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Subharmonic functions in a circle

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1. Introduction

Let $u(z)$ be a subharmonic function of a complex variable z, defined in a circular region $|z| < R$. Let

$$m(r) = \inf_{|z|=r} u(z), \quad M(r) = \max_{|z|=r} u(z), \quad M(R) = \sup_{|z|<R} u(z).$$

A condition of the type
$$m(r) \leqslant \cos \pi\lambda M(r), \tag{1}$$

where λ is a number in the interval $0 < \lambda < 1$, has been found to give consequences concerning the variation of $M(r)/r^{\lambda}$. If $u(z)$ is subharmonic in the entire plane and if (1) holds for all $r > 0$, then $M(r)/r^{\lambda}$ has a positive limit when $r \to \infty$ (see [1, 2, 4, 6]). An essential part of the proof of this is to show that, with a given value of $M(R)/R^{\lambda}$, the quotient $M(r)/r^{\lambda}$ must be bounded for $0 < r < R$. We shall here make a closer study of this problem.

The special case $\lambda = \frac{1}{2}$ has long been known, this being the Milloux-Schmidt inequality (see, for example [5], p. 108–109):

$$M(r) \leqslant U_0(r), \quad \text{where } U_0(r) = \frac{4M(R)}{\pi} \arctan \sqrt{\frac{r}{R}}. \tag{2}$$

One consequence of (2) is that

$$\frac{M(r)}{\sqrt{r}} \leqslant \frac{4}{\pi} \frac{M(R)}{\sqrt{R}}. \tag{3}$$

In the general case $0 < \lambda < 1$, we prove the following.

Theorem

Suppose that $u(z)$ is subharmonic for $|z| < R$ and that $0 < M(R) < \infty$. Let λ be a fixed number in the interval $0 < \lambda < 1$ and suppose that condition (1) *is satisfied for $0 < r < R$. Then there is an extremal subharmonic function,*

$$U(z) = \operatorname{Re}\left\{\frac{2M(R)}{\pi} \tan \frac{\pi\lambda}{2} \int_0^{z/R} \frac{t^{\lambda-1} - t^{1-\lambda}}{1-t^2}\, dt\right\}, \quad |\arg z| \leqslant \pi, \tag{4}$$

for which (1) *holds with equality and such that*

$$M(r) \leqslant U(r). \tag{5}$$

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The inequality (3) *corresponds to*

$$\frac{M(r)}{r^{\lambda}} \leqslant \frac{\tan\frac{\pi\lambda}{2}}{\frac{\pi\lambda}{2}} \frac{M(R)}{R^{\lambda}}, \tag{6}$$

where the constant is best possible.

Condition (1) is trivially satisfied if $M(R) \leqslant 0$; hence it is only in the case $M(R) > 0$ that consequences of (1) can be proved.

Notice that we must have $u(0) \leqslant 0$, because, if $u(0) = \limsup_{z\to 0} u(z) = a$, it follows from (1) and (2) that the same lim sup must be less than or equal to $a \cos \pi\lambda$, which implies that $a \leqslant 0$.

In the first version of the manuscript of this paper (by Hellsten and Kjellberg) only estimates of $U(r)$ and of the constant in (6) were given. The explicit formula (4) and the exact value of the constant (see section 7) are a later contribution by Norstad.

2. An associated function

In many problems on analytic functions, it is often advantageous to form an auxiliary function by making a circular projection of the zero points upon a certain radius. The new function takes its minimum on this radius and its maximum on the opposite radius. Here, we shall make the analogous transformation from $u(z)$ to an associated subharmonic function $u^*(z)$. A subharmonic function which is bounded above for $|z| < R$ can be written in the form (concerning this section, see, for example [7], IV.10):

$$u(z) = u_1(z) + u_2(z), \tag{7}$$

where

$$u_1(z) = \iint_{|\zeta|<R} \log\left|\frac{R(z-\zeta)}{R^2 - z\bar{\zeta}}\right| d\mu(\zeta), \quad u_2(z) = M(R) - \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\, d\nu(Re^{i\theta}).$$

The functions $\mu(\zeta)$ and $\nu(Re^{i\theta})$ correspond to positive mass-distributions over $|z| < R$ and $|z| = R$, respectively; $u_2(z)$ is harmonic for $|z| < R$.

We now construct an associated subharmonic function

$$u^*(z) = u_1^*(z) + u_2^*(z), \tag{8}$$

where

$$u_1^*(z) = \iint_{|\zeta|<R} \log\left|\frac{R(z+|\zeta|)}{R^2 + z|\zeta|}\right| d\mu(\zeta), \quad u_2^*(z) = M(R) - \frac{1}{2\pi}\frac{R^2 - |z|^2}{|R+z|^2}\int_{-\pi}^{+\pi} d\nu(Re^{i\theta}).$$

The potential function $u_1^*(z)$ has its whole mass concentrated on the segment $-R < z \leqslant 0$, while $u_2^*(z)$ has its mass at the point $z = -R$. On $|z| = R$, $z \neq -R$, we have $u_1^*(z) = 0$ and $u_2^*(z) = M(R)$. The function $u^*(z)$ is harmonic in the region D_R which is obtained by cutting $|z| < R$ along $(-R, 0)$. For $|z| = r$, $0 < r < R$, we have $u^*(-r) \leqslant u^*(z) \leqslant u^*(r)$.

3. The connections between $u(z)$ and $u^*(z)$

From the definition of $u^*(z)$ it follows that for $0<r<R$

$$u^*(-r) \leqslant m(r) \leqslant M(r) \leqslant u^*(r) \leqslant M(R), \tag{9}$$

(see an analogous derivation in [5], for example).

As is usual in such cases, we require here a further relation, namely

$$u^*(-r) + u^*(r) \leqslant m(r) + M(r), \tag{10}$$

for $0<r<R$. We begin by showing that

$$u^*(-r) + u^*(r) \leqslant u(-z) + u(z), \tag{11}$$

for any z on $|z| = r$. Let us put $z = re^{i\varphi}$. We prove the relation by dividing up u and u^* according to (7) and (8) and deriving separate inequalities, which together give (11). We consider first

$$\begin{aligned} &u_1(-z) + u_1(z) - u_1^*(-r) - u_1^*(r) \\ &= \iint_{|\zeta|<R} \left\{ \log \left| \frac{R^2(z^2-\zeta^2)}{R^4 - z^2\bar{\zeta}^2} \right| - \log \left| \frac{R^2(r^2 - |\zeta|^2)}{R^4 - r^2|\zeta|^2} \right| \right\} d\mu(\zeta) \geqslant 0, \end{aligned}$$

where the inequality follows from a well-known elementary property of the mapping function $w(z) = [\varrho(z-a)]/(\varrho^2 - z\bar{a})$. Next

$$\begin{aligned} &u_2(-z) + u_2(z) - u_2^*(-r) - u_2^*(r) \\ &= \frac{R^4 - r^4}{\pi} \int_{-\pi}^{+\pi} \left\{ \frac{1}{(R^2+r^2)^2 - 4R^2r^2} - \frac{1}{(R^2+r^2)^2 - 4R^2r^2\cos(\theta-\varphi)} \right\} d\nu(Re^{i\theta}) \geqslant 0. \end{aligned}$$

The proof of (11) is then complete. Since $u(-z)$ can be made sufficiently near $m(r)$ by suitable choice of z and $u(z) \leqslant M(r)$, (10) follows.

Finally, it is seen from (9) and (10) that

$$\begin{aligned} u^*(-r) - \cos\pi\lambda\, u^*(r) &= u^*(-r) + u^*(r) - (1+\cos\pi\lambda)\, u^*(r) \\ &\leqslant m(r) + M(r) - (1+\cos\pi\lambda)\, M(r) = m(r) - \cos\pi\lambda\, M(r) \leqslant 0, \end{aligned} \tag{12}$$

by (1).

Observe that, just as (1) implies that $u(0) \leqslant 0$, (12) implies that $u^*(0) \leqslant 0$.

4. Representation formulae

We now require representation formulae in the simple case of harmonic functions which are bounded from above and are representable as integrals of their boundary values. Let $H(z)$ be such a harmonic function in the half-disc $|z|<R$, $\operatorname{Im} z>0$. Its value for $z = ir$ is (see, for example [3], p. 2)

$$\begin{aligned} H(ir) &= \int_{-R}^{+R} K(r,t)\, H(t)\, dt + \int_0^{\pi} S(r,\varphi)\, H(Re^{i\varphi})\, d\varphi \\ &= \int_0^R K(r,t)\, \{H(t) + H(-t)\}\, dt + \int_0^{\pi} S(r,\varphi)\, H(Re^{i\varphi})\, d\varphi, \end{aligned} \tag{13}$$

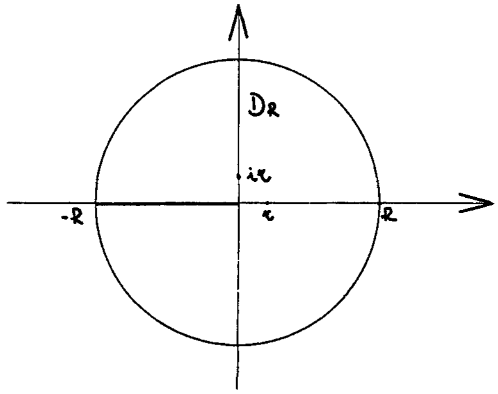

Fig. 1.

where
$$K(r,t)=\frac{r}{\pi}\left\{\frac{1}{t^2+r^2}-\frac{R^2}{R^4+t^2r^2}\right\}$$

and
$$S(r,\varphi)=\frac{2Rr(R^2-r^2)\sin\varphi}{\pi(R^4+r^4+2R^2r^2\cos 2\varphi)}.$$

Consider next, the region D_R consisting of the circle $|z|<R$ cut along $(-R,0)$. In what follows, we shall only be interested in the symmetric case when $H(z)=H(\bar{z})$. In particular $H(z)$ then has the same limit $H(-t)$ whether z approaches the cut $(-R,0)$ from above or below. By means of a simple square root transformation, we obtain from (13):

$$H(r)=\int_0^R Q(r,t)\,H(-t)\,dt+\int_{-\pi}^{\pi} T(r,\varphi)\,H(Re^{i\varphi})\,d\varphi, \tag{14}$$

where
$$Q(r,t)=\frac{\sqrt{r}}{\pi\sqrt{t}}\left\{\frac{1}{t+r}-\frac{R}{R^2+rt}\right\}$$

and
$$T(r,\varphi)=\frac{\sqrt{Rr}\,(R-r)\cos(\varphi/2)}{\pi(R^2+r^2-2Rr\cos\varphi)}.$$

We shall also require a further representation formula for $H(z)$ in D_R. This is obtained by first applying the counterpart of (13) in the half-disc $|z|<R$, $\operatorname{Re} z>0$.

$$H(r)=2\int_0^R K(r,\tau)\,H(i\tau)\,d\tau+\int_{-\pi/2}^{+\pi/2} S\left(r,\psi+\frac{\pi}{2}\right)H(Re^{i\psi})\,d\psi. \tag{15}$$

Then r is replaced by τ in the formula (13) and the resulting expansion for $H(i\tau)$ is inserted in (15). This gives

$$H(r)=\int_0^R L(r,t)\,\{H(t)+H(-t)\}\,dt$$
$$+\int_0^{\pi} N(r,\varphi)\,H(Re^{i\varphi})\,d\varphi+\int_{-\pi/2}^{+\pi/2} S\left(r,\psi+\frac{\pi}{2}\right)H(Re^{i\psi})\,d\psi, \tag{16}$$

where $$L(r,t)=2\int_0^R K(r,\tau)\,K(\tau,t)\,d\tau$$

and $$N(r,\varphi)=2\int_0^R K(r,\tau)\,S(\tau,\varphi)\,d\tau.$$

We observe that the functions K, S, Q, T, L and N above are non-negative.

5. Integral inequalities for $u^*(r)$

We now return to our consideration of the function $u^*(z)$, which is subharmonic for $|z|<R$ and bounded above by $M(R)$. It is harmonic in D_R and has a constant value, $M(R)$, on $|z|=R$ except for the point $z=-R$. By (12), $u^*(-t)\leqslant\cos\pi\lambda\, u^*(t)$. On combining this with (14), we obtain the integral inequality

$$u^*(r)\leqslant\cos\pi\lambda\int_0^R Q(r,t)\,u^*(t)\,dt+h(r),\tag{17}$$

where $$h(r)=M(R)\int_{-\pi}^{+\pi}T(r,\varphi)\,d\varphi=\frac{4M(R)}{\pi}\arctan\sqrt{\frac{r}{R}}.$$

We also need an integral inequality in which $\cos\pi\lambda$ is replaced by a factor which is positive in the whole interval $0<\lambda<1$. For this we use (16) instead of (14) and we obtain

$$u^*(r)\leqslant(1+\cos\pi\lambda)\int_0^R L(r,t)\,u^*(t)\,dt+k(r),\tag{18}$$

where $$k(r)=M(R)\int_0^{\pi}N(r,\varphi)\,d\varphi+M(R)\int_{-\pi/2}^{+\pi/2}S\left(r,\psi+\frac{\pi}{2}\right)d\psi.$$

6. Two integral equations

Let us consider the integral equation which corresponds to (17) i.e.

$$U(r)=\cos\pi\lambda\int_0^R Q(r,t)\,U(t)\,dt+h(r).\tag{19}$$

As is clear from the definition in (14), $Q(r,t)$ has a singularity at $t=0$. In spite of this, the usual method of solution by successive approximation works well here. We perform this step by step.

(*a*) Either by direct calculation or by setting $H(z)\equiv 1$ in (14), it is seen that

$$\int_0^R Q(r,t)\,dt<1\tag{20}$$

for any r in the interval $0<r<R$.

(*b*) Let $\varphi(r)$ be continuous and bounded, $|\varphi(r)|<C$ for $0<r<R$. The integral operator

$$\int_0^R Q(r,t)\,\varphi(t)\,dt=\varphi_1(r)$$

gives a function $\varphi_1(r)$ with the same properties. The continuity requires no comment and $|\varphi_1(r)|<C$ follows from (20) and the fact that $Q(r,t)>0$. If one wishes to have continuity in the closed interval $0\leqslant r\leqslant R$ one must define $\varphi_1(0)=\varphi(0)$ and $\varphi_1(R)=0$, since

$$\lim_{r\to 0}\int_0^R Q(r,t)\,dt=1,\quad \lim_{r\to 0}\int_\delta^R Q(r,t)\,dt=0$$

for each δ, $0<\delta<R$, and further

$$\lim_{r\to R}\int_0^R Q(r,t)\,dt=0.$$

(*c*) Denote by $Q^{(1)}, Q^{(2)}, \ldots, Q^{(n)}, \ldots$ the successive kernels:

$$Q^{(1)}(r,t)=Q(r,t)$$

— — — —

$$Q^{(n)}(r,t)=\int_0^R Q^{(n-1)}(r,\tau)\,Q(\tau,t)\,d\tau,\quad n=2,3,\ldots.$$

— — — —

(*d*) Set $\cos\pi\lambda=\mu$ and consider the series

$$U(r)=h(r)+\mu\int_0^R Q(r,t)\,h(t)\,dt+\ldots+\mu^n\int_0^R Q^{(n)}(r,t)\,h(t)\,dt+\ldots. \tag{21}$$

By (*b*) and the definition of $h(r)$ in (17), the terms in this series are continuous and have values smaller than the terms of the series

$$M(R)+M(R)|\mu|+M(R)|\mu|^2+\ldots+M(R)|\mu|^n+\ldots,$$

which converges for $|\mu|<1$ with sum $M(R)/(1-|\mu|)$. The series (21) therefore converges uniformly in r for each μ such that $|\mu|<1$.

Thus, for each μ in $|\mu|<1$, $U(r)$ is defined and continuous in $0\leqslant r\leqslant R$, with $U(0)=h(0)/(1-\mu)=0$ and $U(R)=M(R)$.

(*e*) By inserting the series (21) into (19) in which we may then integrate term by term, we see that $U(r)$, defined by (21), satisfies the integral equation (19). In the usual way (the difference between two solutions satisfies (19) and (21) with $h(r)\equiv 0$) it is seen that the solution is unique within the class of bounded continuous functions.

Finally, we write down the integral equation corresponding to the inequality (18), namely

$$U(r)=(1+\cos\pi\lambda)\int_0^R L(r,t)\,U(t)\,dt+k(r). \tag{22}$$

The existence of a unique solution can be shown in a way analogous to that used with (19). However, this working does not need to be performed here; what is required in what follows is to show that the same function $U(r)$ which satisfies (19) also satisfies (22).

7. Use of Fourier transforms

By the transformations $r = \mathrm{Re}^{-x}$, $t = \mathrm{Re}^{-s}$ the integral equation (19) takes the form

$$\varphi(x) = \int_0^\infty \{K_0(x-s) - K_0(x+s)\}\,\varphi(s)\,ds + g(x), \tag{23}$$

where $\varphi(x) = U(\mathrm{Re}^{-x})$ is to be determined and

$$K_0(u) = \frac{\cos\pi\lambda}{\pi}\,\frac{1}{2\cosh u/2}, \quad g(x) = \frac{4M(R)}{\pi}\arctan e^{-x/2}.$$

We now extend the definition of $\varphi(x)$ and $g(x)$ to negative values of x by prescribing them to be odd functions. The origin turns out to be a point of discontinuity. By analogy with the case for equations of the Wiener-Hopf type the equation (23) then can be written

$$\varphi(x) = \int_{-\infty}^\infty K_0(x-s)\,\varphi(s)\,ds + g(x). \tag{24}$$

Introducing Fourier transforms we obtain

$$\hat{\varphi}(t) = \hat{K}_0(t)\,\hat{\varphi}(t) + \hat{g}(t) \tag{25}$$

The formal solution

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\hat{g}(t)}{1-\hat{K}_0(t)}\,e^{-ixt}\,dt \tag{26}$$

gives us in this case the desired solution. In fact

$$\hat{g}(t) = \frac{2iM(R)}{t}\left(1 - \frac{1}{\cosh\pi t}\right), \quad \hat{K}_0(t) = \frac{\cos\pi\lambda}{\cosh\pi t} \leqslant \cos\pi\lambda < 1.$$

To evaluate the integral by means of residue calculus for $x>0$, an interval on the real axis is completed by a half-circle in the lower half-plane. The denominator $1-\hat{K}_0(t)$ has two sequences of zeros there, $\{(\lambda-2n)i\}_{n=1}^\infty$ and $\{(-\lambda-2n)i\}_{n=0}^\infty$. The result is

$$\varphi(x) = \frac{2M(R)}{\pi}\,\frac{1-\cos\pi\lambda}{\sin\pi\lambda}\left\{\sum_{n=0}^\infty \frac{e^{-x(\lambda+2n)}}{\lambda+2n} - \sum_{n=1}^\infty \frac{e^{-x(-\lambda+2n)}}{-\lambda+2n}\right\}. \tag{27}$$

This gives

$$U(r) = \frac{2M(R)}{\pi}\,\frac{1-\cos\pi\lambda}{\sin\pi\lambda}\left\{\sum_{n=0}^\infty \frac{(r/R)^{2n+\lambda}}{2n+\lambda} - \sum_{n=1}^\infty \frac{(r/R)^{2n-\lambda}}{2n-\lambda}\right\}, \tag{28}$$

or

$$U(r) = \frac{2M(R)}{\pi}\tan\frac{\pi\lambda}{2}\int_0^{r/R}\frac{t^{\lambda-1}-t^{1-\lambda}}{1-t^2}\,dt. \tag{29}$$

The result can also be written

$$U(r) = \frac{2M(R)}{\pi\lambda} \tan\frac{\pi\lambda}{2} \left\{ (r/R)^{\lambda} - \lambda \int_0^{r/R} \frac{t^{1-\lambda} - t^{1+\lambda}}{1-t^2}\, dt \right\}. \tag{30}$$

The integral of the right-hand side is never negative, i.e. we have the inequality

$$\frac{U(r)}{r^{\lambda}} \leqslant \frac{2}{\pi\lambda} \tan\frac{\pi\lambda}{2} \frac{M(R)}{R^{\lambda}}. \tag{31}$$

8. An extremal subharmonic function

The result (29) of the preceding section suggests a study of the function

$$w(z) = \frac{2M(R)}{\pi} \tan\frac{\pi\lambda}{2} \int_0^{z/R} \frac{t^{\lambda-1} - t^{1-\lambda}}{1-t^2}\, dt,$$

which is analytic in D_R. In fact, a straight-forward computation shows that $\operatorname{Re} w(-r) = \cos\pi\lambda \operatorname{Re} w(r)$ and that the variation of $w(z)$ on the arc $|z| = R, z \neq -R$ is purely imaginary, i.e. $\operatorname{Re} w(z)$ is constant on the arc. Hence the function $U(z) = \operatorname{Re} w(z)$ is harmonic in D_R, has constant boundary value $M(R)$ on $|z| = R, z \neq -R$, as $w(R) = M(R)$, and satisfies $U(-r) = \cos\pi\lambda U(r)$. Furthermore, substitute $H(z)$ for $U(z)$ in (16) of section 4 and there results (22), i.e. $U(r)$ satisfies (22) as well as (19).

We shall now show that $U(r)$ majorizes $u^*(r)$, which in turn majorizes $M(r)$, by (9). We use the formulae containing the positive factor $1 + \cos\pi\lambda$. On subtracting (18) from (22), we obtain

$$U(r) - u^*(r) \geqslant (1 + \cos\pi\lambda) \int_0^R L(r, t) \{U(t) - u^*(t)\}\, dt. \tag{32}$$

The function $\psi(r) = U(r) - u^*(r)$ is not necessarily continuous for $0 \leqslant r \leqslant R$, since it can happen that $u^*(0) = -\infty$. However, it is lower semi-continuous and consequently takes a minimum value, m, in the interval. Further $\psi(0) \geqslant 0$ and $\psi(R) = 0$. The minimum m cannot be negative; for assume this were the case. Let r_0, $0 < r_0 < R$, be the value of r which gives the minimum. Substitution in (32) then gives

$$m \geqslant (1 + \cos\pi\lambda) \int_0^R L(r_0, t)\, \psi(t)\, dt \geqslant m(1 + \cos\pi\lambda) \int_0^R L(r_0, t)\, dt. \tag{33}$$

However, by setting $H(z) \equiv 1$ in (16), we see that

$$2\int_0^R L(r_0, t)\, dt < 1, \quad \text{i.e.} \quad \int_0^R L(r_0, t)\, dt < \tfrac{1}{2}.$$

This contradicts the assumption that $m < 0$ in (33). Hence

$$U(r) - u^*(r) \geqslant 0, \quad \text{i.e.} \quad u^*(r) \leqslant U(r).$$

Since $M(r) \leqslant u^*(r)$, we have proved that

$$M(r) \leqslant U(r), \tag{5}$$

and recalling (31), we obtain (6).

It remains to show that $U(z)$ is subharmonic for $|z| < R$. Since $U(z)$ is harmonic in D_R, it remains only to consider $U(z)$ locally on the segment $-R < z \leqslant 0$. A calculation shows that at each point of the segment its inner normal derivatives in both upward and down-ward directions are positive (and of course equal because of the symmetry of $U(z)$). Continuation of $U(z)$ from above the segment gives, in a disc $|z+r| < \delta$, a harmonic function which is less than $U(z)$ in the lower half of the disc. Thus a local condition for subharmonicity of $U(z)$ is satisfied at $z = -r$. A check shows that the mean of $U(z)$ on a circle centred at the origin is positive. Since $U(0) = 0$, a local condition for subharmonicity is satisfied also at the origin.

We have thus found an extremal solution $U(z)$ to the problem, given in the introduction, of finding the maximum value of $M(r)$.

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