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Relatively maximal function algebras generated by polynomials on compact sets in the complex plane

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Introduction

Wermer's maximality theorem states that if J is a Jordan curve in the complex plane the function algebra $P(J)$ generated by polynomials on J is a maximal closed subalgebra of $C(J)$, the algebra of complex-valued continuous functions on J. Wermer's theorem can also be stated in the following form: If $g \in C(J)$ is such that the polynomials and g generate a proper function algebra of $C(J)$ then g has an analytic extension to the interior of J. In this paper we try to extend Wermer's maximality theorem in the following way: Let J be a Jordan curve in the complex plane. We denote by $H(J)$ the compact set bounded by J. Let F be a closed subset of $H(J)$ containing J. Suppose $g \in C(F)$ is such that g and the polynomials generate a proper function algebra of $C(F)$. Now we wish to find out if g has an analytic extension from J into the interior of $H(J)$, i.e. if there exists a function $G \in C(H(J))$ such that $G=g$ on J and G is analytic in the interior of $H(J)$. Of course we need some conditions on F to obtain such results. We say that F satisfies (C) if the following holds:

1. $R(F)=C(F)$, where $R(F)$ is the function algebra on F generated by rational functions with poles outside F.
2. $H(J)-F$ is connected and $\overline{(F-J)} \cap J \neq J$.

We show in Theorem 1 that if F satisfies (C) and $g \in C(F)$ is such that g and the polynomials generate a proper function algebra of $C(F)$ then there exists an analytic function G in $H(J)-F$ such that $\lim G(z)=g(x)$ as $z \in H(J)-F$ tend to $x \in J$. In the final part of this paper we apply theorems 1 and 2 to solve an approximation problem on the unit interval. Let $f \in C(I)$, where I is the unit interval. Assume $f(\frac{1}{4})=f(\frac{3}{4})$ while $f(x) \neq f(y)$ for all other pairs of distinct points $x, y \in I$. If $g \in C(I)$ is such that $g(\frac{1}{4}) \neq g(\frac{3}{4})$ we wish to find out if the function algebra on I generated by f, g and the constant functions is $C(I)$. This problem has been discussed in several papers, see for example [1, 2 and 4]. The best result is contained in [2] where it is shown that we get $C(I)$ if f and g are continuously differentiable. A famous example in [3] indicates that some smoothness on f and g is necessary. The example consists of a Jordan arc K in C^3 such that K is not polynomially convex. This Jordan arc is used to construct a proper function algebra of $C(I)$. Let us now put $J=\{f(x) \mid x \in I\}$. We see that J has one of the following three forms:

The case when J has the form (3) is easy, we get $C(I)$ with no extra assumptions on f and g. Also case (2) can be easily reduced to case (1) so we only consider that case. To prove that we now get $C(I)$ we need some conditions on J. Obviously J satisfies the condition (C) if $R(J)=C(J)$. We do not know if this alone is sufficient

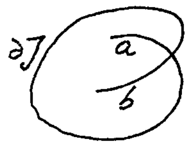

Fig. 1

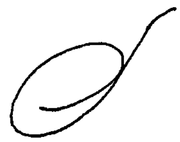

Fig. 2

Fig. 3

to guarantee that we get $C(I)$. In order to prove that we get $C(I)$ we shall need some smoothness of the two Jordan arcs a and b in Fig. 1. It is for example sufficient to have a and b continuously differentiable. Notice that we need no extra condition on g. We shall later introduce a condition on a and b which guarantees that we get $C(I)$. This condition is related to difficult problems on analytic extensions using reflection principles.[1]

Before we state the following results we make the following useful remark: Let F be a compact set satisfying (C). Now we can use a conformal map of $H(J)$ onto the unit disc. Hence we may assume that J is the unit circle T. We point out that all extra conditions we make are invariant under this conformal map. So when we now say that a compact set satisfies (C) it is understood that the Jordan curve J is T.

Theorem 1. *Let K be a compact set of the unit disc D containing T and satisfying (C). Let $g \in C(K)$ be such that g and the polynomials, generate a proper function algebra B of $C(K)$. If now m is a non zero measure on K annihilating B then*

$$G(z)=\int \frac{g(x)\,dm(x)}{x-z} \Big/ \int \frac{dm(x)}{x-z}$$

is a bounded analytic function in $D-K$ with

$$\sup\{|G(z)|\,|\,z \in D-K\} \leqslant \sup\{|g(x)|\,|\,x \in K\}.$$

Also $\lim G(z)=g(e^{i\alpha})$ *exists uniformly as $z \in D-K$ tend to $e^{i\alpha} \in T$.*

Before we prove Theorem 1 we wish to state Theorem 2. Let $G(z)$ be as in Theorem 1, hence $G(z)$ is an analytic function in $D-K$. If $z_0 \in K$ is such that there exists a Jordan arc $J \subset D$ with $J \cap K=\{z_0\}$ and $\lim G(z)=a$ exists as $z \in J$ tend to z_0, then we say that G has the asymptotic value a at z_0. If $z_0 \in K$ is such that $\lim G(z)=a$ exists as $z \in D-K$ tend to z_0, then we say that G has the unrestricted limit value a at z_0.

Theorem 2. *If G has an asymptotic value at some point $z_0 \in K$ which is different from $g(z_0)$ then there exists an open neighborhood V of z_0 such that the restriction of B to $K-V$ generates a proper function algebra of $C(K-V)$. There exists a smallest closed subset F of K such that the restriction of B to F generates a proper function algebra of $C(F)$. The function G is analytic in $D-F$ and if G has an asymptotic value at some point $z_0 \in F$ then G has an unrestricted limit value at z_0 which equals $g(z_0)$.*

Since the proofs are not very short we shall first give some preliminary results. Let M_B be the maximal ideal space of B. As usual we identify K with a closed sub-

[1] In a forthcoming paper by H. S. Shapiro and Shields it is shown that we get $C(I)$ without any extra conditions on f and g. We also remark here that Mergelyan's Theorem shows that $R(J)=C(J)$ is always verified. In a forthcoming paper "Analyticity in the maximal ideal space of a function algebra" by the present author essentiell improvements have been obtained which indicate that f and g generate $C(I)$ in the case where J is only assumed to be a curve with finitely many self-intersections.

set of M_B and then K contains the Shilov boundary of B. If $x \in M_B$ there exists a point $\pi(x) \in D$ such that $P(x) = P(\pi(x))$ for every polynomial P. We say that x lies above $\pi(x)$ and that $\pi(x)$ lies below x. If V is a subset of D we put $\pi^{-1}(V) = \{x \in M_B \mid \pi(x) \in V\}$. The set $\pi^{-1}(V)$ is called the fiber of V in M_B. The correspondence between points $z \in D$ and the fibers $\pi^{-1}(z)$ is continuous in the following way: Let W be an open neighborhood of $\pi^{-1}(z)$ in M_B, then there exists an open neighborhood V of z in D such that $\pi^{-1}(V)$ is contained in W. Since $R(K) = C(K)$ and $D-K$ is connected we see that if $z_0 \in D-K$ then the element $P = z - z_0$ in B cannot be invertible. Hence there exists a point $x \in M_B$ such that $P(x) = z_0$ and it follows that x lies above z_0. We have now proved that the fibers $\pi^{-1}(z)$ are not empty when $z \in D-K$. We shall later prove that the fiber $\pi^{-1}(z)$ is reduced to a single point when $z \in D-K$. If $z \in K$ the fiber $\pi^{-1}(z)$ contains a trivial point, namely z itself. If $z \in K$ and if $\pi^{-1}(z)$ only consists of this trivial point we say that $\pi^{-1}(z)$ is a trivial fiber. If $z \in T$ is is easily seen that $\pi^{-1}(z)$ is a trivial fiber. For suppose that $x \in \pi^{-1}(z)$. Now we can find a positive measure v on K such that $g(x) = \int g\,dv$ for all $g \in B$. In particular $P(z) = \int P\,dv$ for every polynomial. It follows that v is the unit point mass at z and hence $g(x) = g(z)$ for all $g \in B$ which proves that $x = z$. Assume now that we have proved that $\pi^{-1}(z)$ consists of one point $x(z)$ when $z \in D-K$. The function $G(z) = g(x(z))$ is then well defined in $D-K$. We shall later prove that

$$G(z) = \int \frac{g(x)\,dm(x)}{x-z} \Big/ \int \frac{dm(x)}{x-z}$$

if m is an arbitrary non zero measure on K annihilating B. It follows that G is analytic in $D-K$. Since K contains the Shilov boundary of B in M_B we have $|g(x(z)|\leqslant \sup\{|g(x)|\,|x \in K\} = |g|_K$ when $z \in D-K$. If $z \in D-K$ and $\lim z = e^{i\alpha} \in T$ we see that $\lim x(z) = e^{i\alpha}$ holds in M_B too, because $\pi^{-1}(e^{i\alpha})$ is a trivial fiber. Hence $\lim G(z) = \lim g(x(z)) = g(e^{i\alpha})$ as $z \in D-K$ tend to $e^{i\alpha}$. If $z \in K-T$ the fiber $\pi^{-1}(z)$ may be non trivial and then we get troubles. An important result which we shall prove is the following: If G has an asymptotic value at $z_0 \in K-T$ which is different from $g(z_0)$ then $\pi^{-1}(z_0)$ contains exactly two points. Let x_1 be the non trivial point in $\pi^{-1}(z_0)$. We shall later prove that $\lim G(z) = g(x_1)$ as $z \in D-K$ tend to z_0, hence G has an unrestricted limit value at z_0. We can use this to prove that the point $z_0 \in M_B$ has an open neighborhood W in M_B such that $\pi(W)$ is contained in K. Let then V be an open neighborhood of z_0 in D such that $\pi(W)$ is contained in $K \cap V$. We can use this fact to prove that the restriction of B to $F = K - V$ generates a proper function algebra of $C(F)$. If G has an asymptotic value at $z_0 \in K$ which equals $g(z_0)$ we can prove that $\pi^{-1}(z_0)$ is trivial. It follows that if $z \in D-K$ tend to z_0 in D then $x(z)$ tend to z_0 in M_B. Hence $\lim G(z) = \lim g(x(z)) = g(z_0)$, i.e. G has an unrestricted limit value at z_0. We shall freely use results about function algebras. We refer to [5] and [6] for a discussion about these. Here we state some results which are used in the following proofs. Let A be a function algebra with the maximal ideal space M_A and the Shilov boundary S_A. The set $D_A = M_A - S_A$ is called the interior of M_A. The Local Maximum Principle is here used in the following form: Let W be a subset of D_A and let bW be the topological boundary of W in M_A, then $|f(x)| \leqslant \sup\{|f(y)|\,|y \in bW\}$ for every $x \in W$. In particular there exists a positive measure m_x carried on bD_A for each point $x \in D_A$ such that $f(x) = \int f\,dm_x$ for all $f \in A$. It follows from this that if D_A is not empty then the restriction of A to bD_A generates a proper function algebra of $C(bD_A)$. A closed subset F of M_A is A-convex if for every point $x \in M_A - F$ there exists $f \in A$

such that $f(x) > \sup \{|f(y)| \, y \in F\}$. If F is an A-convex subset of M_A the function algebra A_F on F generated by restricting A to F has F as its maximal ideal space.

Proof of Theorem 1. Let m be a non zero measure on K annihilating B. Let us put $W(z) = \int g(x) dm(x)/x - z$ and $R(z) = \int dm(x)/x - z$. Obviously W and R are analytic functions in $D - K$. Because m annihilates B we get $\int \bar{z} g(x) dm(x)/1 - \bar{z}x = 0$ for $z \in D - K$ and hence $W(z) = \int (1 - |z|^2) g(x) dm(x)/(x - z)(1 - \bar{z}x)$ when $z \in D - K$. Let us put $K_1 = (\overline{K - T})$. By assumption there exists a closed arc $L = \{e^{it} \mid a \leqslant t \leqslant b\}$ such that $K_1 \cap L$ is empty. Hence there exists $r_0 < 1$ such that if $r \geqslant r_0$ and $a \leqslant t \leqslant b$ then $re^{it} \notin K_1$. From now on we always assume that $r \geqslant r_0$. We also put $K - T = S$.

Lemma 1. $\overline{\lim} \displaystyle\int_a^b |W(re^{it})| dt < \infty$ as r tends to 1.

Proof. We have $\int_a^b |W(re^{it})| dt \leqslant \int_a^b dt \int_S |g(x)| \, |dm(x)| (1 - r^2)/|x - re^{it}| \, |1 - re^{-it}x| + \int_T |g(x)| \, |dm(x)| \int_a^b (1 - r^2)|/(x - re^{it}|^2 dt = A(r) + B(r)$. *Obviously* $\lim A(r) = 0$ *as* r *tends to* 1 *because* $K_1 \cap L$ *is empty, also* $B(r) \leqslant 2\pi \int_T |g(x)| \, |dm(x)|$ *holds.*

Using Lemma 1 we can now choose two different rays $\{re^{ic}\}$ and $\{re^{id}\}$ where $a < c < d < b$ such that $\lim W(re^{ic})$, $\lim W(re^{id})$, $\lim R(e^{ic})$ and $\lim R(e^{id})$ all exist finitely as r tends to 1. We shall need the following elementary result:

Lemma 2. *Let J be a Jordan curve in D such that $J \cap T = \{e^{it} \mid c \leqslant t \leqslant d\} = J_1$. Also J approaches T along the two rays $\{re^{ic}\}$ and $\{re^{id}\}$, i.e. J contains the two sets $\{re^{ic} \mid r_1 \leqslant r < 1\}$ and $\{re^{id} \mid r_1 \leqslant r < 1\}$ for some $r_1 < 1$. Let z be a point in the interior of J. Let v be the unique positive measure on J such that $P(z) = \int P dv$ for every polynomial P. Then $dv(e^{ix}) = h(e^{ix}) dx$ when $c < x < d$. Here dx is the Haar measure on T and h is bounded on (c, d).*

Lemma 3. *With J and v as in Lemma 2 we have* $\lim C(r) = \lim \int_c^d |R(re^{it}) g(e^{it}) - W(re^{it})| dv(e^{it}) = 0$ *as r tends to* 1.

Proof. We have

$$C(r) \leqslant \int_c^d |dv(e^{it})| \int_{S_1} |g(re^{it}) - g(x)| (1 - r^2) |dm(x)| / |x - re^{it}| \, |1 - xre^{-it}|$$

$$+ \int_T |dm(x)| \int_c^d |g(re^{it}) - g(x)| (1 - r^2) h(e^{it}) / |x - re^{it}|^2 dt = A(r) + B(r).$$

As in Lemma 1 we see that $A(r)$ tends to zero as r tends to 1 and $B(r)$ tends to zero because $h(e^{it})$ is bounded on (c, d) and g is a continuos function.

Let M_B be the maximal ideal space of B. Let $z \in D - K$ and choose $x(z) \in \pi^{-1}(z)$ in M_B. Now we have:

Lemma 4. $R(z) g(x(z)) = W(z)$.

Proof. Choose a Jordan curve J as in Lemma 2 which contains z in its interior. This is possible since $D - K$ is connected. If $r < 1$ is sufficiently close to 1 the functions $W_r(z) = W(rz)$ and $R_r(z) = R(rz)$ are analytic in a neighborhood of the closed set $H(J)$ bounded by J. Hence Runge's theorem shows that we can approximate W_r and R_r uniformly by polynomials on $H(J)$. Let us put $\hat{J} = \pi^{-1}(H(J))$. If $x \in \hat{J}$ then $\pi(x) \in H(J)$. We define $\hat{W}_r(x) = W_r(\pi(x))$ and $\hat{R}_r(x) = R_r(\pi(x))$ on $\hat{J}$. If $\{P_n\}$ are polynomials such that $\lim |P_n - W_r|_{H(J)} = 0$ then we see that $\lim |P_n - \hat{W}_r|_{\hat{J}} = 0$. Hence we can approximate $\hat{W}_r$ and $\hat{R}_r$ uniformly on $\hat{J}$ by functions from B. Assume now that the lemma is false. Then we can find $d > 0$ such that $\lim |R(rz)g(x(z)) - W(rz)| \geq d$. Let $M \geq \sup \{|W_r|_{J-T} + |R_r|_{J-t} | r_0 \leq r < 1\}$. We can find M here because $\lim W(re^{ic}), \dots$ exist finitely. Choose now a polynomial Q such that $Q(z) = 1$ while $|Q|_{J-T} < d/2(|g|_K + 1)M$. Let us consider $\pi^{-1}(J)$. Obviously $\pi^{-1}(J)$ contains the topological boundary of $\hat{J}$ in M_B. Because $\hat{J} - \pi^{-1}(J)$ lies off the Shilov boundary the Local Maximum Principle shows that $|f(x(z))| \leq \sup \{|f(x)| \,|\, x \in \pi^{-1}(J)\}$ for all $f \in B$. Hence we can also find a positive measure λ on $\pi^{-1}(J)$ such that $f(x(z)) = \int f d\lambda$ for all $f \in B$. In particular $P(z) = \int P d\lambda$ for every polynomial P and since $\pi^{-1}(z)$ is trivial when $z \in T$ it follows that the restriction of λ to $\pi^{-1}(J) \cap T$ is identical to the measure v considered in Lemma 2. It follows from Lemma 3 that

$$\lim \int_{\pi^{-1}(J) \cap T} |Q| |\hat{R}_r g - \hat{W}_r| d\lambda = 0$$

as r tends to 1. We also have

$$\int_{\pi^{-1}(J) - T} |Q| |\hat{R}_r g - \hat{W}_r| d\lambda < d/2.$$

Now we obtain a contradiction since

$$|Q(\hat{R}_r g - \hat{W}_r)(x(z))| \geq d.$$

Lemma 4 shows that if $z \in D - K$ is such that $R(z) \neq 0$ then $g(x(z)) = W(z)/R(z)$ for all $x(z) \in \pi^{-1}(z)$. It follows that $\pi^{-1}(z)$ consists of one point denoted by $x(z)$. Since $g(x(z))$ is bounded when $z \in D - K$ it follows that the meromorphic function $G(z) = W(z)/R(z)$ is analytic in $D - K$. Now it is also easy to prove that even if $z \in D - K$ is such that $R(z) = 0$ then $\pi^{-1}(z)$ consists of one point $x(z)$ and $g(x(z)) = G(z)$. Theorem 1 is proved.

Before we prove Theorem 2 we need the following lemma.

Lemma 5. *Let F be a compact subset of* $(D - J) \cup \{0\}$ where J *is a Jordan arc in* D *having* 0 *and* 1 *as endpoints. If now v is a positive measure on F such that* $P(0) = \int P dv$ *for every polynomial, then v is the unit point mass at* 0.

Proof of Theorem 2. Suppose that G has an asymptotic value at some point $z_0 \in K$, then we shall prove that G has an unrestricted limit value at z_0. We may assume that $z_0 \in K - T$ since if $z_0 \in T$ we have already proved that $\lim G(z) = \lim g(x(z)) = g(z_0)$ as $z \in D - K$ tends to z_0. By assumption there exists a Jordan arc J such that $J \cap K = \{z_0\}$ and $\lim G(z)$ exists as $z \in J - \{z_0\}$ tends to z_0. Let us first assume that the asymptotic value is different from $g(z_0)$. Because $g(x(z)) = G(z)$ when $z \in D - K$ we see that $\lim x(z) = x_1$ exists in M_B as $z \in J - \{z_0\}$ tends to z_0 in D. Let z_0 be the trivial point in $\pi^{-1}(z_0)$. Now $x_1 \neq z_0$ because $g(x_1)$ is assumed to be different from $g(z_0)$ here. Suppose now that $x_2 \in \pi^{-1}(z_0)$ is such that $x_2 \neq x_1$ and z_0. Let us put $F = (Z - J) \cup \{z_0\}$ if Z is a closed disc around z_0 in D such that J intersects the boundary of Z. Now we choose Z so small that $|g(x_2) - g(x(z))| \geq d > 0$ when $z \in (Z \cap J) - \{z_0\}$. Now we choose a closed neighborhood W of x_2 in M_B such that W lies off the Shilov boundary and W is contained in $\pi^{-1}(F)$. Let bW be the topological boundary of W in M_B. It follows that $f(x_2) = \int f dv$ for all $f \in B$, where v is a positive measure on bW. In particular $P(z_0) = \int P dv$ for every polynomial. Because bW is contained in $\pi^{-1}(F)$ Lemma 5 shows that the support of v is contained in $\pi^{-1}(z_0)$. It follows that x_2 cannot be a peak point of the function algebra $B(z_0)$ on $\pi^{-1}(z_0)$ generated by the restriction of B to $\pi^{-1}(z_0)$. Hence the Shilov boundary of $B(z_0)$ only contains x_1 and z_0. It follows that $\pi^{-1}(z_0)$ only consists of x_1 and z_0. We now investigate the neighborhoods of x_1 in M_B. Let W be a closed B-convex neighborhood of x_1 such that W lies off the Shilov boundary and $z_0 \notin W$. Suppose now that there exist $y_n \in D-K$ such that y_n tend to z_0 in D while $W \cap \pi^{-1}(y_n)$ are empty. For every n we consider the function $f_n = (z-y_n)$ on W. We see that $f_n \in B_W$, where B_W is the function algebra on W generated by restricting B to W. Because W is B-convex we know that W is the maximal ideal space of B_W. Now f_n is different from zero on W and hence there exists $g_n \in B$ such that $\lim_n |g_n f_n - 1|_W = 0$. Let bW be the topological boundary of W in M_B. Let $S = \pi^{-1}(bW)$, obviously S is a closed subset of D and since $\pi^{-1}(z_0)$ only consists of x_1 and z_0 we see that $z_0 \notin S$. Because y_n tends to z_0 in D we may assume that $|f_n(x)| \geqslant \inf\{|z-y_n| \,|\, z \in S\} \geqslant d > 0$ when $x \in bW$. We may also assume that $|g_n f_n - 1|_W < 1$ for all n. It follows that $|g_n|_{bW} < 2/d$ and hence also $|g_n|_W < 2/d$ for every n. Now we get a contradiction since $\lim_n f_n(x_1) g_n(x_1) = 0$ follows while $\lim_n |f_n(x_1) g_n(x_1) - 1| = 0$ also holds. This shows that there exists a neighborhood V of z_0 in D such that $\pi^{-1}(V-K)$ is contained in W. Since $\pi^{-1}(z)$ contains only one point when $z \in D-K$ it follows that the trivial point z_0 of $\pi^{-1}(z_0)$ has a neighborhood U in M_B such that $\pi(U)$ is contained in K. If now $y_n \in D-K$ tend to z_0 in D it follows that $x(y_n) \in \pi^{-1}(y_n)$ must converge to x_1 in M_B. Hence $\lim G(y_n) = \lim g(x(y_n)) = g(x_1)$ which proves that G has an unrestricted limit value at z_0. We must finally consider the case when G has an asymptotic value at z_0 which equals $g(z_0)$. This case is simpler than the previous and we can prove that $\pi^{-1}(z_0)$ is trivial. It follows as above that G has an unrestricted limit value at z_0 in this case too. Now we complete the proof of Theorem 2. Let S_B be the Shilov boundary of B. Let us put $W_1 = \{x \in K-T \,|\, G$ has an analytic extension to a neighborhood of x and $G(x) \neq g(x)\}$. Clearly W_1 is a relatively open subset of K. If $z_0 \in W_1$ we can choose a neighborhood U of z_0 in D such that $U \cap K$ is contained in W_1. Now the previous results show that $\pi(U \cap K) = U \cap K$ and since $R(K) = C(K)$ it follows easily that $U \cap K$ lies in the interior of S_B. Then the local maximum principle implies that the restriction of B to the set $K_1 = K - W_1$ generates a proper function algebra B_1 of $C(K_1)$. From now on we work with B_1 instead of B. We can define G with respect to B_1 and clearly G is the same function as that defined with respect to B, i.e. we can represent G with a non zero measure on K_1 which annihilates B_1. If we now define $W_1 = W_1(B_1)$ with respect to B_1, i.e. we put $W_1(B_1) = \{x \in K_1 - T \,|\, G$ has an analytic extension to a neighborhood of x and $G(x) \neq g(x)\}$, then $W_1(B_1)$ is empty. So now we assume that B and K are such that W_1 is empty. Let us now put $W_2 = \{x \in K-T \,|\, G$ has an analytic extension to a neighborhood of $x\}$. Clearly $x \in W_2$ implies that $G(x) = g(x)$ (since W_1 is empty) and it follows easily that $W_2 \cap S_B$ is empty. It follows that the restriction of B to the set $K_2 = K - W_2$ generates a proper function algebra of $C(K_2)$. Since we can represent G with any non-zero measure on K annihilating B we see that K_2 is the smallest subset of K such that the restriction of B to K_2 generates a proper function algebra of $C(K_2)$.

We shall now discuss how Theorem 1 can be applied to the approximation problem on the unit interval.

Definition. A Jordan arc J in the complex plane satisfies the reflection principle if the following holds: If $z_0 \in J$ there exists an open disc Z around z_0 such that if G is any bounded analytic function in $Z-J$ with the property that G has an unrestricted limit value at a point $z \in J \cap Z$ when G has an asymptotic value at z, then it follows that G has an analytic continuation to Z.

Definition. A Jordan arc is almost smooth if J satisfies the reflection principle and if $R(J)=C(J)$, i.e. the rational functions with poles outside J generate $C(J)$.

We remark here that every smooth Jordan arc is almost smooth. We do not know if the condition $R(J)=C(J)$ implies that J is almost smooth. Let us now consider a function $f \in C(I)$ such that $J=\{f(x) \mid x \in I\}$ is of the form in Fig. 1. Let us assume that the two Jordan arcs a and b in (1) are almost smooth. Now we can prove that if $g \in C(I)$ is such that $g(\frac{1}{4}) \neq g(\frac{3}{4})$ then the function algebra generated by f, g and the constant functions is $C(I)$. We may assume that $f(\frac{1}{4})=f(\frac{3}{4})=0$ while $g(\frac{1}{4})=1$ and $g(\frac{3}{4})=-1$. Let us put $f_1=f$, $f_2=g^2$ and $f_3=fg$. On J we define $\hat{f}_j(z)=f_j(x)$ where $x \in I$ is such that $f(x)=z$. Obviously $\hat{f}_j$ are well defined on J. Suppose that the function algebra on J generated by $\hat{f}_1, \hat{f}_2, \hat{f}_3$ and the constant functions is different from $C(J)$. Now our previous results show that $\hat{f}_2$ and $\hat{f}_3$ have analytic extensions from ∂J into the interior of J. Here ∂J is the outer boundary of J (see Fig. 1). Call these extensions H_2 and H_3. On ∂J we have the relation $z^2H_2=H_3$ and it follows that H_3/z is a bounded analytic function in the interior of J. Now we can approach 0 along ∂J in two different ways. We get $\lim H_3/z=g(\frac{1}{4})$ from one way and $\lim H_3/z$ $g(\frac{3}{4})$ from the other way. Now Montel's theorem (see [7], p. 170) gives a contradiction. It follows that $\hat{f}_1, \hat{f}_2, \hat{f}_3$ and the constant functions generate $C(J)$ and then it is clear that f, g and the constant functions generate $C(I)$ too.

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