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Pseudo-lattices: Theory and applications

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The notion of a partially ordered set is well-known. It is also known that a quasi-ordered (pre-ordered) set is a system consisting of a set X and a binary relation $\geqslant$ satisfying the following laws:

P_1: For all x in X, $x \geqslant x$ (Reflexive); P_2: If $x \geqslant y$ and $y \geqslant z$, then $x \geqslant z$ (Transitive).

In a quasi-ordered set if a least upper bound or a greatest lower bound of some subset exists it may not exist uniquely, since we do not necessarily have antisymmetry for the quasi-ordering. This motivates the following:

Definition 1. *A quasi-ordered set is called a pseudo-lattice iff any two elements have at least one least upper bound and at least one greatest lower bound.*

Before we construct new pseudo-lattices from given ones, we need more definitions:

Definition 2. *Let $\geqslant$ and $\gg$ be two quasi-orderings on a given set X, then $\gg$ is stronger than $\geqslant$ iff $x \geqslant y$ implies $x \gg y$.*

Definition 3. *Let $(X, \geqslant)$ and $(Y, \geqslant)$ be two quasi-ordered sets, $f: X \to Y$ a mapping. f is order-preserving iff $a \geqslant b$ implies $f(a) \geqslant f(b)$. f is called bi-order-preserving iff*

(1) *$a \geqslant b$ implies $f(a) \geqslant f(b)$ and*
(2) *$f(a) \geqslant f(b)$ implies $a \geqslant b$.*

Definition 4. *Two quasi-ordered sets $(X, \geqslant)$ and $(Y, \geqslant)$ are called isomorphic iff there exists a bijective bi-order-preserving mapping f of X onto Y, i.e., iff there exists a one-to-one-mapping f of X onto Y such that $f(a) \geqslant f(b)$ iff $a \geqslant b$.*

Theorem 1. *Let X be a set, $(Y, \gg)$ a quasi-ordered set and $f: X \to Y$ a mapping. Then there exists a strongest quasi ordering $\geqslant_f$ on X under which f preserves ordering. Furthermore, $(X, \geqslant_f)$ is a pseudo-lattice if $(Y, \gg)$ is a pseudo-lattice and f an onto mapping.*

Proof A binary relation $\geqslant_f$ on X is defined by setting $a \geqslant_f b$ iff $f(a) \gg f(b)$. Evidently $\geqslant_f$ is a quasi-ordering on X under which f preserves ordering. Suppose f preserves ordering under a quasi-ordering $\geqslant$ on X. Then $a \geqslant b$ implies $f(a) \gg f(b)$. This in turn implies $a \geqslant_f b$. Thus $\geqslant_f$ is the strongest quasi-ordering on X under which f preserves ordering.

Suppose $(Y, \gg)$ is a pseudo-lattice and f is an onto mapping. Let a and b be any two elements in X. Let y be a l.u.b. of $f(a)$ and $f(b)$, then there exists c in X such that $f(c)=y$ and c is a l.u.b. of a and b. Since $f(c) \gg f(a)$ and $f(c) \gg f(b)$, c is an upper bound of a and b. Suppose d is an upper bound of a and b. Then $f(d) \gg f(a)$, $f(d) \gg f(b)$ and $f(d) \gg f(c)$, because $f(c)$ is a l.u.b. of $f(a)$ and $f(b)$. This implies $d \geqslant_f c$ and c is therefore a l.u.b. of a and b. The existence of a g.l.b. of a and b can be proved similarly. Thus $(X, \geqslant_f)$ is a pseudolattice.

Definition 5. *Let X be a set, Y a quasi-ordered set and $f: X \to Y$ a mapping. The strongest quasi-ordering on X under which f preserves ordering is called the quasi-ordering induced by f.*

Theorem 2. *Let X, Y, Z be quasi-ordered sets and $f: X \to Y$, $g: Y \to Z$ be mappings. Suppose further that Y has the induced quasi ordering relative to g. Then f is order-preserving iff $g \circ f$ is order-preserving.*

Proof. Suppose that f preserves ordering, then $g \circ f$ preserves ordering, since g preserves ordering. Conversely suppose that $g \circ f$ preserves ordering. Assume that a and b are in X with $a \geqslant b$. Then $(g \circ f)(a) \geqslant (g \circ f)(b)$, i.e., $g(f(a)) \geqslant g(f(b))$. Hence $f(a) \geqslant f(b)$, since Y has the quasi-ordering induced by g. Thus f preserves ordering.

Corollary 1. *Suppose that Y has the induced quasi-ordering relative to $g: Y \to Z$. Then the quasi-ordering induced on X by $f: X \to Y$ coincides with the quasi-ordering induced by $g \circ f$.*

Proof. This corollary follows directly from Theorem 2.

More theorems on constructing quasi-ordered sets will be given after the following

Definition 6. *Given quasi-ordered sets $(Z, \geqslant)$ and $(W, \gg)$. Let $F: Z \to W$ be an onto mapping. F^{-1}, as a set function, is called orderpreserving iff $x \geqslant y$ whenever $x \in F^{-1}(u) \equiv F^{-1}(\{u\})$, $y \in F^{-1}(v) \equiv F^{-1}(\{v\})$ and $u \gg v$.*

Theorem 3. *Let $(Z, \geqslant)$ be a quasi-ordered set and $F: Z \twoheadrightarrow W$ an onto mapping. Then there exists a strongest quasi-ordering $\gg$ on W under which F^{-1} preserves ordering. Further, $(W, \gg)$ is a lattice if (1) $(Z, \geqslant)$ is a pseudo-lattice and (2) $F(x)=F(y)$ iff $x \geqslant y$ and $y \geqslant x$.*

Proof. Define a binary relation $\gg$ on W by setting $u \gg v$ iff $x \geqslant y$ whenever $x \in F^{-1}(u)$ and $y \in F^{-1}(v)$. Clearly $\gg$ is a quasi-ordering on W under which F^{-1} preserves ordering. Suppose F^{-1} preserves ordering under a quasi-ordering $\gg_0$ on W. If $u \gg_0 v$, then $x \geqslant y$ whenever $x \in F^{-1}(u)$ and $y \in F^{-1}(v)$. This implies $u \gg v$. Thus $\gg$ is the strongest quasi-ordering on W under which F^{-1} preserves ordering.

We shall now prove that $(W, \gg)$ is a lattice under the further assumptions (1) and (2). First, we notice that the antisymmetry of $\gg$ follows from (2). Let u and v be any two elements in W. Then there exist x and y in Z such that $F(x)=u$ and $F(y)=v$. There also exists z, a l.u.b. of x and y, since $(Z, \geqslant)$ is a pseudo-lattice. Let $t=F(z)$, then clearly $t \gg u$, $t \gg v$ and t is an upper bound of u and v. Suppose w is also an upper bound of u and v. Then $\zeta \geqslant x$, $\zeta \geqslant y$ and $\zeta \geqslant z$ whenever $\zeta \in F^{-1}(w)$ and $z \in F^{-1}(t)$. Therefore, $F(\zeta)=w \gg t=F(z)$ and t is the l.u.b. of u and v. Similarly we can prove the unique existence of the g.l.b. of u and v. Thus $(W, \gg)$ is a lattice.

Definition 7. *Let $(Z, \geqslant)$ be a quansi-ordered set, W a set and $F: Z \twoheadrightarrow W$ an onto mapping. The strongest quasi-ordering on W under which F^{-1} preserves ordering is called the identification quasi-ordering relative to F. If W is considered to have this quasi-ordering, then F is called an identification mapping.*

Theorem 4. *Let Z, W, S be quasi-ordered sets, $F: Z \twoheadrightarrow W$ an identification mapping, and $G: W \to S$ a mapping. Then G^{-1} preserves ordering iff $(G \circ F)^{-1}$ preserves ordering.*

Proof. Let s and t be in S with $s \geqslant t$. Let $u \in G^{-1}(s)$ and let $v \in G^{-1}(t)$. Then $u \geqslant v$ iff $\forall x$, $x \in F^{-1}(u)$; $\forall y$, $y \in F^{-1}(v)$, $x \geqslant y$. That is to say G^{-1} preserves ordering iff $\forall x$, $x \in F^{-1}(G^{-1}(s))$, $\forall y$, $y \in F^{-1}(G^{-1}(t))$, $x \geqslant y$, whenever $s \geqslant t$. That means G^{-1} preserves ordering if and only if $(G \circ F)^{-1}$ preserves ordering.

Corollary 1. *Let Z be a quasi-ordered set and $F: Z \twoheadrightarrow W$ be an identification mapping. The identification quasi-ordering on S relative to $G: W \twoheadrightarrow S$ coincides with the identification quasi-ordering relative to $G \circ F$.*

Proof. The proof follows directly from Theorem 4.

Theorem 5. *Suppose that $(X, \geqslant)$ is a pseudo-lattice and Y is a set. Let $F: X \twoheadrightarrow Y$ be an onto mapping such that $F(a) = F(b)$ iff $a \geqslant b$ and $b \geqslant a$. Then the lattice $(Y, \gg)$ is isomorphic to the lattice $(X/\sim, \geqslant/\sim)$ where $\gg$ is the identification partial ordering on Y relative to F, $X/\sim$ is the quotient set of X over the equivalence relation $\sim$, $a \sim b$ iff $a \geqslant b$ and $b \geqslant a$, and $\geqslant/\sim$ is the identification partial ordering on $X/\sim$ relative to the quotient mapping from X onto $X/\sim$.*

Proof. By Theorem 3, it is clear that both $(Y, \gg)$ and $(X/\sim, \geqslant/\sim)$ are lattices. To prove that they are isomorphic, define $\bar{F}: X/\sim \to Y$ by setting $\bar{F}(\bar{a}) = F(a)$. It is well-known that $\bar{F}$ is a bijection .Apply Theorem 4 twice, to infer that both $\bar{F}$ and $\bar{F}^{-1}$ preserve ordering. Therefore, $\bar{F}$ is a lattice isomorphism.

To trace the correlation between the induced quasi-ordering and the identification quasi-ordering, we present the following:

Theorem 6. *A quasi-ordered set $(X, \geqslant)$ is a pseudo-lattice iff there exists a surjective bi-order-preserving mapping F from $(X, \geqslant)$ onto some lattice $(Y, \gg)$.*

Proof. For necessity, the quotient lattice $(X/\sim, \geqslant/\sim)$ and the quotient mapping $\varphi \equiv F: X \twoheadrightarrow X/\sim$ will apparently serve the purpose. To prove the sufficiency, assume F is a surjective bi-order-preserving mapping from $(X, \geqslant)$ onto some lattice $(Y, \gg)$. If we can prove that the quasi-ordering $\geqslant$ on X coincides with the quasi-ordering induced by F, then by Theorem 1, we know that $(X, \geqslant)$ is a pseudo-lattice. Let $\geqslant_F$ be the induced quasi-ordering on Y relative to F, then clearly $a \geqslant b$ implies $a \geqslant_F b$, since $\geqslant_F$ is stronger than $\geqslant$. Suppose $a \geqslant_F b$, then $F(a) \geqslant F(b)$. This implies $a \geqslant b$, since F is bi-order-preserving. Thus $\geqslant_F$ coincides with $\geqslant$ and the theorem is proved.

Corollary 1. *A quasi-ordered set $(X, \geqslant)$ is a lattice iff there exists a bijective bi-order-preserving mapping F from $(X, \geqslant)$ onto some lattice $(Y, \gg)$.*

Corollary 2. *Suppose that there exists a surjective bi-order-preserving mapping F from a quasi-ordered set $(X, \geqslant)$ onto some lattice $(Y, \gg)$. Then there exists a unique* *lattice-isomorphism* $G:(X/\sim, \geqslant/\sim)\to(Y,\gg)$ *such that* $F=G\circ\varphi$, *where* φ *is the quotient mapping from* $(X,\geqslant)$ *onto* $(X/\sim, \geqslant/\sim)$ *and* $\sim$ *is such an equivalence relation that* $a\sim b$ *iff* $a\geqslant b$ *and* $b\geqslant a$.

Proof. Define $G:(X/\sim, \geqslant/\sim)\to(Y,\gg)$ by $G(\bar{a})=F(a)$. It is easy to verify that G is a well-defined onto function. If $G(\bar{a})=G(\bar{b})$, then $F(a)\gg F(b)$ and $F(a)\ll F(b)$. The bi-order-preserving of F implies $a\geqslant b$, $b\geqslant a$ and $\bar{a}=\bar{b}$. Therefore, G is one-to-one. We shall now prove that $\gg$ on Y coincides with $\gg_F$, the identification quasi-ordering on Y relative to F. Since $\gg_F$ is stronger than $\gg$, $F(a)\gg F(b)$ implies $F(a)\gg_F F(b)$. Suppose $F(a)\gg_F F(b)$, then $a\geqslant b$ and in turn $F(a)\gg F(b)$. That means $\gg$ coincides with the identification quasi-ordering $\gg_F$. Apply Theorem 4, to infer that both G and G^{-1} preserve ordering. G is therefore a lattice-isomorphism. The uniqueness of such an isomorphism follows directly from the requirement $F=G\circ\varphi$.

Applications

I. Let $\mathfrak{F}$ be the set of all non-negative real-valued functions on a non-empty set X. Define a binary relation $\geqslant$ on $\mathfrak{F}$ by setting $f\geqslant g$ iff $g(x)=0$ implies that $f(x)=0$. Clearly $\geqslant$ is a quasi-ordering which does not have the antisymmetry property. Notice that f and any positive constant multiple af have the same zeros but $af\neq f$ if $a\neq 1$. To prove $(\mathfrak{F},\geqslant)$ is actually a pseudo-lattice, we give two different methods.

Method I-A. Denote the collection of all subsets of X by 2^X. It is well-known that under set inclusion 2^X is a lattice, therefore, a pseudo-lattice. Define function φ: $\mathfrak{F}\to 2^X$ by setting $\varphi(f)=\{x\,|\,x\in X,\ f(x)=0\}$. Clearly φ is an onto function. It is also clear that the quasi-ordering induced by φ coincides with $\geqslant$. By Theorem 1, $(\mathfrak{F},\geqslant)$ is therefore a pseudo-lattice.

Method I-B. Let f and g be any two elements in $\mathfrak{F}$. Define functions h and j by setting respectively

$$h(x)=\begin{cases}0, & \text{if}\quad f(x)g(x)=0\\ (f+g)(x), & \text{if}\quad f(x)g(x)\neq 0.\end{cases}$$

$$j(x)=\begin{cases}0, & \text{if}\quad f(x)g(x)=0\\ P, & \text{for those } x\text{'s elsewhere, where}\\ & P \text{ is a positive constant.}\end{cases}$$

It can be verified easily that both h and j are least upper bounds of f and g. On the other hand, define k by setting

$$k(x)=\begin{cases}0, & \text{if}\quad f(x)=0\quad\text{and}\quad g(x)=0\\ Q, & \text{for those } x\text{'s elsewhere, where}\\ & Q \text{ is a positive constant.}\end{cases}$$

We can easily verify that both $f+g$ and k are greatest lower bounds of f and g. Therefore, $(\mathfrak{F},\geqslant)$ is a pseudo-lattice. By Corollary 2 to Theorem 6, the quotient lattice $(\mathfrak{F}/\sim, \geqslant/\sim)$ is isomorphic to the lattice $(2^X, \cup, \cap)$.

II. A non-empty set X together with a σ-algebra $\mathfrak{a}$ of subsets of X is called a measurable space. A measure m on $\mathfrak{a}$ is said to be absolutely continuous with respect to a measure n on $\mathfrak{a}$, in symbols, $m \ll n$, iff $E \in \mathfrak{a}$ and $n(E)=0$ imply $m(E)=0$.

Let $\mathfrak{M}$ denote the set of all finite non-negative measures on $\mathfrak{a}$. Evidently $(\mathfrak{M}, \ll)$ is a quasi-ordered set without antisymmetry, since $m \ll \alpha m$, $\alpha m \ll m$ but $m \neq \alpha m$ if α is a positive real number different from 1. Given m and n in $\mathfrak{M}$. Different least upper bounds of m and n can be constructed by two distinct methods.

Method II-A. Let m and n be any two elements of $\mathfrak{M}$. Since $(m+n)(E)=0$ iff $m(E)=0=n(E)$, it can be verified easily that $m+n$ is a l.u.b. of m and n, that any linear combination $am+bn$, with positive coefficients a and b, is also a l.u.b. of m and n.

Method II-B. Given m and n in $\mathfrak{M}$. Clearly $m \ll m+n$ and $n \ll m+n$. Put $m+n=\nu$. By Radon-Nikodym Theorem, there exist non-negative finite-valued measurable functions f and g such that for every $E \in \mathfrak{a}$

$$m(E)=\int_E f d\nu \quad \text{and} \quad n(E)=\int_E g d\nu.$$

Let $h(x)=\sup \{f(x), g(x)\}$. Then the measure β defined on $\mathfrak{a}$ by

$$\beta(E)=\int_E h d\nu \quad \forall E \in \mathfrak{a}$$

is finite, since $h(x) \leqslant f(x)+g(x)$.

It is well-known that $\beta(E)=0$ implies $h=0$ ν-a.e. on E. This in turn implies $f=0$ ν-a.e. on E, $g=0$ ν-a.e. on E, and $m(E)=0=n(E)$. Therefore $m \ll \beta$, $n \ll \beta$ and β is an upper bound of m and n. It follows from $0 \leqslant h(x) \leqslant f(x)+g(x)$ that $\beta \ll m+n$. In Method II-A, we have shown that $m+n$ is a l.u.b. of m and n. Hence β must be equivalent to $m+n$, i.e., $\beta \ll m+n$ and $m+n \ll \beta$. Later on an example will show that β is not equal to any positive linear combination of m and n.

We shall also give two methods of constructing a g.l.b. for m and n.

Method II-C. By one version of the Lebesgue Decomposition Theorem [1], for any two finite measures m and n on the same σ-algebra $\mathfrak{a}$, there exists a decomposition of X into mutually disjoint measurable sets A, B, C such that $m_A=0$, $n_B=0$; $m_C \ll n_C$, $n_C \ll m_C$; where m_A is a measure on $\mathfrak{a}$ defined by $m_A(E)=m(A \cap E)$ for all $E \in \mathfrak{a}$, n_B, m_C and n_C are defined similarly.

Define a finite measure λ on $\mathfrak{a}$ by $\lambda(E)=(m+n)(C \cap E)$ for all $E \in \mathfrak{a}$. If $m(E)=0$, then $m_C(E)=n_C(E)=0$ and $\lambda(E)=(m+n)(C \cap E)=0$. Hence $\lambda \ll m$, similarly $\lambda \ll n$. Suppose that l is a lower bound of m and n. Also suppose $\lambda(F)=(m+n)(C \cap F)=0$, then $m(C \cap F)=0=n(C \cap F)$. This implies $l(C \cap F)=0$. Furthermore,

$$\begin{aligned} l(F) &= l(F \setminus C)+l(C \cap F) \\ &= l[(F \setminus C) \cap A]+l[(F \setminus C) \cap B]. \end{aligned}$$

It follows from $m_A=0$ that $m[(F \setminus C) \cap A]=0$. This in turn implies $l[(F \setminus C) \cap A]=0$, since $l \ll m$. Similarly, we have $l[(F \setminus C) \cap B]=0$. Therefore, $l(F)=l[(F \setminus C) \cap A]+l[(F \setminus C) \cap B]=0$, $l \ll \lambda$, and λ is a g.l.b. of m and n.

Method II-D. Our second method will show its importance in some later result. Given finite measures m and n, then by Radon-Nikodym Theorem, there exist non-negative finite-valued measurable functions f and g such that for every $E \in \mathfrak{a}$.

$$m(E) = \int_E f d\nu \quad \text{and} \quad n(E) = \int_E g d\nu$$

where $\nu = m + n$. Let $k(x) = \inf \{f(x), g(x)\}$, then the measure γ defined on $\mathfrak{a}$ by

$$\gamma(E) = \int_E k d\nu \quad \forall E \in \mathfrak{a}$$

is obviously finite. If $m(E) = 0$, then $f = 0$ ν-a.e. on E and $k = 0$ ν-a.e. on E. Hence $\gamma(E) = \int_E k d\nu = 0$ and $\gamma \ll m$. Similarly, $\gamma \ll n$. To prove γ is actually a g.l.b. of m and n, let l be a lower bound of m and n. Then $l \ll m$, $l \ll n$ and $l \ll m + n = \nu$. Apply Radon-Nikodym Theorem a gain, to infer the existence of some non-negative measurable function j such that for every $E \in \mathfrak{a}$

$$l(E) = \int_E j d\nu.$$

If $\gamma(E) = \int_E k d\nu = 0$, then $k = 0$ ν-a.e. on E, i.e., $\nu\{x \in E \mid k(x) > 0\} = 0$. Since $k(x) = \inf \{f(x), g(x)\}$, $\{x \in E \mid k(x) > 0\} = \{x \in E \mid f(x) > 0\} \cap \{x \in E \mid g(x) > 0\}$. Put $G = \{x \in E \mid f(x) > 0\}$, $H = \{x \in E \mid g(x) > 0\}$. Evidently G and H are measurable sets with $\nu(G \cap H) = 0$, i.e., $(m + n)(H \cap G) = 0$. This gives $m(G \cap H) = 0 = n(G \cap H)$ and $l(G \cap H) = 0$, since $l \ll m$. Noticing $G \cap H \subset E$ and

$$E = (G \cap H) \cup [E \setminus (G \cap H)] = (G \cap H) \cup (E \setminus G) \cup (E \setminus H),$$

we have

$$l(E) \leqslant l(G \cap H) + l(E \setminus G) + l(E \setminus H)$$

$$\leqslant l(E \setminus G) + l(E \setminus H).$$

By the construction of G and H, $f(x) = 0$ $\forall x \in E \setminus G$, $g(x) = 0$ $\forall x \in E \setminus H$. Consequently, $m(E \setminus G) = 0 = n(E \setminus H) = 0$. In turn, $l(E \setminus G) = 0 = l(E \setminus H)$, since $l \ll m$ and $l \ll n$. Therefore, $l(E) = 0$, $l \ll \gamma$, and γ is a g.l.b. of m and n.

So we know that $(\mathfrak{M}, \ll)$ is a pseudo-lattice. Let us define an equivalence relation $\sim$ on $\mathfrak{M}$ by setting $m \sim n$ iff $m \ll n$ and $n \ll m$. Then by Theorem 3, $\mathfrak{M}/\sim$ together with the identification ordering $\ll/\sim$ relative to the quotient mapping is a lattice.

It should be pointed out that in [5] there is an indirect proof of the existence of the l.u.b. and the g.l.b. of any two elements $\bar{m}$ and $\bar{n}$ in $\mathfrak{M}/\sim$.

In our proof, we have both $m + n$ and β, $\beta(E) = \int_E \sup \{f, g\}\, d\nu$, as least upper bounds for m and n. We are ready to give a negative answer to the following natural question: Is β always a positive linear combination of m and n?

Let $X = [0, 1]$, $\mathfrak{a}$ = the set of all Lebesgue measurable sets on $[0, 1]$. Let measures m and n be defined by

$$m(E) = \int_E \varphi(x)\, dx \quad \forall E \in \mathfrak{a} \quad \text{where} \quad \varphi(x) = \begin{cases} 0 & 0 \leqslant x < \frac{1}{2} \\ 1 & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

$n(E)=\int_E \psi(x)\,dx \quad \forall E\in \mathfrak{a}$ where

$$\psi(x)=\begin{cases}0 & 0\leqslant x<\frac{1}{3}\\ 1 & \frac{1}{3}\leqslant x\leqslant \frac{2}{3}\\ 0 & \frac{2}{3}<x\leqslant 1.\end{cases}$$

By Radon-Nikodym Theorem, there exist non-negative measurable functions f and g such that

$$m(E)=\int_E f d\nu \quad \forall E\in \mathfrak{a} \quad \text{where} \quad \nu=m+n,$$

$$n(E)=\int_E g d\nu \quad \forall E\in \mathfrak{a}.$$

Let $h=\sup\{f,g\}$, $\beta(E)=\int_E h d\nu\ \forall E\in\mathfrak{a}$. On $(\frac{2}{3},1]$, $\psi(x)=0$, $n([\frac{2}{3},1])=\int_{[\frac{2}{3},1]}\psi dx=0$. On the other hand, $n([\frac{2}{3},1])=\int_{[\frac{2}{3},1]} g d\nu=0$ hence $g=0$ $\nu-$a.e. on $[\frac{2}{3},1]$. This gives rise to $h=\sup\{f,g\}=f$ $\nu-$a.e. on $[\frac{2}{3},1]$. By a similar argument we obtain that $h=g\nu-$a.e. on $[0,\frac{2}{3}]$. Suppose $\beta=am+bn$ for some $a\geqslant 0, b\geqslant 0$. $\beta([0,\frac{2}{3}])=am([0,\frac{2}{3}])+bn([0,\frac{2}{3}])$, i.e., $\frac{1}{3}=(a/6)+(b/3)$, $2=a+2b$. On the other hand, $\beta([\frac{2}{3},1])=am([\frac{2}{3},1])+bn([\frac{2}{3},1])$, i.e., $\frac{1}{3}=(a/3)$. We have $a=1$, $b=\frac{1}{2}$. But $\beta([0,\frac{1}{2}])=am([0,\frac{1}{2}])+bn([0,\frac{1}{2}])$, i.e., $\beta([0,\frac{1}{2}])=(b/6)=\int_{[0,\frac{1}{2}]}h d\nu\geqslant\int_{[0,\frac{1}{2}]} g d\nu=n([0,\frac{1}{2}])=\frac{1}{6}$. Therefore, $b\geqslant 1$ which contradicts $b=\frac{1}{2}$. This shows that $\beta\neq am+bn$ for any $a\geqslant 0$, $b\geqslant 0$.

III. Given a measurable space $(X,\mathfrak{a})$ together with a finite measure μ on $\mathfrak{a}$. $(X,\mathfrak{a},\mu)$ is called a measure space. Let $\mathfrak{X}$ be the set of all non-negative integrable functions f such that $\int_X f d\mu<\infty$. A binary relation $\geqslant$ on $\mathfrak{X}$ is defined by $f\geqslant g$ iff $E\in\mathfrak{a}$ and $f=0$ $\mu-$a.e. on E imply $g=0$ $\mu-$a.e. on E. Clearly $(\mathfrak{X},\geqslant)$ is a quasi-ordered set without antisymmetry. We have two ways to prove that $(\mathfrak{X},\geqslant)$ is actually a pseudo-lattice, one is suggested by Theorem 1, the other is probably more constructive.

Method III-A. Let $\mathcal{N}$ be the set of all finite measures which are absolutely continuous with respect to the given measure μ on $\mathfrak{a}$, i.e., $\mathcal{N}=\{m\,|\,m,\text{ finite measure}, m\ll\mu\}$. Then as a direct consequence of the results in II, $(\mathcal{N},\ll)$ is also a pseudo-lattice. A function Φ from $\mathfrak{X}$ to $\mathcal{N}$ can be defined as follows:

$\Phi(f)=m_f$ where m_f is such a measure on $\mathfrak{a}$ that $m_f(E)=\int_E f d\mu\ \forall E\in\mathfrak{a}$. By Radon-Nikodym Theorem, Φ is an onto function. If we can prove that $\geqslant$ on $\mathfrak{X}$ coincides with the induced quasi-ordering $\geqslant_\Phi$, then by Theorem 1, $(\mathfrak{X},\geqslant)$ is a pseudo-lattice. First, $f\geqslant g$ implies $f\geqslant_\Phi g$, since $\geqslant_\Phi$ is stronger than $\geqslant$. Secondly, assume $f\geqslant_\Phi g$, then by the construction of the induced quasi-ordering $\Phi(g)=m_g\ll m_f=\Phi(f)$. If $f=0$ $\mu-$a.e. on E, then $m_f(E)=\int_E f d\mu=0$. And $g=0$ $\mu-$a.e. on E is implied by $m_g\ll m_f$. Therefore, $f\geqslant g$. The induced quasi-ordering $\geqslant_\Phi$ is exactly the same as $\geqslant$ and $(\mathfrak{X},\geqslant)$ is a pseudo-lattice. Furthermore, it is easy to see that $g\leqslant f$ iff $m_g\ll m_f$. This shows that Φ is bi-order-preserving. Consequently, $P\circ\Phi$ is surjective and bi-order-preserving, where P is the quotient mapping from $\mathcal{N}$ onto $\mathcal{N}/\sim$. By Corollary 2 to Theorem 6, the lattice $(\mathfrak{X}/\sim,\geqslant/\sim)$ is isomorphic to the lattice $(\mathcal{N}/\sim,\ll/\sim)$. We also have the following commutative diagram:

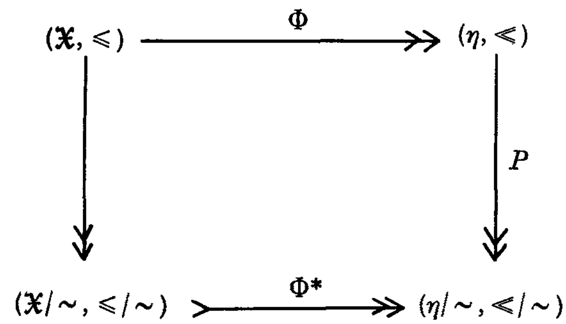

Fig. 1

Method III-B. To exhibit explicitly a l.u.b. and a g.l.b. of any two elements f and g in $\mathfrak{X}$, we first prove the following

Lemma. *In* $(\mathfrak{X}, \geq)$, $f \geq g$ *iff there exists a finite-valued measurable function* φ *such that* $g = \varphi f$ μ*-a.e. on* X.

Proof. The sufficiency is immediate. For necessity, suppose $f \geq g$. Define measures m and n by

$$m(E) = \int_E f d\mu \quad \forall E \in \mathfrak{a}$$

$$n(E) = \int_E g d\mu \quad \forall E \in \mathfrak{a}.$$

It is clear that $m \ll \mu$ and $n \ll \mu$. Furthermore, $g \leq f$ implies that $n \ll m \ll \mu$. Under the condition $n \ll m \ll \mu$, a theorem on the Radon-Nikodym derivative [4] guarantees the existence of a non-negative finite-valued measurable function φ such that $g = \varphi f$ μ-a.e. on X, where φ is such a function that

$$n(E) = \int_E \varphi dm \quad \forall E \in \mathfrak{a}.$$

We shall now prove that $(\mathfrak{X}, \geq)$ is a pseudo-lattice. Let $h(x) = \sup\{f(x), g(x)\}$ for any two elements f and g in $\mathfrak{X}$. Evidently h is in $\mathfrak{X}$. Using the fact that $h = 0$ μ-a.e. on E, $E \in \mathfrak{a}$, iff $f = 0$ μ-a.e. on E and $g = 0$ μ-a.e. on E, we can easily verify that h is a l.u.b. of f and g. Let $k(x) = \inf\{f(x), g(x)\}$, then k is in $\mathfrak{X}$. If $f = 0$ μ-a.e. on E, $E \in \mathfrak{a}$, then $k = 0$ μ-a.e. on E. Thus $f \geq k$. Similarly, $g \geq k$. Suppose that j is a lower bound of f and g, i.e., $f \geq j$ and $g \geq j$. By the preceding Lemma there exist finite-valued measurable functions φ and ψ such that

$$j = \varphi f \ \mu\text{-a.e. on } X \quad \text{and}$$

$$j = \psi g \ \mu\text{-a.e. on } X.$$

If $k = 0$ μ-a.e. on E, then $f = 0$ μ-a.e. on E or $g = 0$ μ-a.e. on E. This implies $j = 0$ μ-a.e. on E. Hence $k \geq j$ and k is a g.l.b. of f and g. We complete the proof that

$(\mathfrak{X}, \geqslant)$ is a pseudo-lattice. One final remark: Let us look back at the proof of Theorem 1. Under suitable assumptions, we proved that $(X, \geqslant_f)$ together with the quasi-ordering $\geqslant_f$ induced by f is a pseudo-lattice. We found that c is a sup of a, b in X where c has the property that $f(c)$ is a sup of $f(a)$ and $f(b)$ in Y. Therefore, it is not surprising at all that Method II-B and Method III-B are closely related by the following equality:

$$\beta(E) = \int_E \sup\{f, g\}\, d\nu = \int_E h d\nu = \Phi_h(E) = \Phi_{\sup\{f,g\}}(E).$$

IV. Let X be a Hausdorff, completely regular topological space. (f, Y) is called a Hausdorff compactification of X iff

(1) Y is a compact Hausdorff space.

(2) $f: X \to Y$ is a homeomorphism onto $f(X)$ and $f(X)$ is dense in Y.

Let $K(X) = \{(f, Y) \mid (f, Y)$ a Hausdorff compactification of $X\}$. A binary relation $\geqslant$ on $K(X)$ can be defined as follows: $(f, Y) \geqslant (g, Z)$ iff there exists a continuous surjection $h: Y \to Z$ such that $g = h \circ f$ i.e., the following diagram is commutative.

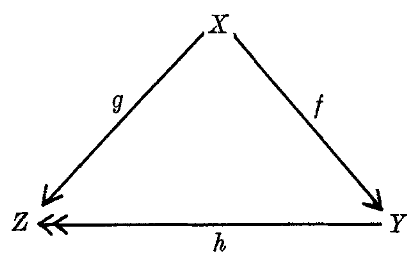

Fig. 2

It can be proved easily that $(K(X), \geqslant)$ is a quasi-ordered set without antisymmetry [8]. Using Stone-Čech compactification and assuming that X is a locally compact Hausdorff space, we are able to prove that $(K(X), \geqslant)$ is a pseudo-lattice. If (f, Y) is a Hausdorff compactification of X, we frequently identify X with $f(X) \subset Y$. Now let $(i, \beta(X))$ be the Stone-Čech compactification of X, where $i: X \to \beta(X)$ is the inclusion mapping. Then we have the following well-known facts [3]:

(1) For each compact Hausdorff space Y and each continuous $f: X \to Y$, there exists a unique continuous $\beta f: \beta(X) \to Y$ such that $f = \beta f \circ i$.
(2) $\beta(X)$ is the "largest" Hausdorff compactification of X: if Z is any Hausdorff compactification of X, then Z is a quotient space of $\beta(X)$.

Given (f, Y) and (g, Z) in $K(X)$. In order to find a l.u.b. of (f, Y) and (g, Z), an equivalence relation on βX is suggested by fact (2). Define an equivalence relation $\sim$ on βX as follows: $a \sim b$ iff $\beta f(a) = \beta f(b)$ and $\beta g(a) = \beta g(b)$, where $\beta f: \beta X \twoheadrightarrow Y$ and $\beta g: \beta X \twoheadrightarrow Z$ are the continuous surjections extended by f and g respectively. Let $\varphi: \beta X \twoheadrightarrow \beta X/\sim$ be the quotient mapping onto the quotient space. Let $h: X \to \beta X/\sim$ be defined by $h = \varphi \circ i$. We claim that $(h, \beta X/\sim)$ is a l.u.b. of (f, Y) and (g, Z). Clearly, $(h, \beta X/\sim)$ is a compactification of X. That $\beta X/\sim$ is a Hausdorff space is implied by X being a Hausdorff locally compact space. There exists a surjection ψ such that

$\beta f = \psi \circ \varphi$, since $\beta f : \beta X \twoheadrightarrow Y$ is compatible with the equivalence relation $\sim$ on βX. (i.e., βf is relation-preserving.) A theorem on quotient space [7] implies that ψ is continuous. Furthermore, $f = \beta f \circ i = \psi \circ \varphi \circ i = \psi \circ h$. Thus $(h, \beta X/\sim) \geqslant (f, Y)$. Similarly, we can prove $(h, \beta X/\sim) \geqslant (g, Z)$. The following commutative diagrams may illystrate how $(h, \beta X/\sim)$ is constructed.

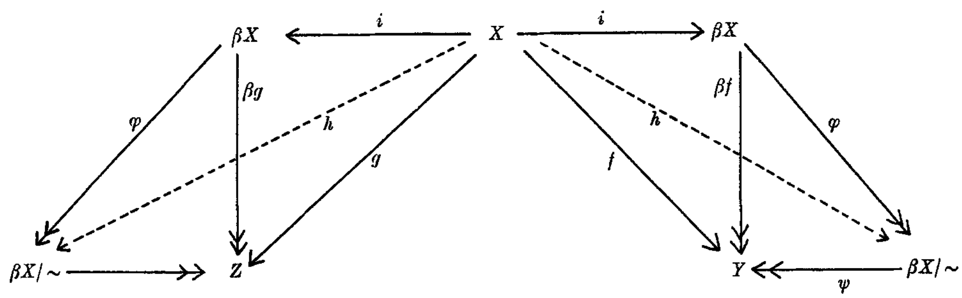

Fig. 3

To prove that $(h, \beta X/\sim)$ is a l.u.b. of (f, Y) and (g, Z), assume that (j, U) is a Hausdorff compactification of X such that $(j, U) \geqslant (f, Y)$ and $(j, U) \geqslant (g, Z)$. Then there exist continuous surjections $\xi : U \twoheadrightarrow Y$ and $\eta : U \twoheadrightarrow Z$ such that $f = \xi \circ j$ and $g = \eta \circ j$. We now define a mapping $\zeta : U \twoheadrightarrow \beta X/\sim$ as follows: $\zeta(\beta j(x)) = \varphi(x)$. It follows from $f = \xi \circ j$, $g = \eta \circ j$ and a theorem on quotient space [7] that ζ is well-defined and continuous. Therefore, $h = \varphi \circ i = \zeta \circ \beta j \circ i = \zeta \circ j$ and $(j, U) \geqslant (h, \beta X/\sim)$ which complete the proof that $(h, \beta X/\sim)$ is a l.u.b. of (f, Y) and (g, Z). The following commutative diagram may indicate what was going on.

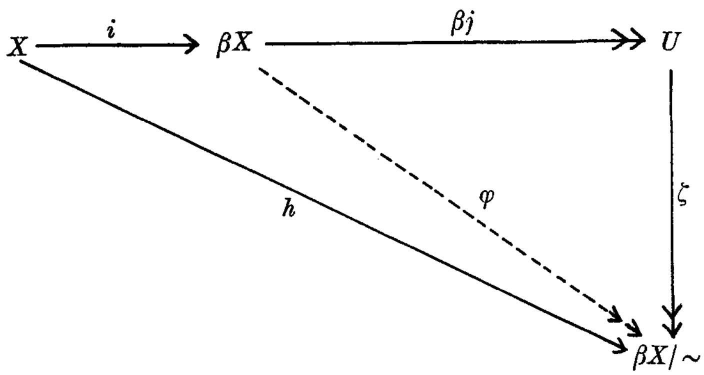

Fig. 4

The construction of a g.l.b. of (f, Y) and (g, Z) is quite similar to the preceding work. The equivalence relation is defined this time by $a \sim b$ iff $\beta f(a) = \beta f(b)$ or $\beta g(a) = \beta g(b)$.

We omit the rest of the details of the proof that $(K(X), \geqslant)$ is a pseudo-lattice. On the other hand, we raise the following open question: Is $(E(X), \geqslant)$ a pseudo-lattice? Where $E(X)$ is the collection of all extensions of a given topological space X and $\geqslant$ is defined similarly as in $(K(X), \geqslant)$. By an extension of X we mean a pair (f, Y) such that (1) Y is a topological space; (2) $f: X \to Y$ is a homeomorphism onto $f(X)$ and $f(X)$ is dense in Y.

V. Given two quasi-ordered sets $(X, \gg)$ and $(Y, \geqslant)$. Let $\mathfrak{F}$ be the set of all functions f from X to Y. A binary relation $\geqslant$ on $\mathfrak{F}$ is defined by setting $f \geqslant g$ iff for every a in X there exists b in X such that $a \gg b$ and $f(a) \geqslant f(b)$. Apparently, $(\mathfrak{F}, \geqslant)$ is a quasi-ordered set.

Following this general idea, we are able to ask lots of open questions. For example, let $\mathfrak{F}$ be the set of all real-valued functions defined on the real line R which has the usual order. Open question: Is $(\mathfrak{F}, \geqslant)$ a pseudo-lattice? We give a related result in the following

Theorem 7. *Suppose that $(X, \gg)$ is a given quasi-ordered set. Let $\mathfrak{F}$ be the set of all real-valued functions defined on X, and suppose that the quasi-ordering $\geqslant$ on $\mathfrak{F}$ is defined by setting $f \geqslant g$ iff for every x in X there exists y in X such that $x \gg y$ and $f(x) \geqslant f(y)$. Finally, let*

$$\mathcal{G} = \left\{ f \in \mathfrak{F} \,\middle|\, \begin{array}{l} \text{for every } x \text{ in } X \text{ there exists } t \text{ in } X \text{ such that } t \ll x \text{ and} \\ f(t) = \inf_{y \ll x} f(y) \end{array} \right\}.$$

Then

(1) *$(\mathcal{G}, \geqslant)$ is a pseudo-lattice with $k(x) = \min\{f(x), g(x)\}$ as a g.l.b. of f and g; with $h(x) = \max\{\inf_{y \ll x} f(y), \inf_{y \ll x} g(y)\}$ as a l.u.b. of f and g.*

(2) *For every $f \in \mathcal{G}$, there is one and only one $\varphi \in \mathcal{G}$ such that $f \leqslant \varphi \leqslant f$ and φ is a monotone decreasing function.*

Proof. Let f and g be in $\mathcal{G}$. Define function k by $k(x) = \min\{f(x), g(x)\}$. If x is in X, then there exist r and t in X such that $r \ll x$, $t \ll x$, $f(r) = \inf_{y \ll x} f(y)$ and $g(t) = \inf_{y \ll x} g(y)$. To prove $k \in \mathcal{G}$, we consider the following:

Case 1: $f(r) \leqslant g(t)$. We claim $k(r) = \inf_{y \ll x} k(y)$. Since $r \ll x$, $\inf_{y \ll x} k(y) \leqslant k(r)$. On the other hand, $y \ll x$ implies $f(r) \leqslant g(t) \leqslant g(y)$ and $f(r) \leqslant f(y)$. This gives $k(r) \leqslant k(y)$, since $k(y) = \min\{f(y), g(y)\}$. Thus $k(r) \leqslant \inf_{y \ll x} k(y)$. We have $k(r) = \inf_{y \ll x} k(y)$.

Case 2: $f(r) \geqslant g(t)$. We claim $k(t) = \inf_{y \ll x} k(y)$. We omit the proof which can be carried out as similarly as in Case 1.

Combine Case (1) and Case (2), to infer that k is an element of $\mathcal{G}$. Furthermore, it is clear that k is a g.l.b. of f and g. Now let function h be defined by $h(x) = \max\{\inf_{y \ll x} f(y), \inf_{y \ll x} g(y)\}$. We shall now prove that h is monotone decreasing and therefore an element of $\mathcal{G}$. If $z \ll x$, then

$$\inf_{y \ll x} f(y) \leqslant \inf_{y \ll z} f(y) \leqslant h(z)$$

and

$$\inf_{y \ll x} g(y) \leqslant \inf_{y \ll z} g(y) \leqslant h(z).$$

Thus $z \ll x$ implies $h(x) \leqslant h(z)$, i.e., h is monotone decreasing. Further, $\inf_{y \ll x} h(y) = h(x)$, hence h is an element of $\mathcal{G}$.

If x is in X, then there esists t in X such that $t \ll x$ and $f(t) = \inf_{y \ll x} f(y)$. Thus $f(t) \leqslant h(x)$ and $f \leqslant h$. Similarly, $g \leqslant h$. Let j be an upper bound of f and g. Then for every

x in X there exist y and z in X such that $y \ll x$, $f(y) \leqslant j(x)$, $z \ll x$ and $g(z) \leqslant j(x)$. This implies

$$\inf_{u \ll x} f(u) \leqslant f(y) \leqslant j(x)$$

and

$$\inf_{u \ll x} g(u) \leqslant g(z) \leqslant j(x).$$

Therefore, $h(x) \leqslant j(x)$. Since $x \leqslant x$, whe have $h \leqslant j$ and h is a l.u.b. of f and g.

To prove part (2) of our theorem, for every f in $\mathcal{G}$ let function φ be defined by $\varphi(x) = \inf_{y \ll x} f(y)$. Clearly, φ is monotone decreasing and is therefore an element of $\mathcal{G}$. It follows from the definition of $\mathcal{G}$ and $\geqslant$ that $f \geqslant \varphi \geqslant f$. To prove the uniqueness of such a function, let ψ be a monotone decreasing function in $\mathcal{G}$ such that $f \geqslant \psi \geqslant f$. By the transitivity of the quasi-ordering $\geqslant$, we have $\varphi \geqslant \psi \geqslant \varphi$. If x is in X, then there exists y in X such that $x \gg y$ and $\psi(x) \geqslant \varphi(y)$. Also, there exists z in X such that $x \gg z$ and $\varphi(x) \geqslant \psi(z)$. Since φ and ψ are monotone decreasing, $\varphi(x) \geqslant \psi(z) \geqslant \psi(x)$ and $\psi(x) \geqslant \varphi(y) \geqslant \varphi(x)$. Therefore for every x in X $\varphi(x) = \psi(x)$ and $\varphi = \psi$.

One last remark: Let an equivalence relation $\sim$ be defined on $\mathcal{G}$ by setting $f \sim g$ iff $f \geqslant g$ and $g \geqslant f$. Then by Corollary 2 to Theorem 6, $(\mathcal{G}/\sim, \geqslant/\sim)$ is a lattice which is isomorphic to $(\mathcal{L}, \geqslant)$ where $\mathcal{L}$ is the set of all monotone decreasing functions in $\mathcal{G}$ and $(\mathcal{L}, \geqslant)$ is a lattice under the same quasi-ordering (in $\mathcal{L}$ it becomes a partial ordering) defined on $\mathcal{G}$.

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REFERENCES

1. BERBERIAN, S. K., Measure and Integration, pp. 158–159. The Macmillan Co., New York, 1965.
2. BIRKHOFF, G., Lattice Theory (3rd ed.). Am. Math. Soc., Providence, R. I. (1967).
3. DUGUNDJI, J., Topology, p. 243. Allyn and Bacon, Inc., Boston, 1965.
4. HALMOS, P. R., Measure Theory, pp. 128–129, p. 133. D. Van Nostrand Co., Princeton, N. J., 1950.
5. HALMOS, P. R., Introduction to Hilbert Space (2nd ed.), pp. 79–80. Chelsea Publishing Co., New York, 1957.
6. HSU, I., Pseudo-lattices whith Applications, Dissertation. The University of New Mexico, 1969.
7. KELLEY, J. L., General Topology, p. 95. D. Van Nostrand Co., Princeton, N. J., 1955.
8. THRON, W. J., Topological Structures, pp. 135–136. Holt, Rinehart and Winston, New York, 1966.

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