus we get $h_1 = 1$. (20) gives

$$fN_1^3 \pm 1 = 3 \cdot 2^\alpha N_3^6, \quad fN_1^3 \mp 1 = 2^\beta N_4^6,$$

which gives

$$\pm 2 = 3 \cdot 2^\alpha N_3^6 - 2^\beta N_4^6.$$

We have either $\alpha = 1$, $\beta = 3$, or $\alpha = 3$, $\beta = 1$. Hence we get the two equations

$$\pm 1 = 3 N_3^6 - 4 N_4^6, \tag{21}$$

$$\pm 1 = 12 N_3^6 - N_4^6, \tag{22}$$

where the upper signs are impossible modulo 3. (22) may be written

$$(2N_3^3 + 1)^3 - (2N_3^3 - 1)^3 = 2(N_4^2)^3,$$

so that this equation is impossible. (21) may be written

$$4(N_4^2)^3 - 3(N_3^2)^3 = 1,$$

and since the equation $4x^3 - 3y^3 = 1$ has the only solution $x = y = 1$, we get $|N_4| = |N_3| = |N_2| = |N_1| = 1$ and $f = 7$. This gives $a = 7$, $b = 11$, $c = 2$, $d = 1$, $M = 1$, $N = 2$, or $a = 7$, $b = 571$, $c = 2$, $d = 1$, $M = -1$, $N = 2$.

The former solution corresponds to the equation

$$28x^3 + 11y^3 = 3,$$

which has the solution $x = 52$, $y = -71$. However, the number

$$\tfrac{1}{3}(52\sqrt[3]{28} - 71\sqrt[3]{11})^3$$

is not a biquadrate of the field $\boldsymbol{K}(\sqrt[3]{28 \cdot 11^2})$. If it were, system (7) would have a solution in integers X, Y, and Z. In this case, (7) may be written

$$\begin{cases} 77Z^2 + 2XY = 2, & (23) \\ Y^2 + 28XZ = 1, & \\ 2X^2 + 22YZ = -4. & (24) \end{cases}$$

It follows from (23) that Z is divisible by 2. If we put $Z = 2Z_1$, we may write (24)

$$X^2 + 22YZ_1 = -2,$$

so that X is divisible by 2. If we put $X = 2X_1$, we get from (23)

$$77 \cdot 2Z_1^2 + 2X_1Y = 1;$$

but this equation is impossible in integers X_1, Y, and Z_1.

The latter solution corresponds to the equation

$$28x^3 + 571y^3 = 3,$$

which has the solution $x = -724$, $y = 265$. However, the number

$$\tfrac{1}{3}(-724\sqrt[3]{28} + 265\sqrt[3]{571})^3$$

is not a biquadrate of the field $\boldsymbol{K}(\sqrt[3]{28 \cdot 571^2})$. If it were, system (7) would have a solution in integers X, Y, and Z. In this case the system may be written

$$\begin{cases} 7 \cdot 571\, Z^2 + 2\, X Y = -2, & (25) \\ Y^2 + 28\, X Z = 1, & \\ 2\, X^2 + 2 \cdot 571\, Y Z = -4. & (26) \end{cases}$$

It follows from (25) that Z, and from (26) that X is divisible by 2. If we put $Z = 2\, Z_1$ and $X = 2\, X_1$, we get from (25)

$$7 \cdot 571 \cdot 2\, Z_1^2 + 2\, X_1 Y = -1;$$

but this equation is impossible in integers X_1, Y, and Z_1.

3) fg is odd.

If we put $f = 1$ in (17), it may be written

$$(g N_2^3 + 1)^3 - (g N_2^3 - 1)^3 = 2\,(N_1^2)^3,$$

and we can see that this equation is impossible. Hence $f \neq 1$. N_1 is odd in (17); otherwise we would get from (17) the congruence

$$3\,(g N_2^3)^2 + 1 \equiv 0 \pmod 8,$$

which is impossible. Hence N_2 is an even integer, and we get

$$f N_1^3 \pm 1 = 3 \cdot 2^\alpha h^2 N_3^6, \quad f N_1^3 \mp 1 = 2^\beta k^2 N_4^6,$$

which gives

$$\pm 2 = 3 \cdot 2^\alpha h^2 N_3^6 - 2^\beta k^2 N_4^6. \tag{27}$$

We have either $\alpha = 1$, $\beta = 5$, or $\alpha = 5$, $\beta = 1$.

In the former case we get

$$\pm 1 = 3\, h^2 N_3^6 - 16\, k^2 N_4^6,$$

which is impossible modulo 8.

In the latter, we get

$$\pm 1 = 3 \cdot 16\, h^2 N_3^6 - k^2 N_4^6, \tag{28}$$

where the upper sign is impossible modulo 3. If $k = 1$, (28) may be written

$$(4\, h N_3^3 + 1)^3 - (4\, h N_3^3 - 1)^3 = 2\,(N_4^2)^3;$$

but this equation has no integral solution when N_3 and N_4 are $\neq 0$. Hence $k \neq 1$. Since fg is supposed to contain at most two distinct prime factors and since $(f, g) = 1$, $f \neq 1$, and $g = hk \neq 1$, g evidently contains only prime factors of the same kind, and we may put $g = p^n$, where p is an odd prime and $n = 1$, or $n = 2$. According to (28), we have $(h, k) = 1$, which implies $h = 1$. Hence the equation (28) may be written

$$1 = p^{2n} N_4^6 - 48\, N_3^6,$$

which gives

$$p^n N_4^3 \pm 1 = 3 \cdot 2^\gamma N_5^6, \quad p^n N_4^3 \mp 1 = 2^\varepsilon N_6^6,$$

and

$$\pm 2 = 3 \cdot 2^\gamma N_5^6 - 2^\varepsilon N_6^6. \tag{29}$$

We have either $\gamma = 1$, $\varepsilon = 3$, or $\gamma = 3$, $\varepsilon = 1$. Hence we get from (29) the two equations

$$\pm 1 = 3 N_5^6 - 4 N_6^6, \tag{30}$$

$$\pm 1 = 12 N_5^6 - N_6^6, \tag{31}$$

where the upper signs are impossible modulo 3. The equation (31) is identical with the equation (22) and thus impossible. (30) is identical with (21) and has the only solution $|N_6| = |N_5| = 1$, which gives $|N_4| = |N_3| = 1$, $|N_2| = 2$, $|N_1| = 1$, $g = 7$, and $f = 97$. We get $a = 679$, $b = 2131$, $c = d = M = 1$, $N = 4$, or $a = 679$, $b = 110771$, $c = d = 1$, $M = -1$, $N = 4$.

The former solution corresponds to the equation

$$679 x^3 + 2131 y^3 = 3,$$

which has the solution $x = 20168$, $y = -13775$. However, the number

$$\tfrac{1}{3}(20168 \sqrt[3]{679} - 13775 \sqrt[3]{2131})^3$$

is not a biquadrate of the field $\boldsymbol{K}(\sqrt[3]{679 \cdot 2131^2})$. If it were, system (7) would have a solution in integers X, Y, and Z. In this case, the system may be written

$$\begin{cases} 679 \cdot 2131 Z^2 + 2XY = 4, & (32) \\ Y^2 + 2 \cdot 679 XZ = 1, & (33) \\ X^2 + 2 \cdot 2131\, YZ = -8. & (34) \end{cases}$$

From (32) and (34) we see that Z and X, respectively, are even. It follows from (33) that Y is odd. We put $Z = 2Z_1$ and $X = 2X_1$ and get from (32) and (34)

$$679 \cdot 2131 Z_1^2 + X_1 Y = 1, \tag{35}$$

$$X_1^2 + 2131\, Y Z_1 = -2. \tag{36}$$

Let us suppose that X_1 is odd. From (35) we get that Z_1 is even, and from (36) that X_1 is even, which contradicts our postulate. Hence X_1 is even, and from (36) we get that Z_1 is even. We put $X_1 = 2X_2$ and $Z_1 = 2Z_2$, and get from (35)

$$679 \cdot 2131 \cdot 4 Z_2^2 + 2 X_2 Y = 1;$$

but this equation is impossible in integers X_2, Y, and Z_2.

The latter solution corresponds to the equation

$$679 x^3 + 110771 y^3 = 3,$$

which has the solution $x = -280904$, $y = 51409$. However, the number

$$\tfrac{1}{3}\left(-280904\sqrt[3]{679} + 51409\sqrt[3]{110771}\right)^3$$

is not a biquadrate of the field $\boldsymbol{K}(\sqrt[3]{679 \cdot 110771^2})$. If it were, system (7) would have a solution in integers X, Y, and Z. In this case, the system may be written

$$\begin{cases} 679 \cdot 110771\, Z^2 + 2\, XY = -4, & (37) \\ Y^2 + 2 \cdot 679\, XZ = 1, & (38) \\ X^2 + 2 \cdot 110771\, YZ = -8. & (39) \end{cases}$$

From (37) and (39) we see that Z and X, respectively, are even. It follows from (38) that Y is odd. We put $Z = 2\,Z_1$ and $X = 2\,X_1$, and get from (37) and (39)

$$679 \cdot 110771\, Z_1^2 + X_1 Y = -1, \tag{40}$$

$$X_1^2 + 110771\, Y Z_1 = -2. \tag{41}$$

Let us suppose that X_1 is odd. From (40) we get that Z_1 is even, and from (41) that X_1 is even, which contradicts our postulate. Hence X_1 is even, and from (41) we get that Z_1 is even. We put $X_1 = 2\,X_2$ and $Z_1 = 2\,Z_2$, and get from (40)

$$679 \cdot 110771 \cdot 4\, Z_2^2 + 2\, X_2 Y = 1;$$

but this equation is impossible in integers X_2, Y, and Z_2.

§ 4.

Let us suppose that equation (3) has a solution in integers and that η is the biquadrate of a unit in $\boldsymbol{R}$ when $C = 1$, and in $\boldsymbol{K}$ when $C = 3$. Then equations (6) and (8), respectively, have a solution in integers f, g, N_1, and N_2. Further, system (7) has a solution in integers X, Y, and Z. We know that $N = 2\,N_1 N_2$ and that $(M, N) = 1$. M is thus odd. The system may be written

$$\begin{cases} ab Z^2 + 2\, X\, Y = 2\, N_1 N_2 M, & (42) \\ d\, Y^2 + 2\, ac X Z = d M^2, & (43) \\ c X^2 + 2\, b d\, Y Z = -2\, c N_1^2 N_2^2. & (44) \end{cases}$$

Let us suppose that $A = ac^2$ and $B = bd^2$ are odd integers. From (42) we get that Z is even, from (43) that Y is odd, and from (44) that X is even. We put $X = 2\,X_1$ and $Z = 2\,Z_1$, and get from (42)

$$2\, ab Z_1^2 + 2\, X_1\, Y = N_1 N_2 M.$$

Thus we have either N_1 or N_2 divisible by 2. If N_1 is even, we get from (6) the congruence

$$3\,(3\, g N_2^3)^2 + 1 \equiv 0 \pmod 8,$$

and from (8) the congruence

$$3(gN_2^3)^2 + 1 \equiv 0 \pmod 8,$$

which are both impossible, so that N_1 must be odd and N_2 even. We put $N_2 = 2N_3$ and can write the system

$$\begin{cases} abZ_1^2 + X_1 Y = N_1 N_3 M, & (45) \\ dY^2 + 8acX_1 Z_1 = dM^2, & \\ cX_1^2 + bdYZ_1 = -2cN_1^2 N_3^2. & (46) \end{cases}$$

Let us suppose that N_3 is odd. If X_1 is odd it follows from (45) that Z_1 is even, and from (46) that X_1 is even, which contradicts our postulate. Hence X_1 is even, and from (46) we get that Z_1 is even, but this is impossible according to (45). Hence N_3 is an even integer, $N_3 = 2N_4$, and we conclude that a necessary condition for η being a biquadrate of $\boldsymbol{R}$ when $C = 1$ and of $\boldsymbol{K}$ when $C = 3$ is that the following equations have a solution in integers f, g, N_1, and N_2:

$$1 = f^2 N_1^6 - 27 \cdot 2^{12} g^2 N_4^6, \tag{47}$$

and

$$1 = f^2 N_1^6 - 3 \cdot 2^{12} g^2 N_4^6, \tag{48}$$

respectively, where $fg = A$ or $fg = B$.

Let us consider the equation (48). If $f = 1$, (48) may be written

$$(2^6 g N_4^3 + 1)^3 - (2^6 g N_4^3 - 1)^3 = 2(N_1^2)^3;$$

but this equation is impossible. Hence $f \neq 1$, and we get from (48)

$$fN_1^3 \pm 1 = 2^\alpha f_1^2 N_5^6, \quad fN_1^3 \mp 1 = 3 \cdot 2^\beta g_1^2 N_6^6,$$

which gives

$$\pm 2 = 2^\alpha f_1^2 N_5^6 - 3 \cdot 2^\beta g_1^2 N_6^6. \tag{49}$$

We have either $\alpha = 1$, $\beta = 11$, or $\alpha = 11$, $\beta = 1$. Hence we get from (49) the two equations

$$\pm 1 = f_1^2 N_5^6 - 3 \cdot 2^{10} g_1^2 N_6^6, \tag{50}$$

$$\pm 1 = 2^{10} f_1^2 N_5^6 - 3 g_1^2 N_6^6, \tag{51}$$

where the lower signs are impossible modulo 3. From (51) we get the congruence

$$3(g_1 N_6^3)^2 + 1 \equiv 0 \pmod 8,$$

so that this equation is impossible.

If the equation (48) has a solution in integers, so has equation (50). We have $f_1 g_1 = g$, and as before we can see that $f_1 \neq 1$, which implies $g \neq 1$. Since $(f, g) = 1$, fg contains at least two distinct prime factors.

Continuing this process, we arrive at the equations

$$1 = f_2^2 N_7^6 - 3 \cdot 2^8 g_2^2 N_8^6,$$

$$1 = f_3^2 N_9^6 - 3 \cdot 2^6 g_3^2 N_{10}^6,$$

$$1 = f_4^2 N_{11}^6 - 3 \cdot 2^4 g_4^2 N_{12}^6, \tag{52}$$

where $f_2 g_2 = g_1$, $f_3 g_3 = g_2$, and $f_4 g_4 = g_3$. Further we have $f_2 \neq 1$, $f_3 \neq 1$, and $f_4 \neq 1$, which implies $g_1 \neq 1$, $g_2 \neq 1$, and $g_3 \neq 1$. Since $(f_1, g_1) = 1$, $(f_2, g_2) = 1$, and $(f_3, g_3) = 1$, it follows that

$$f g = f f_1 f_2 f_3 f_4 g_4$$

must have at least five distinct prime factors if (48) is to have a solution in integers. We have thus proved

Theorem 3. *Let a, b, c, and d denote positive integers, relatively prime in pairs, and possessing no square factors. If the equation*

$$A x^3 + B y^3 = 3,$$

where $A = a c^2$ and $B = b d^2$ are odd integers not divisible by 3, containing at most four distinct prime factors each, has a solution in integers x and y, then

$$\eta = \tfrac{1}{3} (x \sqrt[3]{A} + y \sqrt[3]{B})^3$$

is the fundamental unit of the field $\mathbf{K}(\sqrt[3]{a c^2 b^2 d})$, or the square of this unit.

The reasoning will be quite analogous if we start from equation (47). We have only to substitute the coefficient 27 for 3 everywhere. (52) may then be written

$$1 = f_4^2 N_{11}^6 - 27 \cdot 2^4 g_4^2 N_{12}^6,$$

which gives

$$f_4 N_{11}^3 \pm 1 = 2^\gamma f_5^2 N_{13}^6, \quad f_4 N_{11}^3 \mp 1 = 27 \cdot 2^\varepsilon g_5^2 N_{14}^6,$$

and

$$\pm 2 = 2^\gamma f_5^2 N_{13}^6 - 27 \cdot 2^\varepsilon g_5^2 N_{14}^6.$$

We have either $\gamma = 1$, $\varepsilon = 3$, or $\gamma = 3$, $\varepsilon = 1$.

In the former case, we get

$$\pm 1 = f_5^2 N_{13}^6 - 27 \cdot 4 g_5^2 N_{14}^6, \tag{53}$$

where the lower sign is impossible modulo 3. If $f_5 = 1$, (53) may be written

$$(6 g_5 N_{14}^3 + 1)^3 - (6 g_5 N_{14}^3 - 1)^3 = 2 (N_{13}^2)^3;$$

but this equation is impossible. Hence $f_5 \neq 1$, and we get from (53)

$$f_5 N_{13}^3 \pm 1 = 2 f_6^2 N_{15}^6, \quad f_5 N_{13}^3 \mp 1 = 2 \cdot 27 g_6^2 N_{16}^6,$$

which gives

$$\pm 1 = f_6^2 N_{15}^6 - 27\, g_6^2 N_{16}^6,$$

where the lower sign is impossible modulo 3. However, as was shown by T. NAGELL [6], this equation has no solution in integers when the number of distinct prime factors in $g_5 = f_6 g_6$ is $\leqq 2$.

In the latter case, we get

$$\pm 1 = 4 f_5^2 N_{13}^6 - 27\, g_5^2 N_{14}^6, \tag{54}$$

where the lower sign is impossible modulo 3. If $f_5 = 1$, (54) may be written

$$1 = 4 N_{13}^6 - 27\, g_5^2 N_{14}^6;$$

but this equation is impossible modulo 9. Hence $f_5 \neq 1$, and we get from (54)

$$2 f_5 N_{13}^3 \pm 1 = f_6^2 N_{15}^6, \quad 2 f_5 N_{13}^3 \mp 1 = 27\, g_6^2 N_{16}^6,$$

which gives

$$\pm 2 = f_6^2 N_{15}^6 - 27\, g_6^2 N_{16}^6. \tag{55}$$

In (55) we have $f_6 \neq 1$, otherwise we would get

$$\pm 2 = N_{15}^6 - 27\, g_6^2 N_{16}^6,$$

which is impossible modulo 9.

Hence we have $f_5 g_5 = g_4$, $f_6 g_6 = g_5$, $f_5 \neq 1$, and $f_6 \neq 1$, which implies $g_4 \neq 1$, and $g_5 \neq 1$. Since $(f_4, g_4) = 1$ and $(f_5, g_5) = 1$,

$$f g = f f_1 f_2 f_3 f_4 f_5 f_6 g_6$$

must have at least seven distinct prime factors if (47) is to have a solution in integers. We have thus proved

Theorem 4. *Let* a, b, c, *and* d *denote positive integers, relatively prime in pairs, and possessing no square factors. If the equation*

$$A x^3 + B y^3 = 1,$$

where $A = ac^2$ *and* $B = bd^2$ *are odd integers* > 1, *containing at most six distinct prime factors each, has a solution in integers* x *and* y, *then*

$$\eta = (x\sqrt[3]{A} + y\sqrt[3]{B})^3$$

is the fundamental unit of the ring $\boldsymbol{R}(1, \sqrt[3]{ac^2b^2d}, \sqrt[3]{a^2cbd^2})$, *or the square of this unit.*

§ 5.

Let us now suppose that $A = ac^2$, or $B = bd^2$, is even and divisible by 4.

Let us further suppose that equation (3) has an integral solution and that η is a biquadrate of $\boldsymbol{R}$ when $C = 1$, and of $\boldsymbol{K}$ when $C = 3$. Let A be even and divisible by 4. Then $c = 2c_1$, where c_1 is odd.

We first consider the case $C = 1$.

If η is a square of $\boldsymbol{R}$, the following equation has a solution in integers x_1, y_1, and z_1 (Cp. NAGELL [4], p. 253):

$$x_1^3 a^2 c + y_1^3 b^2 d + z_1^3 a c^2 b d^2 - 3 x_1 y_1 z_1 a b c d = 1. \tag{56}$$

Further, we have either

$$\frac{2 x_1}{c} = \pm N^2, \quad \frac{y_1}{d} = \mp M^2, \quad z_1 = M N, \tag{57}$$

or

$$\frac{x_1}{c} = \pm N^2, \quad \frac{2 y_1}{d} = \mp M^2, \quad z_1 = M N. \tag{58}$$

When η is a biquadrate of $\boldsymbol{R}$, it follows from (56) and (57) that N is even, and that the following equation has a solution in integers f, g, N_1, and N_2:

$$1 = f^2 N_1^6 - 27 g^2 N_2^6, \tag{59}$$

where $f g = A$, and $2 N_1 N_2 = N$; it follows from (56) and (58) that M is even and that equation (59) has a solution in integers f, g, N_1, and N_2, where $f g = B$, and $2 N_1 N_2 = M$.

Let us regard the relations (58). Since M is even, y_1 and z_1 are even. Since c is even, x_1 is even; but this is impossible according to (56). In the present case we can thus only use relations (56) and (57). It is hence sufficient to consider (59) when $f g = A = 4 a c_1^2$.

Since η is supposed to be a biquadrate of $\boldsymbol{R}$, system (7) has a solution in integers X, Y, and Z. The system may be written

$$\begin{cases} a b Z^2 + 2 X Y = 2 N_1 N_2 M, & (60) \\ d Y^2 + 4 a c_1 X Z = d M^2, & (61) \\ 2 c_1 X^2 + 2 b d Y Z = -4 c_1 N_1^2 N_2^2. & (62) \end{cases}$$

Since $(M, N) = 1$, M is odd. Further a, b, c_1, and d are odd integers. It follows from (60) that Z is even, from (61) that Y is odd, and from (62) that X is even. We put $Z = 2 Z_1$ and $X = 2 X_1$, and get from (60)

$$2 a b Z_1^2 + 2 X_1 Y = N_1 N_2 M,$$

so that N_1 or N_2 must be even. As before, we conclude that N_1 is odd and N_2 even. We put $N_2 = 2 N_3$ and may write (59)

$$1 = f^2 N_1^6 - 27 \cdot 2^6 g^2 N_3^6. \tag{63}$$

It is easily seen that f is odd. Hence $g = 4 g_1$, and (63) may be written

$$1 = f^2 N_1^6 - 27 \cdot 2^{10} g_1^2 N_3^6, \tag{64}$$

where $f g_1 = \frac{A}{4}$ contains odd prime factors only. Equation (64) is analogous to (50), and exactly the same reasoning as in § 4 may now be applied. We have thus proved

Theorem 5. *Let a, b, c, and d denote positive integers, relatively prime in pairs, and possessing no square factors. If the equation*

$$Ax^3 + By^3 = 1,$$

where $A = ac^2$ *and* $B = bd^2$ *are* > 1, *and where one of the numbers A and B is divisible by* 4, *and contains at most five distinct odd prime factors, has a solution in integers x and y, then*

$$\eta = (x\sqrt[3]{A} + y\sqrt[3]{B})^3$$

is the fundamental unit of the ring $\boldsymbol{R}(1, \sqrt[3]{ac^2b^2d}, \sqrt[3]{a^2cbd^2})$, *or the square of this unit.*

We regard the case $C = 3$, and suppose that η is a biquadrate of $\boldsymbol{K}$. The reasoning is altogether the same as before. We have only to substitute the number 9 for 1 in the right member of (56), and the coefficient 3 for 27 in (59), (63), and (64). We obtain the following result:

Theorem 6. *Let a, b, c, and d denote positive integers, relatively prime in pairs, and possessing no square factors. If the equation*

$$Ax^3 + By^3 = 3,$$

where AB is not divisible by 3, *and where one of the numbers* $A = ac^2$ *and* $B = bd^2$ *is divisible by* 4, *and contains at most three distinct odd prime factors, has a solution in integers x and y, then*

$$\eta = \tfrac{1}{3}(x\sqrt[3]{A} + y\sqrt[3]{B})^3$$

is the fundamental unit of the field $\boldsymbol{K}(\sqrt[3]{ac^2b^2d})$, *or the square of this unit.*

Remark. Theorems 5 and 6 express a somewhat more general result than the one given in § 1. It is not necessary to postulate anything as to the number of distinct prime factors in the odd one of the integers A and B.

BIBLIOGRAPHY. [1] **B. Delaunay,** Journal Charkow Math. Soc. 1915 (in Russian), see also *Comptes rendus,* t. 162, 1916, p. 150. — [2] **T. Nagell,** Vollständige Lösung einiger unbestimmten Gleichungen dritten Grades, Videnskapsselskapets Skrifter, I. Mat.-Naturv. Klasse, no. 14, Kristiania 1922. — [3] ——, Über die Einheiten in reinen kubischen Zahlkörpern, Videnskapsselskapets Skrifter, I. Mat.-Naturv. Klasse, no. 11, Kristiania 1923. — [4] ——, Solution complète de quelques équations cubiques à deux indéterminées, Journal de Mathématiques, t. IV, 9^e^ sér., Paris 1925. — [5] ——, Einige Gleichungen von der Form $ay^2 + by + c = dx^3$, Avh. utgitt av Det Norske Videnskaps-Akademi i Oslo, I. Mat.-Naturv. Klasse, no. 7, Oslo 1930. — [6] ——, Zahlentheoretische Notizen VII—IX, Norsk Matematisk Forenings Skrifter. Serie I, no. 17, Oslo 1927. — [7] ——, Zahlentheoretische Sätze, Avh. utgitt av Det Norske Videnskaps-Akademi i Oslo, I. Mat.-Naturv. Klasse, no. 5, Oslo 1930.

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