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# Analyticity of fundamental solutions

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## Introduction

Trèves and Zerner [1] have studied analyticity domains of fundamental solutions of linear partial differential operators with constant coefficients. They formulate a general criterion that ensures the existence of a fundamental solution which is analytic in the complement of a certain algebraic conoid and this criterion shows that an operator with real principal part and simple real characteristics has a fundamental solution which is analytic in the complement of the bicharacteristic conoid. They also show that a semielliptic operator has a fundamental solution which is analytic outside a certain linear subspace of $R^n$.

In section 1 of this paper we give a criterion (announced in [2]) different from that of [1], geared to the classical method of constructing fundamental solutions by integrating over suitable chains in complex space where the Fourier kernel is small and the characteristic polynomial does not vanish. In section 2 this criterion is applied to hypoelliptic, in particular semielliptic, operators and to operators with real principal part and simple real characteristics. Trèves and Zerner remark (l.c. p. 156) that their method does not seem to work in the case of complex coefficients. In section 3 we give a simple result for such operators. Finally, we generalize this result to products of operators. In an appendix we have gathered some simple facts, used in section 3, concerning convergence of distributions.

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## 1. Vectorfields and fundamental solutions

Points in $R^n$ will usually be denoted by $x$, $y$ or $\xi$, $\eta$ and when $z=x+iy\in \mathbf{C}^n$ we write $\operatorname{Re} z=x$, $\operatorname{Im} z=y$ and $\bar z=x-iy$. On $\mathbf{C}^n$ we use the duality $z\cdot\zeta=z\zeta=z_1\zeta_1+\ldots+z_n\zeta_n$ and the norm $|z|=(z\cdot\bar z)^{\frac{1}{2}}$. When $P(\xi)$ is a polynomial let $P(D)$ be the associated differential operator, where $D_k=\partial/i\partial x_k$ and $D=(D_1, \ldots, D_n)$. Let $P_k(\xi)$ be the part of $P$ of homogeneity $k$ so that $P=P_m+P_{m-1}+\ldots$, where $m$ is the degree of $P$ and hence $P_m(D)$ the principal part of $P(D)$.

Given a differential operator $P(D)$, we denote by $V=V(P)$ the family of vectorfields

$$R^n\in\xi\to w(\xi)\in \mathbf{C}^n$$

such that (*a*) $w(\xi)\in C^1$, (*b*) $w(\xi)$ and $dw(\xi)/d\xi$ are bounded, (*c*) there are positive constants $c$, $k$, $\delta$ such that

$$|\xi| \geqslant c \Rightarrow |P(\xi + w(\xi))| \geqslant k|\xi|^{-\delta}$$

When $w\in V$ put $E_{w,c}(\varphi)=\int_{|\xi|\geqslant c}\hat{\varphi}(\zeta)P(\zeta)^{-1}d\zeta$, $\zeta=\xi+w(\xi)$, where $d\zeta=d\zeta_1\wedge\ldots\wedge d\zeta_n$, $\varphi\in C_0^\infty(R^n)$ and $\hat{\varphi}(\zeta)=(2\pi)^{-n}\int e^{ix\zeta}\varphi(x)dx$. This defines a distribution $E_w(x)$ which differs from a fundamental solution of $P(D)$ by an entire function (see [1]).

**Definition 1.** *Two elements $w_0$, $w_1$ of $V$ are said to be homotopic, $w_0\sim w_1$, if there is a bounded $C^1$-vectorfield $[0,1]\times R^n\ni(t,\xi)\to w(t,\xi)\in\mathbf{C}^n$ satisfying (c) above uniformly in $t$, such that $dw(t,\xi)/d(t,\xi)$ is bounded and $w(t,\xi)$ reduces to $w_0$, $w_1$ when $t=0,1$ and $|\xi|$ large.*

**Lemma 1.1.** *If $w_0\sim w_1$ then $E_{w_0}-E_{w_1}$ is an entire function.*

*Proof.* When $0\leqslant t\leqslant 1$, $c\leqslant|\xi|\leqslant N$ the vectorfield $w(t,\xi)$ generates an $(n+1)$-chain $M$ in $R^{2n}$ whose boundary $\partial M$ has four parts, $M_0$ and $M_1$ corresponding to $t=0,1$ and $M_3$, $M_4$ corresponding to $|\xi|=c$ and $|\xi|=N$. Since $d(\hat{\varphi}(\zeta)P(\zeta)^{-1}d\zeta)=0$, Stokes formula gives $\int_{\partial M}\hat{\varphi}(\zeta)P(\zeta)^{-1}d\zeta=0$, provided that $\partial M$ is suitably oriented. Since $\hat{\varphi}(\zeta)$ decreases faster than any $(1+|\operatorname{Re}\zeta|)^{-q}$ as $|\operatorname{Re}\zeta|\to\infty$, while $\operatorname{Im}\zeta$ is bounded, an easy estimate shows that the integral over $M_4$ tends to zero as $N\to\infty$. The integral over $M_3$ obviously corresponds to an entire function. This proves the lemma.

**Definition 2.** *Let $d=(d_1,\ldots,d_n)$ be a vector whose components are $\geqslant 1$ and put $|\xi|_d=\sum_1^n|\xi_k|^{1/d_k}$. A vectorfield $w\in V(P)$ is said to belong to $W^d=W^d(P)$ if there are positive constants $c$, $\varrho$, $\delta$, $k$ such that $|P(\xi+\tau w(\xi))|\geqslant k|\xi|^{-\delta}$ when $|\xi|\geqslant c$, $1\leqslant\tau\leqslant\varrho|\xi|_d$.*

Next we shall define Gevrey classes. When $\alpha=(\alpha_1,\ldots,\alpha_n)$ is a multi-index and $d$ as above, we put $|\alpha|=\Sigma\alpha_k$ and $\alpha^{\alpha d}=\Pi\alpha_k^{\alpha_k d_k}$.

**Definition 3.** *If $\Omega\subseteq R^n$ is open, we denote by $\Gamma^d(\Omega)$ the set of all $u\in C^\infty(\Omega)$ such that $\sup_K|D^\alpha u(x)|\leqslant C^{|\alpha|+1}\alpha^{\alpha d}$, for all $\alpha$, where $K\subseteq\Omega$ is any compact set and $C=C(u,K)$ is a constant depending on $u$ and $K$. When $y\in R^n$ we say that $u\in\Gamma^d$ at $y$ if $u\in\Gamma^d(\Omega)$ for some neighbourhood $\Omega$ of $y$.*

*Remark.* Definition 3 makes sense also if the components of $d$ are $\geqslant 0$, but we will always assume that they are $\geqslant 1$.

**Lemma 1.2.** *If $w\in W^d$ and if there is a constant $c>0$, such that $y\cdot\operatorname{Im}w(\xi)\geqslant c$ for all sufficiently large $\xi$, then $E_w\in\Gamma^d$ at $y$.*

*Proof.* Let us first notice that if $\varphi\in C_0^\infty$, then $|\hat{\varphi}(\zeta)|\leqslant c_q(1+|\zeta|)^{-q}e^{-h(\operatorname{Im}\zeta)}$ where $q>0$ is any integer, $c_q$ a constant depending on $q$ and $h(\operatorname{Im}\zeta)=\min x\cdot\operatorname{Im}\zeta$ when $x$ belongs to the support of $\varphi$. Choose $0<\tau(\xi)\in C^1(R^n)$ such that $\tau(\xi)=1$ for small $\xi$ and $\tau(\xi)=\varrho_1\sum_k(1+\xi_k^2)^{1/2d_k}$ for large $\xi$. Here $0<\varrho_1<\varrho$ is small. Put $w(t,\xi)=tw(\xi)$ and let $M$ be the $(n+1)$-chain given by the vectorfield $\xi+w(t,\xi)$, when $1\leqslant t\leqslant\tau(\xi)$ and $c\leqslant|\xi|\leqslant N$. By Stokes' formula the integral of $\hat{\varphi}(\zeta)P(\zeta)^{-1}d\zeta$ over $\partial M$ vanishes. Further, if the support of $\varphi$ is close to $y$, the exponential factor in the estimate for $\hat{\varphi}(\zeta)$ is $\leqslant 1$ and it follows that the integral over the part of $\partial M$ where $|\xi|=N$ tends to zero as $N \to \infty$. Hence, if $v(\xi) = \tau(\xi) w(\xi)$, $E_v - E_w$ is holomorphic close to $y$ and an easy argument shows that

$$E_v(x) = (2\pi)^{-n} \int_{|\xi| \geq c} P(\xi + v(\xi))^{-1} e^{ix(\xi + v(\xi))} d(\xi + v(\xi)) \quad \text{when } x \text{ is close to } y.$$

In fact, since $y \cdot \operatorname{Im} v(\xi) \geq c_0 \tau(\xi)$, the integral is absolutely convergent when $x$ is close to $y$ and represents an infinitely differentiable function. Multiplying by $\varphi(x)$ and integrating, we get $\int E_v(x) \varphi(x) dx = E_v(\varphi)$. If $d = (1, 1, ..., 1)$ we have

$$-\operatorname{Re} i(x + ix')(\xi + \tau(\xi) w(\xi)) = x'(\xi + \tau(\xi) \operatorname{Re} w(\xi)) + x\tau(\xi) \operatorname{Im} w(\xi) > 0$$

when $x$ is close to $y$, $x'$ is real and small and $\xi$ is large. Hence $E_v(x)$ has a holomorphic continuation obtained by replacing $x$ by $x + ix'$ in the integral. When $d$ is arbitrary one has to estimate the derivatives of $E_v(x)$ in a neighbourhood of $y$. It is well known (see [3] p. 28) that it suffices to verify the Gevrey-estimates when we only take derivatives with respect to an arbitrary $x_i$. When $x$ is close to $y$ we have

$$|D_i^\alpha E_v(x)| = (2\pi)^{-n} \left| \int_{|\xi| \geq c} P(\xi + v(\xi))^{-1} e^{ix \cdot (\xi + \tau(\xi) \operatorname{Re} w(\xi)) - x \cdot \tau(\xi) \operatorname{Im} w(\xi)} (\xi + v(\xi))_i^\alpha \right.$$

$$\left. \det (I + dv(\xi)/d\xi)\, d\xi \right| \leq c_1^\alpha \int e^{-c_2 |\xi|_d} (|\xi_i|^\alpha + \sum |\xi_k|^{\alpha/d_k}) |\xi|^{n+\delta} d\xi,$$

where $c_1$, $c_2$ are suitable positive constants and $\delta$ is the constant from definition 2. Integration by parts with respect to the different $\xi_k$:s now gives the desired estimate. The following theorem sums up the results of Lemmas 1.1 and 1.2.

**Theorem 1.** *If $V \ni w \sim w_0 \in W^d$ and $y \cdot \operatorname{Im} w_0(\xi) \geq c > 0$ for $\xi$ large, then $E_w \in \Gamma^d$ at $y$.*

## 2. Applications of Theorem 1

I. *Hypoelliptic operators.* When $P$ is a polynomial, let $Z(P)$ be the hypersurface given by $P(\zeta) = 0$, $\zeta = \xi + i\eta \in \mathbf{C}^n$, and $Z_k(P)$ the part of $Z(P)$ where $\eta_i = 0$ for $i \neq k$. The following definition is due to Gorin [4].

**Definition 4.** *$P$ is said to be $(j, k)$-hypoelliptic with exponent $a_{jk} \geq 1$ if there is a constant $c > 0$, such that $|\xi_j| \leq c(1 + |\eta_k|)^{a_{jk}}$ on $Z_k(P)$.*

The following lemma states some simple properties of $(j, k)$-hypoelliptic polynomials.

**Lemma 2.1.** *The best possible $a_{jk}$ in definition 4 is rational. Further, if $P$ is $(j, k)$ hypoelliptic for all $j$, there are positive constants $c$, $\varrho$, $\delta$, $k$ such that $|P(\xi + i\tau e_k)| \geq k|\xi|^{-\delta}$ when $|\xi| \geq c$, $0 \leq \tau \leq \varrho |\xi|_d$. Here $e_k$ is the $k$th coordinatevector and $d = (a_{1k}, ..., a_{nk})$.*

*Proof.* Let $\mu(h)$ be the supremum of $\xi_j^2$ when $|P(\zeta)|^2 = 0$ and $(\eta_k^2 - h)^2 \leq 0$. Then Lemma 2.1 in the appendix of [6] gives that $\mu(h) = Ah^a(1 + o(1))$, $h \to \infty$, with rational $a$, and the first part of the lemma is proved. Choose an integer $m$ such that $m/a_{jk}$ is an integer for every $j$. We notice that if $a_1, ..., a_n$ are positive numbers then $\Sigma a_k^m \leq (\Sigma a_k)^m \leq n^m \Sigma a_k^m$, This together with the $(j, k)$-hypoellipticity of $P$ implies that there is a positive constant $\varrho$, such that $P(\xi+i\tau e_k)\neq 0$ for large $\xi$ if $\tau^{2m}\leqslant\varrho\sum|\xi_j|^{2m/a_{jk}}$ and that we only need to prove the estimate under this assumption. Let $\mu(R)$ be the supremum of $-|P(\xi+i\tau e_k)|^2$ when $|\xi_j|^2=R$ and $\tau^{2m}\leqslant\sum|\xi_j|^{2m/a_{jk}}$ and apply Lemma 2.1 in [6].

**Theorem 2** (compare Grušin [5]). *If $P$ is $(j, k)$-hypoelliptic for $k=k_1, ..., k_r$ and all $j$ then every tempered fundamental solution of $P(D)$ belongs to $\Gamma^d(\Omega)$, where $\Omega=R^n-\{x; x_{k_1}=...=x_{k_r}=0\}$ and $d_j=\max a_{jk}$ when $k=k_1, ..., k_r$.*

*Proof.* Let $E$ be an arbitrary tempered fundamental solution of $P(D)$. The Fourier-transform $\tilde{E}$ of $E$ then satisfies $P(\xi)\tilde{E}=1$. If $|\xi|\geqslant c$, where $c$ is the constant from Lemma 2.1, we may multiply by the $C^\infty$ function $P(\xi)^{-1}$ and get $\tilde{E}=P(\xi)^{-1}=\tilde{E}_w$. Here $w(\xi)\equiv 0$ and $E_w=E_{w,c}$. Write $E=E_w+T$, where $T=E-E_w$. $\tilde{T}=\tilde{E}-\tilde{E}_w$ has compact support, so that $\tilde{T}=D^{\alpha_0}f$ for some continuous $f$ with compact support and some $\alpha_0$. Hence $T(x)=x^{\alpha_0}(2\pi)^{-n}\int f(\xi)e^{ix\xi}d_\xi$ is analytic.

When $y\in\Omega$ then $y_p\neq 0$ for some $p=k_l$. Put $w_p(\xi)=i(\operatorname{sign} y_p)\tau(\xi)e_p$, where $0\leqslant\tau(\xi)\in C^1(R^n)$, $\tau(\xi)=0$ for $|\xi|\leqslant c$ and $\tau(\xi)=\varrho_1\sum_k(1+\xi_k^2)^{\frac{1}{2}d_k}$ for large $\xi$. In view of Lemma 2.1 the arguments of the proof of Lemma 1.2 show that $E_w=E_{w_p}$ in a neighbourhood of $y$ and that $E_{w_p}\in\Gamma^d$ at $y$.

As a corollary of Theorem 2 we get the converse part of the following well-known proposition (see [6]). The operator $P(D)$ has a fundamental solution in $\Gamma^d(R^n-0)$ if and only if there is a $c>0$ such that $c|\xi|_d\leqslant 1+|\eta|$ on $Z(P)$. Such an operator is said to be hypoelliptic with exponent $d$. Combining Theorem 2 with this proposition we also have the following result by Gorin [4]. If $P$ is $(j, k)$-hypoelliptic for all $(j, k)$ with exponents $a_{jk}\geqslant 1$, then $P$ is hypoelliptic with exponent $d$, where $d_j=\max a_{jk}$. In fact Theorem 2 shows that such a $P$ has a fundamental solution in $\Gamma^d(R^n-0)$. We can also specialize to semielliptic operators $P(D)=\sum a_\alpha D^\alpha$ characterized by the property that there exists an $n$-tuple $m=(m_1, ..., m_n)$ of integers $>0$ such that $a_\alpha=0$ except when $|\alpha:m|=\sum\alpha_j/m_j\leqslant 1$ and that $\sum_{|\alpha:m|=1}a_\alpha\xi^\alpha$ does not vanish when $\xi\neq 0$. Here $m$ is uniqely determined by $P$ and $m_i$ is the degree of $P(D)$ with respect to $D_i$. Examples of semielliptic operators are elliptic operators, the heat operator and more generally the $p$-parabolic operators of Petrowsky. Semielliptic operators are $(j, k)$-hypoelliptic for all $(j, k)$ with exponents $a_{jk}=\max(m_k/m_j, 1)$ (see [5]). Hence, by Theorem 2, every tempered fundamental solution of $P(D)$ is analytic outside the intersection of the coordinate planes $x_k=0$ where $m_k=\min m_j$.

II. *Operators with real principal part and simple real characteristics.* Let $m$ be the degree of $P$. We assume that $P_m$ is real and that $P$ has simple real characteristics, i.e. $\partial P_m(\xi)\neq 0$ when $\xi\neq 0$ is real and $P_m(\xi)=0$. Here $\partial$ means the gradient. The set of all real $\xi$ such that $P_m(\xi)=0$ will be denoted, by $N=N(P)$ and the set of $\partial P_m(\xi)$, where $\xi\in N$, by $N'=N'(P)$. A subset $A$ of $R^n$ is said to be conical if $\lambda A=A$ for all $0\neq\lambda\in R$. The set $A-\{0\}$ will be denoted by $\dot{A}$.

**Lemma 2.2.** *For every $y\notin N'$ and $k>0$ there is a conical neighbourhood $\mathcal{U}=\mathcal{U}_y$ of $\dot{N}$, a real vectorfield $v\in C^\infty(\dot{R}^n)$, homogeneous of degree zero, and a constant $c>0$ such that*

(i) $v(\xi)\cdot y\geqslant c$ *in* $\dot{R}^n$, (ii) $|v(\xi)\cdot\partial P_m(\xi)|\geqslant k|\partial P_m(\xi)|$ *in* $\dot{\mathcal{U}}$

*Proof.* Because $P_m$ is homogeneous, we have $mP_m(\xi)=\partial P_m(\xi)\cdot\xi$. Hence $\partial P_m(\xi)\neq 0$ if $\xi\in\dot{R}^n$. Put $w(\xi)=\partial P_m(\xi)/|\partial P_m(\xi)|$. Since $y\notin N'$, there is an $\varepsilon>0$ and a conical neighbourhood $\mathcal{U}'$ of $N$ such that (1) $|y\cdot w(\xi)|\leqslant(1-\varepsilon)|y|$ when $\xi\in\mathcal{U}'$. Let $\mathcal{U}$ be a an entire function. We suppose that $c$ is chosen so large that everything in the proof that is true only when $\xi$ is large is true when $|\xi| \geq c$. We also often omit the phrase $|\xi| \geq c$.

Put $P^1(\zeta) = (\operatorname{Re} P)(\zeta)$ and $P^2(\zeta) = (\operatorname{Im} P)(\zeta)$. Given $y \in R^n - N'$, there is a conical $C^n$-neighbourhood $\mathcal{U}'$ of $N$ where $y$, $\partial P^1(\zeta)$ and $\partial P^2(\zeta)$ are linearly independent. Hence locally in $\mathcal{U}'$ we can find a holomorphic solution $F$ to the linear system

$$\partial P^1(\zeta) \cdot F(\zeta) = \partial P^2(\zeta) \cdot F(\zeta) = 0,\ u \cdot F(\zeta) = 1.$$

A suitable partition of unity gives a global $C^\infty$-solution $F$ in $\mathcal{U}'$ such that $F(\zeta)$ and $|\zeta|\partial F(\zeta)$ are bounded. We solve the system $du(t, \xi)/dt = i|\xi| F(u(t, \xi))$, $u(0, \xi) = \xi$ for $t$ real and $\xi$ in a sufficiently small conical $R^n$-neighbourhood $\mathcal{U}$ of $N$. There is a $t_0 > 0$ such that $u(t, \xi)$ is a $C^\infty$-function defined on $[0, t_0] \times \mathcal{U}$. Let $\alpha \in C^\infty$ be homogeneous of degree zero, $\alpha = 1$ in a neighbourhood of $N$, $\alpha = 0$ outside $\mathcal{U}$ and $0 \leq \alpha \leq 1$ otherwise and extend $u$ by $v(t, \xi) = \alpha(\xi) u(t, \xi) + (1 - \alpha(\xi))(\xi + it|\xi|\eta)$, where $\eta \in R^n$ satisfies $\eta \cdot y = 1$. Finally we put $w(t, \xi) = v(t\beta(\xi), \xi)$, where $\beta \in C^\infty$ is choosen so that $\beta(\xi) = 0$ when $|\xi| \leq c + 2$ and $\beta(\xi) = 1$ when $|\xi| \geq c + 3$.

One easily verifies that $w(t, \xi)$ and $dw(t, \xi)/d(t, \xi)$ are bounded by some constant times $|\xi|$, if $t_0$ is small enough. Further the following properties are satisfied

(i) $w(0, \xi) = \xi$ if $|\xi| \geq c$ and $w(t, \xi) = \xi$ if $|\xi| \leq c + 2$,

(ii) $y \cdot \operatorname{Im} w(t, \xi) \geq t|\xi|$ if $|\xi| \geq c + 3$,

(iii) $P^i(w(t, \xi)) = P^i(\xi)$, $i = 1, 2$, in a conical neighbourhood of $N$ and

$$|P(w(t, \xi))| \geq k|\xi|^m > 0$$

otherwise.

(i) is trivial and for (ii) it is sufficient to note that

$$y \cdot u(t, \xi) = y \cdot \xi + i|\xi| \int_0^t y \cdot F(u(s, \xi))\, ds = y \cdot \xi + it|\xi|.$$

To verify (iii) we observe that

$$dP^i(u(t, \xi))/dt = (\partial P^i)(u(t, \xi)) \cdot (du(t, \xi)/dt) = i|\xi|(\partial P^i)(u(t, \xi) \cdot F(u(t, \xi)) = 0.$$

Outside a conical neighbourhood of $N$ we know that

$$|P_m(\xi)| \geq k_1|\xi|^m > 0 \quad \text{and} \quad |P_m(w(t, \xi)) - P_m(\xi)| \leq |(\partial P_m)(w(\tau, \xi))(dw(\tau, \xi)/dt)|t,$$

where $0 \leq \tau \leq t$. Thus (iii) is satisfied if $t_0$ is small enough.

Define $E_t(\varphi) = \int_{|\xi| \geq c} \hat{\varphi}(\zeta) P(\zeta)^{-1} d\zeta$, $\zeta = w(t, \xi)$. (iii) implies that $E_t \in \mathcal{S}'$. Using (ii) we find, just as in Lemma 1.2, that $E_{t_0}$ is analytic at $y$. It remains to prove that $E_{t_0} = E_0$ near $y$. Integrate $\hat{\varphi}(\zeta) P(\zeta)^{-1}$ over the boundary of the $(n+1)$-chain $M$ given by $w(t, \xi)$, when $0 \leq t \leq t_0$, $c \leq |\xi| \leq N$ and $|P(\xi)| \geq \varepsilon$. Because of (iii) $\hat{\varphi}(\zeta) P(\zeta)^{-1}$ is holomorphic on $M$ and Stokes' formula may be used. (ii) and (iii) give that the integral over the part of $\partial M$ corresponding to $|\xi| = N$ tends to zero when $N \to \infty$. Further $d\zeta$ vanishes on the part of $\partial M$ given by $|P(\xi)| = \varepsilon$, because in $\mathcal{U}'$ we may locally define holomorphic bijections $z=\theta(\zeta)$ such that $\theta_i(\zeta)=P^i(\zeta)$, $i=1, 2$. The fact that $\theta^{-1}$ is holomorphic implies that

$$d\zeta=\theta_0^*(\theta^{-1})^*d\zeta=\theta^*(G_1(z)\,dz)=G_2(\zeta)\,dP^1(\zeta)\wedge dP^2(\zeta)\wedge\;\dots\;\wedge d\theta_n(\zeta),$$

where $G_1$ and $G_2$ are holomorphic functions. But this gives that $d\zeta=0$ on the boundary-piece corresponding to $|P(\xi)|=\varepsilon$, because here we have, according to (ii), that $P^i(w(t,\xi))=P^i(\xi)$, $i=1, 2$ and $P^1(\xi)$, $P^2(\xi)$ are connected by $P^1(\xi)^2+P^2(\xi)=\varepsilon^2$. We end the proof by letting $N\to\infty$ and after that $\varepsilon\to 0$.

Finally we shall generalize Theorem 4 to the case of "regular" products. If $P$ is a polynomial $\tilde{P}$ will denote the principal part of $P$. We are going to suppose that $P=P_1^{p_1}\dots P_k^{p_k}Q_1^{q_1}\dots Q_l^{q_l}$, where the $P_i$s and $Q_j$s have real and complex coefficients respectively. Further the following condition shall be satisfied.

If $\qquad \xi\in R^n \quad$ and $\quad \tilde{P}_{i_1}(\xi)=\dots=\tilde{P}_{i_r}(\xi)=\tilde{Q}_{j_1}(\xi)=\dots=\tilde{Q}_{j_s}(\xi)=0$

then $\quad \partial\tilde{P}_{i_1}(\xi),\dots,\partial\tilde{P}_{i_r}(\xi),\partial\operatorname{Re}\tilde{Q}_{j_1}(\xi),\dots,\partial\operatorname{Im}\tilde{Q}_{j_s}(\xi)$ are linearily independent. $\quad$ (1)

When $\xi\in\dot{N}(P)$ and $\tilde{P}_{i_1},\dots,\tilde{P}_{i_r},\tilde{Q}_{j_1},\dots,\tilde{Q}_{j_s}$ are precisely the factors of $\tilde{P}_1\dots\tilde{P}_k\tilde{Q}_1\dots\tilde{Q}_l$ which vanish at $\xi$, then $N'_\xi$ is defined as the real vector space spanned by the vectors on the right side of (1). $N'=N'(P)$ is the union of 0 and all the $N'_\xi$.

**Theorem 5.** *If $P$ satisfies the conditions above, then $P(D)$ has a fundamental solution which is analytic outside $N'(P)$.*

*Proof.* In order to define some distribution $P(\xi)^{-1}$ it suffices to do it locally. Further we are only interested in the restriction to $\{\xi;\ |\xi|\geq c\}$, $c$ large, so it is sufficient to consider division by $x_1^{p_1}\dots x_k^{p_k}\cdot(y_1+iz_1)^{q_1}\dots(y_l+iz_l)^{q_l}$, where $x_1,\dots,x_{n'},y_1,\dots,z_{n''}$ are coordinates in $R^{n'+2n''}$ and $k\leq n'$, $l\leq n''$. This division is always done as in the appendix. Let $\psi\in C^\infty$ be a fixed function such that $\psi(\xi)=0$ if $|\xi|\leq c+1$ and $\psi(\xi)=1$ if $|\xi|\geq c+2$. Then $\psi(\xi)\cdot P(\xi)^{-1}\in S'$ and $E(\varphi)=\langle P(\xi)^{-1},\psi(\xi)\hat{\varphi}(\xi)\rangle$ defines a fundamental solution of $P$, modulo an entire function. Put $P_{i,\Lambda_i}(\xi)=(P_i(\xi)-\lambda_1)\cdots(P_i(\xi)-\lambda_{p_i})$ and $Q_{j,\Pi_j}(\xi)=(Q_j(\xi)-\pi_1)\cdots(Q_j(\xi)-\pi_{q_j})$, where the $\lambda$s and $\pi$s are different real numbers. It is a consequence of the results in the appendix that $\psi\cdot P_{\Lambda,\Pi}^{-1}=\psi\cdot(P_{1,\Lambda_1}\cdots Q_{l,\Pi_l})^{-1}\to\psi\cdot P^{-1}$ in $S'$, when $\Lambda,\Pi\to 0$, and that $P_{\Lambda,\Pi}^{-1}$ is defined as a Cauchy principal value.

Given $y\in R^n-N'$ and $\xi\in N$ suppose that $\tilde{P}_{i_1},\dots,\tilde{P}_{i_r},\tilde{Q}_{j_1},\dots,\tilde{Q}_{j_s}$ are exactly the factors in $\tilde{P}_1\cdots\tilde{P}_k\tilde{Q}_1\cdots\tilde{Q}_l$ which vanishes at $\xi$. Solve locally for $|\zeta|\geq c$, $c$ large, the following linear system, in a conical $\mathbf{C}^n$-neighbourhood of the line through 0 and $\xi$: $\partial P_{i_1}(\zeta)\cdot F(\zeta)=\cdots=\partial P_{i_r}(\zeta)\cdot F(\zeta)=\partial(\operatorname{Re}Q_{j_1})(\zeta)\cdot F(\zeta)=\cdots=\partial(\operatorname{Im}Q_{j_s})(\zeta)\cdot F(\zeta)=0$, $y\cdot F(\zeta)=1$. Proceed then as in the proof of Theorem 4 to find a $w$ satisfying (i), (ii) and (iii) with the first part of (iii) modified to: $P_i(w(t,\xi))=P_i(\xi)$ in a conical neighbourhood of $N(P_i)$ and $(\operatorname{Re}Q_j)(w(t,\xi))=\operatorname{Re}Q_j(\xi)$, $(\operatorname{Im}Q_j)(w(t,\xi))=\operatorname{Im}Q_j(\xi)$ in a conical neighbourhood of $N(Q_j)$. This modified (iii) implies that the following scalar products are defined $E_t^p(\varphi)=\langle P(w(t,\xi))^{-1},\psi(\xi)\hat{\varphi}(w(t,\xi))\det(dw(t,\xi)/d\xi)\rangle$, $0\leq t\leq t_0$. Suppose that the support of $\varphi$ is so close to $y$ that $x\cdot w(t,\xi)\geq t|\xi|/2$ when $x\in\operatorname{supp}\varphi$. Then the Fubini law for distributions (see e.g. [7] p. 109) gives that

$$E_{t_0}^p(\varphi)=\int_{\operatorname{supp}\varphi}\langle P(w(t_0,\xi))^{-1},\psi(\xi)\,e^{ix\cdot w(t_0,\xi)}\det(dw(t_0,\xi)/d\xi)\rangle\,\varphi(x)\,dx.$$

But $$P(w(t_0, \xi))^{-1}\psi(\xi) \in S' \quad \text{and} \quad e^{i(x+ix')\cdot w(t_0,\xi)} \in S$$

when $x \in \operatorname{supp} \varphi$ and $|x'|$ is small. Furthermore, we may differentiate with respect to $(x+ix')$ under the integral. Hence $E^p_{t_0}$ is analytic at $y$.

To show that $E^p_{t_0}(\varphi) = E^p_0(\varphi)$ it suffices to show that $E^{p\Lambda,\Pi}_{t_0}(\varphi) = E^{p\Lambda,\Pi}_0(\varphi)$ for all $\Lambda$, $\Pi$ and then take limits. We therefore integrate $\hat{\varphi}(\zeta)P_{\Lambda,\Pi}(\zeta)^{-1}$ over the boundary of the $(n+1)$-chain $M$ given by $w(t, \xi)$, when $0 \leqslant t \leqslant t_0$, $c+2 \leqslant |\xi| \leqslant N$ and $|P_1(\xi) - \lambda_1| \geqslant \varepsilon, \ldots, |Q_l(\xi) - \pi^a| \geqslant \varepsilon$. Use Stokes' formula and let $N$ tend to infinity. Exactly as in the proof of Theorem 4 we note that $d\zeta$ vanishes on the parts of $\partial M$ which corresponds to a condition $|Q_j(\xi) - \pi| = \varepsilon$. In the case when the boundarypiece is given by $|P_i(\xi) - \lambda| = \varepsilon$ the situation is even simpler because here we have $d\zeta = G_1(\zeta)\,dP_i(\zeta)\Lambda \cdot\cdot \Lambda d\theta_n(\zeta)$ and $P_i(w(t,\xi)) = P_i(\xi) = \lambda \pm \varepsilon$. We can now make $\varepsilon \to 0$ because $P_{\Lambda,\Pi}(w(t,\xi))^{-1}$ is defined as a Cauchy principal value and the proof is finished.

## 4. Appendix

$x_1, \ldots, x_{n'}, y_1, \ldots, y_{n''}, z_1, \ldots, z_{n''}$ are coordinates in $R^{n'+2n''}$. We shall give a definition of the distribution

$$P^{-1} = x_1^{-p_1} \cdot \cdot x_k^{-p_k} (y_1 + iz_1)^{-q_1} \cdot \cdot (y_l + iz_l)^{-q_l},\ k \leqslant n',\ l \leqslant n''.$$

If $\lambda_0, \ldots, \lambda_p$ are different real numbers and $\varphi \in C_0^\infty(R)$, put

$$[(x-\lambda_0)\cdot\cdot(x-\lambda_p)]^{-1}(\varphi) = \lim_{\varepsilon\to 0} \int_{M_\varepsilon} \varphi(x)\,[(x-\lambda_0)\cdot\cdot(x-\lambda_p)]^{-1}dx,$$

where $M_\varepsilon$ is the set given by $|x - \lambda_r| \geqslant \varepsilon$, $0 \leqslant r \leqslant p$. Define also

$$x^{-p-1} = \lim_{\varepsilon\to 0}(p!)^{-1}\int_{|x|\geqslant\varepsilon} x^{-1}(d/dx)^p\varphi(x)\,dx.$$

It is easily verified that $x^{p+1}\cdot x^{-p-1} = 1$. Furthermore, we have that

$$[(x-\lambda_0)\cdot\cdot(x-\lambda_p)]^{-1} \to x^{-p-1} \text{ in } S',$$

if $\lambda_0, \ldots, \lambda_p$ are different real numbers tending to zero.

*Proof* (*due to Wim Nuij*). We have

$$\int_{s_1=0}^{1}\int_{s_2=0}^{s_1}\int_{s_p=0}^{s_{p-1}} \varphi^{(p)}(x + A_0 + A_1 s_1 + \ldots + A_p s_p)\,d_{s_p}\ldots d_{s_1}$$
$$= \sum_{k=0}^{p} \varphi(x+\lambda_k)\,[\prod_{i\neq k}(\lambda_k - \lambda_i)]^{-1},$$

where $A_0 = \lambda_0$, $A_1 = \lambda_1 - \lambda_0$, $A_p = \lambda_p - \lambda_{p-1}$. Hence

$$\lim_{\varepsilon\to 0}\int_{M_\varepsilon} \varphi(x)\,[(x-\lambda_0)\cdot\cdot(x-\lambda_p)]^{-1}dx =$$
$$= \lim_{\varepsilon\to 0}\int_{|x|\geqslant\varepsilon} x^{-1}\sum_{k=0}^{p}\varphi(x+\lambda_k)\,[\prod_{i\neq k}(\lambda_k-\lambda_i)]^{-1}dx = \lim_{\varepsilon\to 0}\int_{|x|\geqslant\varepsilon} x^{-1}\cdot(p!)^{-1}\varphi^{(p)}(x)\,dx.$$

In the same way we define

$$[(y+iz-\pi_0)\cdot\cdot(y+iz-\pi_q)]^{-1}(\varphi)=\int\varphi(y,z)\,[(y+iz-\pi_0)\cdot\cdot(y+iz-\pi_q)]^{-1}dy\,dz,$$

where $\pi_0, ..., \pi_q$ are different real numbers, and

$$(y+iz)^{-q-1}=(p!)^{-1}\int(y+iz)^{-1}(\partial/\partial y)^p\varphi(y,z)\,dy\,dz.$$

One proves the same results as above, by the same argument. Put now

$$P^{-1}=x_1^{-p_1}\otimes\ ...\ \otimes x_k^{-p_k}\otimes(y_1+iz_1)^{-q_1}\otimes(y_l+iz_l)^{-q_l}\otimes 1\otimes\ ...\ \otimes 1.$$

The continuity of the tensorproduct $S'(R^m)\times S'(R^n)\to S'(R^{n+m})$ gives that

$$P^{-1}_{\Lambda,\Pi}=[(x_1-\lambda_1)\cdot\cdot(x_1-\lambda_{p_1})\cdot\cdot(y_l+iz_l-\pi_1)\cdot\cdot(y_l+iz_l-\pi_{q_l})]^{-1}(\varphi)=$$

$$=\lim_{\varepsilon\to 0}\int_{M_\varepsilon}[(x_1-\lambda_1)\cdot\cdot(y_l+iz_l-\pi_{q_l})]^{-1}\varphi(x,y,z)\,dx\,dy\,dz,$$

where $M_\varepsilon$ is the set given by

$$|x_i-\lambda_r|\geqslant\varepsilon,\ 0\leqslant i\leqslant k,\ 0\leqslant r\leqslant p_i,$$

and that $P^{-1}_{\Lambda,\Pi}\to P^{-1}$ in $S'$, if $\Lambda, \Pi\to 0$.

## REMARK

After the setting of this article L. Hörmander pointed out to me the existence of the paper [8], where it is proved that to an operator $P(D)$ with real coefficients and simple real characteristics and to an arbitrary half-one $F$ containing one point on every bicharacteristic there is a fundamental solution $E$ with sing. supp $E\subseteq F$.

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