ARKIV FÖR MATEMATIK Band 8 nr 5

1.68 11 4 Communicated 11 September 1968 by LENNART CARLESON

A note on asymptotic normality of sums of higher-dimensionally indexed random variables

By BENGT ROSÉN

1. Summary and notation

We shall consider asymptotic normality of sums of random variables when the domain of the summation index is a subset of the lattice points in some higher-dimensional space. Our main aim is to point out that the idea used by the author in [6] to treat asymptotic normality of sums of "one-dimensionally" indexed random variables can easily be adapted to the case of higher-dimensionally indexed random variables.

The course of the paper is as follows. In section 2 we state a result about asymptotic normality, which is equivalent to the author's theorem A in [6]. In the following two sections we illustrate the general idea by considering two particular cases. In section 3 we consider general m-dependent random variables, and section 4 is devoted to U-statistics (see [1]) in the case $\zeta_1=0$ (according to Hoeffding's notation [1]).

We use the following notation and conventions throughout the paper. E denotes expectation and σ^2 variance. $\mathcal{L}(X)$ stands for the law of the random variable, or vector, X. $\mathcal{B}(X_1, X_2, \dots, X_n)$ is the σ-algebra of events generated by the random variables $X_1, X_2, \dots, X_n$. $E^{\mathcal{B}}$ denotes the conditional expectation given the σ-algebra $\mathcal{B}$. We usually write E^X instead of $E^{\mathcal{B}(X)}$. $N(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2. Convergence in distribution is denoted by $\Rightarrow$. When we put a non-integer, λ, in a place where there should naturally be an integer we interprete λ as its integral part $[\lambda]$.

2. A general result about asymptotic normality

The following theorem is equivalent to theorem A in [6].

Theorem 1. *Let* $\{S_\alpha^{(n)}, 0\leqslant\alpha\leqslant 1\}_{n=1}^\infty$ *be a sequence of stochastic processes on* $[0,1]$ *which satisfies* $S_0^{(n)}=0$, $n=1, 2, \dots$, *and the following conditions*

(C 1) *There is a function* $\chi(s)$, $0\leqslant s\leqslant 1$, *which tends to* 0 *as* s *tends to* 0, *such that for* $0\leqslant\beta<\alpha\leqslant 1$ *we have*

$$\overline{\lim_{n\to\infty}}\, E(S_\alpha^{(n)}-S_\beta^{(n)})^2\leqslant\chi(\alpha-\beta),\quad 0\leqslant\beta<\alpha\leqslant 1.$$

(C 2) *There is a function* $\varrho(\alpha)$, *continuous on* $0\leqslant\alpha<1$, *such that*

$$\lim_{\Delta\to+0}\frac{1}{\Delta}\,\overline{\lim_{n\to\infty}}\, E\,|E^{S_\alpha^{(n)}}(S_{\alpha+\Delta}^{(n)}-S_\alpha^{(n)})-\Delta\varrho(\alpha)\,S_\alpha^{(n)}|=0,\quad 0\leqslant\alpha<1.$$

(C 3) *There is a function* $\sigma^2(\alpha)$, *continuous on* $0 \leqslant \alpha < 1$, *such that*

$$\lim_{\Delta\to+0} \frac{1}{\Delta} \overline{\lim_{n\to\infty}} E\,|\,E^{S_\alpha^{(n)}} (S_{\alpha+\Delta}^{(n)} - S_\alpha^{(n)})^2 - \Delta\sigma^2(\alpha)\,| = 0, \quad 0 \leqslant \alpha < 1.$$

(C 4) *For every* $\varepsilon > 0$, *we have*

$$\lim_{\Delta\to+0} \frac{1}{\Delta} \overline{\lim_{n\to\infty}} \int_{|x|>\varepsilon} x^2 d\mathfrak{L}(S_{\alpha+\Delta}^{(n)} - S_\alpha^{(n)})(x) = 0, \quad 0 \leqslant \alpha < 1.$$

Then

$$\mathfrak{L}(S_\alpha^{(n)}) \Rightarrow N(0, \psi(\alpha)) \quad \text{as} \quad n\to\infty, \quad 0 \leqslant \alpha \leqslant 1 \tag{2.1}$$

where

$$\psi(\alpha) = \begin{cases} \int_0^\alpha \sigma^2(s) \exp\left(2\int_s^\alpha \varrho(u)\,du\right) ds, & 0 \leqslant \alpha < 1 \\ \lim_{s\nearrow 1} \psi(s), & \alpha = 1. \end{cases}$$

3. Asymptotic normality of sums of stationary m-dependent random variables

The concept of m-dependence in a sequence of random variables was introduced in [2] where it was also shown that, under general conditions, the sum of a large number of m-dependent random variables is approximately normally distributed. A more general central limit theorem for m-dependent random variables was proved by S. Orey in [5]. We also refer to § 13 in [6]. In this section we generalize, in a straightforward way, the concept of m-dependence to higher-dimensionally indexed random variables and we show that the sum of such variables is approximately normally distributed under certain conditions. For simplicity we shall also assume that the random variables are stationary.

Firstly some notation. Z^k is the set of lattice points (points with integral coordinates) in k-dimensional space and N^k is the set of lattice points with positive coordinates. Points in Z^k will be denoted $\nu = (\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(k)})$. We define $\nu_1 + \nu_2 = (\nu_1^{(1)} + \nu_2^{(1)}, \ldots, \nu_1^{(k)} + \nu_2^{(k)})$ and $\nu_1 - \nu_2$ is defined analogously. $\|\nu\| = \max\{|\nu^{(i)}|, i = 1, 2, \ldots, k\}$. For subsets C and D of Z^k we let

$$d(C, D) = \inf\{\|\nu_1 - \nu_2\|;\ \nu_1 \in C, \nu_2 \in D\}.$$

The family $\{X_\nu, \nu \in N^k\}$ is said to be *stationary* if for arbitrary $\nu_1, \nu_2, \ldots, \nu_r$ belonging to N^k and for every $\mu = (\mu^{(1)}, \ldots, \mu^{(k)})$ such that $\mu^{(i)} \geqslant 0, i = 1, 2, \ldots, k$, we have

$$\mathfrak{L}(X_{\nu_1+\mu}, X_{\nu_2+\mu}, \ldots, X_{\nu_r+\mu}) = \mathfrak{L}(X_{\nu_1}, X_{\nu_2}, \ldots, X_{\nu_r}).$$

The random variables in the family $\{X_\nu, \nu \in N^k\}$ are said to be *m-dependent* if for all subsets C and D of N^k for which $d(C, D) > m$, we have that $\{X_\nu, \nu \in C\}$ and $\{X_\nu, \nu \in D\}$ are independent families. (For def. of independent families see Loève [3], Ch. V.)

If the family $\{X_\nu, \nu \in N^k\}$ is stationary and $EX_\nu^2 < \infty$, then there is a function $\sigma(\cdot)$ defined on Z^k, such that

$$E(X_{\nu_1} - EX_{\nu_1})(X_{\nu_2} - EX_{\nu_2}) = \sigma(\nu_1 - \nu_2) = \sigma(\nu_2 - \nu_1).$$

In the sequel $\sigma(\cdot)$ denotes this function.

To get easier writing we shall, in this section, use the notation $\{\ \}\Sigma$, and inside the parenthesis $\{\ \}$ we indicate the domain of the summation.

Theorem 2. *Let $\{X_\nu, \nu \in N^k\}$ be a family of stationary, m-dependent random variables such that $EX_\nu^2 < \infty$, and let*

$$V_n = \begin{Bmatrix} 1 \leqslant \nu^{(i)} \leqslant n \\ i = 1, 2, \ldots, k \end{Bmatrix} \sum X_\nu$$

Then

$$\mathcal{L}((V_n - EV_n)/\sigma(V_n)) \Rightarrow N(0, 1) \quad \text{as} \quad n \to \infty. \tag{3.1}$$

We have

$$\sigma^2(V_n) = n^k \begin{Bmatrix} -m \leqslant \nu^{(i)} \leqslant m \\ i = 1, 2, \ldots, k \end{Bmatrix} \sum \sigma(\nu) \prod_{j=1}^{k} \left(1 - \frac{|\nu^{(j)}|}{n}\right)^+, \tag{3.2}$$

where $a^+ = \max(a, 0)$.

Proof. Without loss of generality we can assume that $EX_\nu = 0$, and we do so in the rest of the proof. We shall need the following formula:

$$E\left(\sum_{\nu^{(1)}=s_1+1}^{s_1+n_1} \cdots \sum_{\nu^{(k)}=s_k+1}^{s_k+n_k} X_\nu\right)^2 = \begin{Bmatrix} -m \leqslant \nu^{(i)} \leqslant m \\ i = 1, 2, \ldots k \end{Bmatrix} \sum \sigma(\nu) \prod_{j=1}^{k} (n_j - |\nu^{(j)}|)^+. \tag{3.3}$$

Formula (3.3) holds generally for a stationary family if the summation to the right is taken over $\{-n_i \leqslant \nu^{(i)} \leqslant n_i,\ i = 1, 2, \ldots, k\}$. Because of the m-dependence we have $\sigma(\nu) = 0$ if $\|\nu\| > m$ and we obtain (3.3). Formula (3.2) is a special case of (3.3).

We shall apply Theorem 1 and we introduce

$$S_\alpha^{(n)} = n^{-k/2} \sum_{\nu^{(1)}=1}^{\alpha n} \begin{Bmatrix} 1 \leqslant \nu^{(i)} \leqslant n \\ i = 2, 3, \ldots, k \end{Bmatrix} \sum X_\nu, \quad 0 \leqslant \alpha \leqslant 1.$$

We first verify (C 1). According to (3.3) we have for $0 \leqslant \beta < \alpha \leqslant 1$

$$E(S_\alpha^{(n)} - S_\beta^{(n)})^2 = n^{-k} E\left(\sum_{\nu^{(1)}=\beta n+1}^{\alpha n} \begin{Bmatrix} 1 \leqslant \nu^{(i)} \leqslant n \\ i = 2, 3, \ldots, k \end{Bmatrix} \sum X_\nu\right)^2$$

$$= \begin{Bmatrix} -m \leqslant \nu^{(i)} \leqslant m \\ i = 1, 2, \ldots, k \end{Bmatrix} \sum \sigma(\nu) \frac{([\alpha n] - [\beta n] - |\nu^{(1)}|)^+}{n} \prod_{j=2}^{k} \left(1 - \frac{|\nu^{(j)}|}{n}\right)^+. \tag{3.4}$$

By letting $n \to \infty$ in (3.4) we deduce that

$$\lim_{n \to \infty} E(S_\alpha^{(n)} - S_\beta^{(n)})^2 = (\alpha - \beta) \begin{Bmatrix} -m \leqslant \nu^{(i)} \leqslant m \\ i = 1, 2, \ldots, k \end{Bmatrix} \sum \sigma(\nu)$$

and (C 1) is verified. Next we shall verify that (C 2) is fulfilled for $\varrho(\alpha) \equiv 0$. By using the fact that $S_\alpha^{(n)}$ and $\{X_\nu, \nu^{(1)} > \alpha n + m\}$ are independent and the stationarity we obtain

$$E\,|E^{S_\alpha^{(n)}}(S_{\alpha+\Delta}^{(n)} - S_\alpha^{(n)})| = E\left|E^{S_\alpha^{(n)}} n^{-k/2} \sum_{\nu^{(1)}=\alpha n+1}^{\alpha n+m} \begin{Bmatrix} 1 \leqslant \nu^{(i)} \leqslant n \\ i = 2, 3, \ldots, n \end{Bmatrix} \sum X_\nu\right|$$

$$\leqslant mn^{-k/2}E\left|\left\{\nu^{(1)}=1, \begin{matrix} 1\leqslant \nu^{(i)} \leqslant n \\ i=2,3,\ldots,k\end{matrix}\right\}\sum X_\nu\right|$$

$$\leqslant mn^{-k/2}\sqrt{E\left(\left\{\nu^{(1)}=1, \begin{matrix} 1\leqslant \nu^{(i)} \leqslant n \\ i=2,3,\ldots,k\end{matrix}\right\}\sum X_\nu\right)^2}. \tag{3.5}$$

Now, $\{X_\nu; \nu^{(1)}=1, (\nu^{(2)}, \nu^{(3)}, \ldots, \nu^{(k)}) \in Z^{k-1}\}$ is a family of stationary m-dependent random variables. Thus, according to (3.3) we have

$$E\left(\left\{\nu^{(1)}=1, \begin{matrix} 1\leqslant \nu^{(i)} \leqslant n \\ i=2,3,\ldots,k\end{matrix}\right\}\sum X_\nu\right)^2 = n^{k-1}\left\{\begin{matrix} -m\leqslant \nu^{(i)} \leqslant m \\ i=2,3,\ldots,k\end{matrix}\right\}\sum \sigma(0, \nu^{(2)}, \ldots, \nu^{(k)}) \cdot \prod_{j=2}^{k}\left(1-\frac{|\nu^{(j)}|}{n}\right)^{+}.$$

By inserting this estimate into (3.5) we obtain that $E\,|E^{S_\alpha^{(n)}}(S_{\alpha+\Delta}^{(n)}-S_\alpha^{(n)})|$ tends to 0 as n tends to infinity. Thus (C 2) is verified. Next we show that (C 3) is fulfilled for

$$\sigma^2(\alpha) = \left\{\begin{matrix} -m\leqslant \nu^{(i)} \leqslant m \\ i=1,2,\ldots,k\end{matrix}\right\}\sum \sigma(\nu), \quad 0 \leqslant \alpha < 1. \tag{3.6}$$

We have

$$E^{S_\alpha^{(n)}}(S_{\alpha+\Delta}^{(n)}-S_\alpha^{(n)})^2 = n^{-k}E^{S_\alpha^{(n)}}\left(\sum_{\nu^{(1)}=\alpha n+1}^{\alpha n+m}\left\{\begin{matrix} 1\leqslant \nu^{(i)} \leqslant n \\ i=2,3,\ldots,k\end{matrix}\right\}\sum X_\nu + \sum_{\nu^{(1)}=\alpha n+m+1}^{(\alpha+\Delta)n}\left\{\begin{matrix} 1\leqslant \nu^{(i)} \leqslant n \\ i=2,3,\ldots,k\end{matrix}\right\}\sum X_\nu\right)^2 = n^{-k}E^{S_\alpha^{(n)}}(U_\alpha^{(n)}+R_{\alpha,\Delta}^{(n)})^2.$$

As $S_\alpha^{(n)}$ and $\{X_\nu, \nu^{(1)} > \alpha n + m\}$ are independent we get

$$E^{S_\alpha^{(n)}}(S_{\alpha+\Delta}^{(n)}-S_\alpha^{(n)})^2 = n^{-k}(E(R_{\alpha,\Delta}^{(n)})^2 + E^{S_\alpha^{(n)}}(U_\alpha^{(n)})^2 + 2E^{S_\alpha}U_\alpha^{(n)}R_{\alpha,\Delta}^{(n)}). \tag{3.7}$$

We have

$$E\,|E^{S_\alpha^{(n)}}U_\alpha^{(n)}R_{\alpha,\Delta}^{(n)}| \leqslant \sqrt{E(U_\alpha^{(n)})^2\cdot E(R_{\alpha,\Delta}^{(n)})^2}. \tag{3.8}$$

By using (3.3) one easily checks that

$$n^{-k}E(U_\alpha^{(n)})^2 \to 0 \quad \text{as} \quad n\to\infty \tag{3.9}$$

$$n^{-k}E(R_{\alpha,\Delta}^{(n)})^2 \to \Delta\left\{\begin{matrix} -m\leqslant \nu^{(i)} \leqslant m \\ i=1,2,\ldots,k\end{matrix}\right\}\sum \sigma(\nu) \text{ as } n\to\infty. \tag{3.10}$$

By combining formulas (3.10), (3.9), (3.8) and (3.7) we see that (C 3) is fulfilled for $\sigma^2(\alpha)$ according to (3.6). Now it remains to verify (C 4). We need some notation. For $\mu \in Z^k$ let

$$H_\mu = \{\nu: \nu\in Z^k, \nu = \mu + (\lambda_1(m+1), \lambda_2(m+1), \ldots, \lambda_k(m+1)), \lambda_i = 0, \pm 1, \pm 2, \ldots\} \quad i=1,2,\ldots k.$$

Furthermore, let

$$T_{\mu}^{(n)} = \left\{ \nu \in H_{\mu}, \begin{matrix} 1 \leqslant \nu^{(i)} \leqslant n \\ i = 1, 2, \ldots, k \end{matrix} \right\} \sum X_{\nu}.$$

Then we have

$$V_n = \left\{ \begin{matrix} 1 \leqslant \mu^{(i)} \leqslant m+1 \\ i = 1, 2, \ldots, k \end{matrix} \right\} \sum T_{\mu}^{(n)}.$$

Now, according to the m-dependence, $T_{\mu}^{(n)}$ is a sum of independent random variables. By applying the results in section 10 in [6], in particular Lemmata 10.1 and 10.7 it is easy to complete the verification of (C4).

Thus, Theorem 1 applies with $\varrho(\alpha) \equiv 0$ and $\sigma^2(\alpha)$ according to (3.6) and (3.1) now follows by putting $\alpha = 1$ in (2.1). Thereby Theorem 2 is proved.

4. Asymptotic normality of U-statistics when $\zeta_1 = 0$

The concept of U-statistic was introduced by W. Hoeffding in [1]. As far as possible we shall follow the notation in [1]. Let $\Phi(x, y)$ be a symmetric function of x and y and let $X, X', X'', X_1, X_2, \ldots$ throughout be independent random variables, all with the same distribution F. The U-statistic generated by the kernel $\Phi(x, y)$ and the sample $X_1, X_2, \ldots, X_n$ from F is defined as

$$U_n(F) = \frac{1}{n(n-1)} \sum_{\nu,\mu=1}^{n}{}' \Phi(X_\nu, X_\mu). \tag{4.1}$$

Here, and in the sequel, Σ' denotes that the summation does not include $\nu = \mu$. Let

$$\theta = E\Phi(X, X'),\ \Psi(X, X') = \Phi(X, X') - \theta,\ \Psi_1(X) = E^X \Psi(X, X')$$

$$\zeta_1 = E\Psi_1(X)^2, \quad \zeta_2 = E\Psi(X, X')^2$$

$$\chi = E(E^{X,X'}\Psi(X, X'')\,\Psi(X', X''))^2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} (\Phi(x, u) - \theta)\,(\Phi(y, u) - \theta)\, dF(u) \right)^2 dF(x)\, dF(y). \tag{4.2}$$

Hoeffding proved in [1] that if F and Φ are fixed, and if $E(\Phi(X, X'))^2 < \infty$, then $\sqrt{n}(U_n - EU_n)$ is asymptotically $N(0, 4\zeta_1)$-distributed. This result, however, becomes trivial in the case when $\zeta_1 = 0$, as the limiting distribution has zero variance in this case. We shall here consider the case $\zeta_1 = 0$ in more detail. We shall also allow F to vary with n. We use notations like $U_n(F_n)$, $\zeta_1(F_n)$ etc. to indicate that the quantity is computed under the assumption that the X-variables have the common distribution F_n.

Theorem 3. *Let $\Phi(x, y)$ be a fixed kernel, and let $F_1, F_2, \ldots$ be a sequence of distribution functions, such that*

$$\zeta_1(F_n) = 0, \quad n = 1, 2, \ldots \tag{4.3}$$

$$\zeta_2(F_n) \geqslant \gamma > 0, \quad n = 1, 2, \ldots \tag{4.4}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(x,y)^4 dF_n(x)\,dF_n(y)\leqslant C<\infty,\quad n=1,2,\ldots \tag{4.5}$$

Let $U_n(F_n)$ *and* $\chi(F_n)$ *be defined according to* (4.1) *and* (4.2). *If*

$$\chi(F_n)\to 0 \quad as \quad n\to\infty \tag{4.6}$$

then

$$\mathfrak{L}\left(\frac{n(U_n(F_n)-EU_n(F_n))}{\sqrt{2\zeta_2(F_n)}}\right)\Rightarrow N(0,1)\quad \text{as}\quad n\to\infty. \tag{4.7}$$

Before we prove this theorem we shall derive some auxiliary results. In Lemmata 4.1–4.6 we assume F to be fixed. Furthermore, we shall use the notation

$$Z_k=\sum_{\nu,\mu=1}^{k}{}'\,\Psi(X_\nu,X_\mu),\quad k=1,2,\ldots.$$

In computing moments of the second order the following formula is crucial (cf. [1], p. 299). For $\nu_1\neq\mu_1$ and $\nu_2\neq\mu_2$: $E\Psi(X_{\nu_1},X_{\mu_1})\Psi(X_{\nu_2},X_{\mu_2})=$

$$=\begin{cases}0 & \text{if}\quad \nu_1,\nu_2,\mu_1,\mu_2 \text{ are all different}\\ \zeta_1 & \text{if}\quad \nu_1,\nu_2,\mu_1,\mu_2 \text{ contains 3 different elements}\\ \zeta_2 & \text{if}\quad \nu_1,\nu_2,\mu_1,\mu_2 \text{ contains 2 different elements.}\end{cases} \tag{4.8}$$

Lemma 4.1.

$$E(Z_{k+m}-Z_k)^2=m[4\zeta_1\{k(k+3m-4)+(m-1)(m-2)\}+2\zeta_2(2k+m-1)].$$

Proof. We have the following identity

$$\begin{aligned}(Z_{k+m}-Z_k)^2&=(\{2\sum_{\nu=1}^{k}\sum_{\mu=k+1}^{k+m}+\sum_{\nu,\mu=k+1}^{k+m}{}'\}\Psi(X_\nu,X_\mu))^2\\ &=\{4\sum_{\nu_1,\nu_2=1}^{k}\sum_{\mu_1,\mu_2=k+1}^{k+m}+4\sum_{\nu_1=1}^{k}\sum_{\mu_1=k+1}^{k+m}\sum_{\nu_2,\mu_2=k+1}^{k+m}{}'\\ &\quad+\sum_{\nu_1,\mu_1=k+1}^{k+m}{}'\sum_{\nu_2,\mu_2=k+1}^{k+m}{}'\}\Psi(X_{\nu_1},X_{\mu_1})\Psi(X_{\nu_2},X_{\mu_2}).\end{aligned} \tag{4.9}$$

Now take expectation termwise in (4.9) and use (4.8) to simplify. Lemma 4.1 then follows.

Lemma 4.2. *If* $\zeta_1=0$, *then*

$$E(Z_{k+m}-Z_k)^4\leqslant Cm^2(k^2+m^2)\,E\Psi(X,X')^4, \tag{4.10}$$

where C is an absolute constant.

Proof. $$E(Z_{k+m}-Z_k)^4=E(\{2\sum_{\nu=1}^{k}\sum_{\mu=k+1}^{k+m}+\sum_{\nu,\mu=k+1}^{k+m}{}'\}\Psi(X_\nu,X_\mu))^4$$

$$\leqslant C[E(\sum_{\nu=1}^{k}\sum_{\mu=k+1}^{k+m})^4+E(\sum_{\nu,\mu=1}^{m}{}')^4]=C[\{\sum_{\nu_1,\nu_2,\nu_3,\nu_4=1}^{k}\sum_{\mu_1,\mu_2,\mu_3,\mu_4=k+1}^{k+m}$$

$$+\sum_{\left\{\substack{\nu_1,\nu_2,\nu_3,\nu_4\\ \mu_1,\mu_2,\mu_3,\mu_4}\right\}=1}^{m}{}'\}E\prod_{i=1}^{4}\Psi(X_{\nu_i},X_{\mu_i}). \tag{4.11}$$

If too many of the ν's and μ's are different, then $E\prod\Psi(X_{\nu_i}, X_{\mu_i})$ vanishes (cf. (4.8)). Upon some thought we realize that the number of non-vanishing terms in (4.11) does not exceed $Cm^2(k^2+m^2)$. For all these terms we have by Schwarz's inequality

$$|E\prod_{i=1}^{4}\Psi(X_{\nu_i},X_{\mu_i})|\leqslant E(\Psi(X,X'))^4$$

and (4.10) follows.

Next we shall consider some conditioning formulas. Let $\mathcal{B}_n=\mathcal{B}(X_1, X_2, \ldots, X_n)$. The proof of the next lemma is quite straightforward, and we omit it.

Lemma 4.3. *When $\zeta_1=0$, we have for $k, m=0, 1, 2, \ldots$*

$$E^{\mathcal{B}_k}(Z_{k+m}-Z_k)=0.$$

Here we add some notation. Let

$$\Lambda(X)=E^X\Psi(X,X')^2$$

and

$$\Omega(X,X')=E^{X,X'}\Psi(X,X'')\Psi(X',X'').$$

Lemma 4.4. *When $\zeta_1=0$, we have for $k, m=0, 1, 2, \ldots$*

$$E^{\mathcal{B}_k}(Z_{k+m}-Z_k)^2=4m\sum_{\nu=1}^{k}\Lambda(X_\nu)+4m\sum_{\nu,\mu=1}^{k}{}'\,\Omega(X_\nu,X_\mu)+2m(m-1)\,\zeta_2. \tag{4.12}$$

Proof. We take the conditional expectation $E^{\mathcal{B}_k}$ in (4.9) and we obtain, noting that $\zeta_1=0$ is equivalent to $E^X\Psi(X,X')\equiv 0$,

$$E^{\mathcal{B}_k}(Z_{k+m}-Z_k)^2=4m\sum_{\nu_1,\nu_2=1}^{k}E^{X_{\nu_1},X_{\nu_2}}\Psi(X_{\nu_1},X)\,\Psi(X_{\nu_2},X)+E(\sum_{\nu,\mu=k+1}^{k+m}{}'\Psi(X_\nu,X_\nu))^2 \tag{4.13}$$

which is easily simplified to (4.12) by using Lemma 4.1.

Lemma 4.5. $E|\sum_{\nu=1}^{k}(\Lambda(X_\nu)-\zeta_2)|\leqslant\sqrt{kE\Psi(X,X')^4},\quad k=1,2,\ldots.$

Proof: $\Lambda(X_1)$, $\Lambda(X_2)$, ... are independent, equally distributed random variables with mean ζ_2. Thus

$$(E|\sum_{\nu=1}^{k}(\Lambda(X_\nu)-\zeta_2)|)^2\leqslant\sigma^2(\sum_{\nu=1}^{k}\Lambda(X_\nu))=k\sigma^2(\Lambda(X))$$

$$\leqslant kE\Lambda(X)^2=kE(E^X\Psi(X,X')^2)^2\leqslant kEE^X\Psi(X,X')^4$$

and the lemma is proved.

Lemma 4.6. *When $\zeta_1 = 0$, we have for $k = 1, 2, \ldots$*

$$E\left|\sum_{\nu,\mu=1}^{k}{}' \Omega(X_\nu, X_\mu)\right| \leqslant \sqrt{2k(k-1)\chi}.$$

Proof. We regard $\Omega(x, y)$ as a kernel. It is easily checked that $E\Omega(X, X') = 0$, and that $\zeta_1(\Omega) = 0$ when $\zeta_1(\Phi) = 0$. Furthermore, we have $\zeta_2(\Omega) = \chi(\Phi)$. From Lemma 4.1 we get

$$\left(E\left|\sum_{\nu,\mu=1}^{k}{}' \Omega(X_\nu, X_\mu)\right|\right)^2 \leqslant E\left(\sum_{\nu,\mu=1}^{k}{}' \Omega(X_\nu, X_\mu)\right)^2 = 2k(k-1)\chi$$

and the lemma is proved.

Proof of Theorem 3. First we shall prove Theorem 3 under the extra assumption

$$\zeta_2(F_n) \to \zeta_2 \quad \text{as} \quad n \to \infty, \quad 0 < \zeta_2 < \infty. \tag{4.14}$$

We shall apply Theorem 1, and we introduce

$$S_\alpha^{(n)} = n(U_{\alpha n}(F_n) - EU_{\alpha n}(F_n)) = Z_{\alpha n}^{(n)}/(n-1), \quad 0 \leqslant \alpha \leqslant 1.$$

From Lemma 4.1 and (4.14) we conclude that

$$\overline{\lim_{n\to\infty}}\, E(S_\alpha^{(n)} - S_\beta^{(n)})^2 = 2(\alpha - \beta)(\alpha + \beta)\zeta_2, \quad 0 \leqslant \beta < \alpha \leqslant 1.$$

Thus (C 1) is verified. In a similar manner (C 4) follows from Lemma 4.2, (4.5) and Lemma 10.3 in [6]. From Lemma 4.3 we conclude that (C 2) is satisfied for $\varrho(\alpha) \equiv 0$. Next we shall verify that (C 3) is satisfied for $\sigma^2(\alpha) = 4\alpha\zeta_2$, where ζ_2 is defined in (4.14). By virtue of Lemmata 4.4, 4.5 and 4.6 we have

$$\begin{aligned}
&\overline{\lim_{n\to\infty}}\, E\left|E^{S_\alpha^{(n)}}(S_{\alpha+\Delta}^{(n)} - S_\alpha^{(n)})^2 - \Delta 4\alpha\zeta_2\right| \\
&\leqslant \overline{\lim_{n\to\infty}}\, E\left|E^{\mathcal{B}_{\alpha n}^{(n)}} \frac{1}{(n-1)^2}(Z_{(\alpha+\Delta)n}^{(n)} - Z_{\alpha n}^{(n)})^2 - \Delta 4\alpha\zeta_2\right| \\
&\leqslant \Delta 4\alpha\, \overline{\lim_{n\to\infty}}\, |\zeta_2(F_n) - \zeta_2| + \Delta\, \overline{\lim_{n\to\infty}}\, E\left|\frac{1}{n}\sum_{\nu=1}^{\alpha n}(\Lambda^{(n)}(X_\nu) - \zeta_2(F_n))\right| \\
&+ \Delta\, \overline{\lim_{n\to\infty}}\, E\left|\frac{1}{n}\sum_{\nu,\mu=1}^{\alpha n}{}' \Omega^{(n)}(X_\nu, X_\mu)\right| + 2\Delta^2\zeta_2 \\
&\leqslant \Delta\left(\overline{\lim_{n\to\infty}}\, \frac{1}{n}\sqrt{\alpha n C} + \overline{\lim_{n\to\infty}}\, \frac{1}{n}\sqrt{2\alpha n(\alpha n - 1)\chi(F_n)}\right) + 2\Delta^2\zeta_2 = 2\Delta^2\zeta_2
\end{aligned} \tag{4.15}$$

according to (4.5) and (4.6). Now (C 3) follows easily from (4.15).

Thus, Theorem 1 applies, and we obtain

$$\mathcal{L}(n(U_n(F_n) - EU_n(F_n))) = \mathcal{L}(S_1^{(n)}) \Rightarrow N\left(0, \int_0^1 4\alpha\zeta_2\, d\alpha\right) = N(0, 2\zeta_2) \quad \text{as} \quad n \to \infty \tag{4.16}$$

Thus, Theorem 3 is proved under the extra assumption (4.14). Our next step is to remove this assumption for the validity of Theorem 3. Put

$$V_n = n(U_n(F_n) - EU_n(F_n))/\sqrt{2\zeta_2(F_n)}.$$

We give an indirect proof. Thus, we assume that the conditions of Theorem 3 are met, but that the conclusion (4.7) does not hold, i.e.

$$\mathcal{L}(V_n) \not\Rightarrow N(0, 1) \quad \text{as} \quad n \to \infty. \tag{4.17}$$

From Lemma 4.1 we conclude that $EV_n^2 \leqslant n/(n-1)$. Thus, in regard of (4.17) we can pick a subsequence such that

$$\mathcal{L}(V_{n_\nu}) \Rightarrow \mathcal{L} \neq N(0, 1) \quad \text{as} \quad \nu \to \infty. \tag{4.18}$$

From (4.4) and (4.5) it follows that, by restricting to a new subsequence (without changing notation), we can obtain that

$$\zeta_2(F_{n_\nu}) \to \zeta_2 \quad \text{as} \quad \nu \to \infty, \quad 0 < \zeta_2 < \infty. \tag{4.19}$$

According to (4.19) and what is already proved, it follows that

$$\mathcal{L}(V_{n_\nu}) \Rightarrow N(0, 1) \quad \text{as} \quad \nu \to \infty. \tag{4.20}$$

Now (4.18) and (4.20) contradict each other. Thereby the proof of Theorem 3 is complete.

We conclude by treating an example, whose first part illustrates Theorem 3. Its second part illustrates the well-known fact (see the paper [4] by von Mieses) that, when $\zeta_1 = 0$, U-statistics can have other limiting distributions than the normal, although $(U_n(F_n) - EU_n(F_n))/\sigma(U_n(F_n))$ have uniformly bounded moments of all orders.

Example. Let $\Phi(x, y) = \sin xy$, and let F_n be the uniform distribution over $[-A_n, A_n]$, $A_n > 0$, $n = 1, 2, \ldots$.

a. If $A_n \to \infty$ as $n \to \infty$, we have

$$\mathcal{L}(nU_n(F_n)) \Rightarrow N(0, 1). \tag{4.21}$$

b. If $A_n \to 0$ as $n \to \infty$, we have

$$P\left(\frac{nU_n(F_n)}{\sqrt{\zeta_2(F_n)}} \leqslant x\right) \to \begin{cases} \int_{-1}^{x} \frac{e^{-(t+1)/2}}{\sqrt{2\pi(t+1)}}\, dt, & x \geqslant -1 \\ 0, & x < -1 \end{cases} \quad \text{as} \quad n \to \infty. \tag{4.22}$$

Verification. First we note that $\zeta_1(F_n) = 0$, $n = 1, 2, \ldots$. We verify (4.21) by applying Theorem 3. We have

$$\zeta_2(F_n) = \left(\frac{1}{2A_n}\right)^2 \int_{-A_n}^{A_n} \int_{-A_n}^{A_n} \sin^2 xy\, dx\, dy = \frac{1}{2} - \frac{1}{4A_n^2} \int_0^{2A_n^2} \frac{\sin u}{u}\, du. \tag{4.23}$$

Thus
$$\zeta_2(F_n) \to \tfrac{1}{2} \quad \text{as} \quad A_n \to \infty. \tag{4.24}$$

Furthermore,

$$\chi(F_n) = \left(\frac{1}{2A_n}\right)^2 \int_{-A_n}^{A_n} \int_{-A_n}^{A_n} \left(\frac{1}{2A_n} \int_{-A_n}^{A_n} \sin xu \cdot \sin yu du\right)^2 dx\, dy$$
$$= \left(\frac{1}{2A_n}\right)^2 \int_{-A_n}^{A_n} \int_{-A_n}^{A_n} \frac{1}{4} \left(\frac{\sin (x-y) A_n}{(x-y) A_n} - \frac{\sin (x+y) A_n}{(x+y) A_n}\right)^2 dx\, dy. \tag{4.25}$$

From (4.25) it is easily deduced that

$$\chi(F_n) \to 0 \quad \text{as} \quad A_n \to \infty. \tag{4.26}$$

(4.24) and (4.26) yield that the conditions (4.4) and (4.6) in Theorem 3 are fulfilled. The verification of (4.5) is straightforward. (4.21) now follows from Theorem 3.

Next we prove (4.22). From (4.23) we easily deduce

$$\zeta_2(F_n) \sim A_n^4/9 \quad \text{as} \quad A_n \to 0. \tag{4.27}$$

Let
$$Q_n(F_n) = \frac{3}{A_n^2(n-1)} \sum_{\nu,\mu=1}^{n}{}' X_\nu X_\mu.$$

According to (4.27) and Lemma 4.1 we have

$$E\left(\frac{nU_n(F_n)}{\sqrt{\zeta_2(F_n)}} - Q_n(F_n)\right)^2 \sim E\left(\frac{3}{A_n^2(n-1)} \sum_{\nu,\mu=1}^{n}{}' (\sin X_\nu X_\mu - X_\nu X_\mu)\right)^2$$
$$= \frac{9 \cdot 2n(n-1)}{A_n^4(n-1)^2} \left(\frac{1}{2A_n}\right)^2 \int_{-A_n}^{A_n} \int_{-A_n}^{A_n} (\sin xy - xy)^2 dx\, dy \to 0 \quad \text{as} \quad A_n \to 0. \tag{4.28}$$

Furthermore,
$$Q_n(F_n) = \left(\frac{1}{\sqrt{n-1}} \sum_{\nu=1}^{n} \left(\frac{X_\nu \sqrt{3}}{A_n}\right)\right)^2 - \frac{1}{n-1} \sum_{\nu=1}^{n} \left(\frac{X_\nu \sqrt{3}}{A_n}\right)^2. \tag{4.29}$$

According to the central limit theorem, the first term to the right in (4.29) converges in distribution to a χ^2-variable with one degree of freedom, and according to the law of large numbers the second term converges in probability to 1. These two facts easily yield that $Q_n(F_n)$ has the distribution in (4.22) as limit distribution. From (4.28) we conclude that $nU_n(F_n)/\sqrt{\zeta_2(F_n)}$ and $Q_n(F_n)$ have the same limiting distribution. Thus (4.22) is verified.

Institute of Mathematics, University of Uppsala, 752 23 Uppsala, Sweden

REFERENCES

1. Hoeffding, W., A class of statistics with asymptotically normal distribution. Ann. Math. Stat. *19*, 293–325 (1948).
2. Hoeffding, W., and Robbins, H., The central limit theorem for dependent random variables. Duke Math. Journ. *15*, 773–780 (1948).
3. Loève, M., Probability Theory. 2nd ed. D. van Nostrand, Princeton 1960.
4. von Mieses, R., On the asymptotic distribution of differentiable statistical functions. Ann. Math. Stat. 18, 309–348 (1947).
5. Orey, S., A central limit theorem for m-dependent random variables. Duke Math. Journ. 25, 543–546 (1958).
6. Rosén, B., On the central limit theorem for sums of dependent random variables. Zeitschr. Wahrscheinlichkeitstheorie verw. Geb. 7, 48–82.

Tryckt den 14 maj 1969

Uppsala 1969. Almqvist & Wiksells Boktryckeri AB