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# On the remainder term in the central limit theorem

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## 1. Summary and notations

Let $X_1, X_2, \ldots, X_n$ be independent random variables. Throughout this paper the following notations will be used. The random variable $X_k$ has the distribution function $F_k(x)$, the characteristic function $f_k(t)$ and $\operatorname{Var}(X_k) = \sigma_k^2$. We shall always assume that $E(X_k) = 0$, $\sigma_k^2 < \infty$. The normalized sum

$$\frac{1}{s_n} \sum_1^n X_k,$$

where $s_n^2 = \sum_1^n \sigma_k^2$ has the distribution function $\bar{F}_n(x)$ and the characteristic function $\bar{f}_n(t)$. By the same $C$ we shall denote generally different positive absolute constants and by the same $\theta$ generally different real or complex quantities such that $|\theta| \leqslant 1$. The standardized normal distribution function is denoted by $\Phi(x)$.

Suppose that $\beta_{3k} = E(|X_k|^3) < \infty$, $k = 1, 2, \ldots, n$. It has been proved by A. C. Berry and the present author, see e.g. Feller [1, p. 515], that

$$|\bar{F}_n(x) - \Phi(x)| \leqslant C \frac{\sum_1^n \beta_{3k}}{s_n^3}, \quad -\infty < x < \infty. \tag{1.1}$$

If the random variables are identically distributed with the same distribution function $F(x)$ then (1.1) becomes

$$|\bar{F}_n(x) - \Phi(x)| \leqslant C \frac{\beta_3}{\sigma^3 \sqrt{n}}, \quad -\infty < x < \infty, \tag{1.2}$$

where $\beta_3 = E(|X_k|^3)$, $\sigma^2 = E(X_k^2)$.

Recently Ibragimov [2] has obtained the following interesting result in the case of identically distributed random variables. In order that

$$\sup_x |\bar{F}_n(x) - \Phi(x)| = O(n^{-1/2}), \quad n \to \infty,$$

it is necessary and sufficient that

$$\int_{-z}^{z} x^3 dF(x) = O(1), \quad z \int_{|x| \geqslant z} x^2 dF(x) = O(1) \quad \text{as} \quad z \to \infty.$$

In this note we shall prove an inequality analogous to (1.1) but valid under weaker conditions, similar to the conditions (1.3) of Ibragimov in the case of identically distributed random variables. The following notations will be used in the sequel:

$$\lambda_k = \sup_{z>0} z \int_{|x|\geq z} x^2 dF_k(x), \tag{1.4}$$

$$\varrho_k = \sup_{z>0} \left( \left| \int_{-z}^{z} x^3 dF_k(x) \right| + z \int_{|x|\geq z} x^2 dF_k(x) \right). \tag{1.5}$$

**Theorem 1.** *If* $\varrho_k < \infty$ *for* $k = 1, 2, \ldots, n$, *then*

$$\sup_x |\overline{F}_n(x) - \Phi(x)| \leq C \frac{\sum_1^n \varrho_k}{s_n^3}. \tag{1.6}$$

The proof of this theorem is for the most part analogous to that of the inequality (1.1). It is based on the use of characteristic functions and the fundamental inequality (see for instance Feller [1, p. 512])

$$\sup_x |\overline{F}_n(x) - \Phi(x)| \leq \frac{1}{\pi} \int_{-T}^{T} |\bar{f}_n(t) - e^{-t^2/2}| \, |t|^{-1} dt + \frac{24}{\pi \sqrt{2\pi} T}, \tag{1.7}$$

where $T > 0$ is an arbitrary parameter.

Since absolute third order moments, however, are not assumed to be finite, we need some new results concerning the behaviour of $\bar{f}_n(t)$ in the vicinity of $t = 0$; these are stated and proved in the next section where the proof of Theorem 1 is also given. This proof is an immediate consequence of the inequality (1.7) once the behaviour of $\bar{f}_n(t)$ about $t = 0$ is known. We shall not aim at getting as small a numerical value of the constant $C$ in (1.6) as possible. If the absolute third order moments are finite then, as is easily seen, the inequality (1.1) is a corollary of Theorem 1.

Finally we shall use Theorem 1 to obtain an estimation of $\sup_x |\overline{F}_n(x) - \Phi(x)|$ only assuming the existence of the variances.

## 2. Proof of Theorem 1

We begin by proving three auxiliary results.

**Lemma 1.** *Let* $X$ *be a random variable with the distribution function* $F(x)$ *and and let* $X'$ *be a random variable independent of* $X$ *and with the same distribution. Denote by* $F^s(x)$ *the distribution function of* $X - X'$. *Then*

$$\int_{|x|\geq z} x^2 dF^s(x) \leq 40 \int_{|x|\geq \frac{z}{2}} x^2 dF(x).$$

If $F(x,y)$ is the distribution function of the random vector $(X, X')$, then

$$\int_{|x|\geqslant z} x^2 dF^s(x) = \iint_{x-y\geqslant z} (x-y)^2 dF(x,y) + \iint_{x-y\leqslant -z} (x-y)^2 dF(x,y) = J_1 + J_2. \tag{2.1}$$

Obviously

$$J_1 \leqslant \iint_{\substack{x-y\geqslant z\\ x\geqslant z}} (x-y)^2 dF(x,y) + \iint_{\substack{x-y\geqslant z\\ y\leqslant -z}} (x-y)^2 dF(x,y)$$

$$+ \iint_{\substack{x-y\geqslant z\\ x<z,\, y>-z}} (x-y)^2 dF(x,y) = J_1' + J_1'' + J_1'''. \tag{2.2}$$

From the inequality $(x-y)^2 \leqslant 2(x^2+y^2)$ we get

$$J_1' \leqslant 2\int_z^\infty \left(\int_{-\infty}^\infty (x^2+y^2)\,dF(y)\right) dF(x) = 2\left(\int_z^\infty x^2 dF(x) + \int_{-\infty}^\infty x^2 dF(x)\int_z^\infty dF(x)\right).$$

The integral $J_1''$ is estimated in a similar way. Hence we have

$$J_1' + J_1'' \leqslant 2\left(\int_{|x|\geqslant z} x^2 dF(x) + \int_{-\infty}^\infty x^2 dF(x)\int_{|x|\geqslant z} dF(x)\right).$$

It is easily seen that

$$\int_{-\infty}^\infty x^2 dF(x)\int_{|x|\geqslant z} dF(x) \leqslant \int_{|x|\geqslant z} x^2 dF(x).$$

Thus

$$J_1' + J_1'' \leqslant 4\int_{|x|\geqslant z} x^2 dF(x). \tag{2.3}$$

In the remaining integral $J_1'''$ we have

$$z \leqslant x - y < 2z.$$

Hence $\quad J_1''' \leqslant 4z^2 P(X - X' \geqslant z, X < z, X' > -z)$

$$\leqslant 4z^2\left(P\left(-z < X' \leqslant -\frac{z}{2}\right) + P\left(\frac{z}{2} \leqslant X < z\right)\right) \leqslant 16\left(\frac{z}{2}\right)^2 P\left(|X| \geqslant \frac{z}{2}\right).$$

But

$$\int_{|x|\geqslant z} x^2 dF(x) \geqslant z^2 P(|X| \geqslant z).$$

Thus

$$J_1''' \leqslant 16\int_{|x|\geqslant \frac{z}{2}} x^2 dF(x). \tag{2.4}$$

From (2.2), (2.3) and (2.4) we get

$$J_1 \leqslant 20 \int_{|x| \geqslant \frac{z}{2}} x^2 dF(x).$$

In a similar way

$$J_2 \leqslant 20 \int_{|x| \geqslant \frac{z}{2}} x^2 dF(x).$$

By (2.1) it follows that

$$J \leqslant 40 \int_{|x| \geqslant \frac{z}{2}} x^2 dF(x)$$

and the lemma is proved.

**Lemma 2.** *Let $F(x)$ be a distribution function,*

$$\alpha_2 = \int_{-\infty}^{\infty} x^2 dF(x), \quad \lambda = \sup_{z>0} z \int_{|x| \geqslant z} x^2 dF(x) < \infty.$$

*Then* $$\alpha_2/\lambda^{2/3} < 2.$$

Denote by $\varepsilon$ a parameter such that $0 < \varepsilon < 1$. Then

$$\alpha_2 = \int_{|x| < \varepsilon \sqrt{\alpha_2}} x^2 dF(x) + \int_{|x| \geqslant \varepsilon \sqrt{\alpha_2}} x^2 dF(x) \leqslant \varepsilon^2 \alpha_2 + \int_{|x| \geqslant \varepsilon \sqrt{\alpha_2}} x^2 dF(x).$$

Thus $$\lambda \geqslant \varepsilon \sqrt{\alpha_2} \int_{|x| \geqslant \varepsilon \sqrt{\alpha_2}} x^2 dF(x) \geqslant \varepsilon (1 - \varepsilon^2) \alpha_2^{3/2}.$$

But $\varepsilon(1-\varepsilon^2)$ takes its maximum $2/3\sqrt{3}$ for $\varepsilon = 1/\sqrt{3}$. Hence

$$\alpha_2/\lambda^{2/3} \leqslant \left(\frac{3\sqrt{3}}{2}\right)^{2/3} < 2$$

and the lemma is proved.

**Lemma 3.** *Let $X$ be a random variable with the distribution function $F(x)$ and the characteristic function $f(t)$. If $E(X) = 0$, $E(X^2) = \sigma^2$ and*

$$\lambda = \sup_{z>0} z \int_{|x| \geqslant z} x^2 dF(x) < \infty,$$

*then* $$|f(t)|^2 \leqslant 1 - \sigma^2 t^2 + 47\lambda |t|^3.$$

Using the distribution function $F^s(x)$ introduced in Lemma 1 we have

$$|f(t)|^2 = 1 - \sigma^2 t^2 + \int_{-\infty}^{\infty} (\cos\, tx - 1 + \tfrac{1}{2} t^2 x^2) dF^s(x). \tag{2.5}$$

Denoting the integral in (2.5) by $J$ we get

$$J=\int_{-1/|t|}^{1/|t|}(\cos tx-1+\tfrac{1}{2}t^2x^2)\,dF^s(x)+\int_{|x|\geqslant 1/|t|}(\cos tx-1+\tfrac{1}{2}t^2x^2)\,dF^s(x).$$

We apply the inequality

$$\cos y-1+\tfrac{1}{2}y^2\leqslant\frac{1}{24}y^4$$

to the first integral, the inequality $\cos y-1\leqslant 0$ to the second integral and obtain

$$J\leqslant\frac{1}{24}t^4\int_{-1/|t|}^{1/|t|}x^4\,dF^s(x)+\tfrac{1}{2}t^2\int_{|x|\geqslant 1/|t|}x^2\,dF^s(x). \tag{2.6}$$

From Lemma 1

$$\tfrac{1}{2}t^2\int_{|x|\geqslant 1/|t|}x^2\,dF^s(x)\leqslant 40|t|^3\frac{1}{2|t|}\int_{|x|\geqslant 1/|2t|}x^2\,dF(x)\leqslant 40\lambda|t|^3. \tag{2.7}$$

Putting

$$\left.\begin{aligned}R(z)&=\int_{|x|\geqslant z}x^2\,dF(x),\\ R^s(z)&=\int_{|x|\geqslant z}x^2\,dF^s(x),\end{aligned}\right\} \tag{2.8}$$

we have, still from Lemma 1,

$$\int_{-1/|t|}^{1/|t|}x^4\,dF^s(x)=\int_0^{1/|t|}x^2\,d(-R^s(x))$$

$$=-|t|^{-2}R^s(|t|^{-1})+2\int_0^{1/|t|}xR^s(x)\,dx\leqslant 160\int_0^{1/|t|}\frac{x}{2}R\left(\frac{x}{2}\right)dx\leqslant 160\,\lambda|t|^{-1}.$$

Thus

$$\frac{1}{24}t^4\int_{-1/|t|}^{1/|t|}x^4\,dF^s(x)\leqslant 7\,\lambda|t|^3.$$

From (2.5), (2,6), (2.7) and (2.9) we obtain the desired inequality.

We now proceed to the behavior of $\bar{f}_n(t)$ about $t=0$.

**Lemma 4.**

$$|\bar{f}_n(t)|\leqslant e^{-t^2/4}\quad \textit{for}\quad |t|\leqslant T_1=\frac{1}{94}\frac{s_n^3}{\sum_1^n\varrho_k}.$$

From Lemma 3

$$\left|f_k\left(\frac{t}{s_n}\right)\right|^2\leqslant 1-\frac{\sigma_k^2}{s_n^2}t^2+47\frac{\lambda_k}{s_n^3}|t|^3\leqslant\exp\left\{-\frac{\sigma_k^2}{s_n^2}t^2+47\frac{\lambda_k}{s_n^3}|t|^3\right\}.$$

Thus, since $\sum_1^n \lambda_k \leqslant \sum_1^n \varrho_k$

$$|f_n(t)| = \prod_1^n \left| f_k\left(\frac{t}{s_n}\right) \right| \leqslant \exp\left\{\frac{-t^2}{2}\left(1 - 47\frac{\sum_1^n \varrho_k}{s_n^3}|t|\right)\right\} \leqslant e^{-t^2/4} \quad \text{for} \quad |t| \leqslant T_1.$$

**Lemma 5.**

$$|f_n(t) - e^{-t^2/2}| \leqslant 4\frac{\sum_1^n \varrho_k}{s_n^3}|t|^3 e^{-t^2/2} \quad \textit{for} \quad |t| \leqslant T_0 = \tfrac{1}{3}\frac{s_n}{\left(\sum_1^n \varrho_k\right)^{1/3}}.$$

From
$$f_k(t) = 1 - \tfrac{1}{2}\sigma_k^2 t^2 + \int_{-\infty}^{\infty} (e^{itx} - 1 - itx + \tfrac{1}{2}t^2x^2)\, dF_k(x)$$

we obtain

$$f_k(t) = 1 - \tfrac{1}{2}\sigma_k^2 t^2 + \int_{-1/|t|}^{1/|t|}\left(-\frac{i}{6}t^3x^3 + \frac{\theta}{24}t^4x^4\right)dF_k(x) + \int_{|x|\geqslant 1/|t|}\left(-\frac{\theta}{2}t^2x^2 + \tfrac{1}{2}t^2x^2\right)dF_k(x).$$

This partitioning of the domain of integration has earlier been used by Ibragimov [2, p. 571]. As in the proof of Lemma 3 we get

$$0 \leqslant \int_{-1/|t|}^{1/|t|} x^4\, dF_k(x) \leqslant 2\int_0^{1/|t|} xR_k(x)\, dx,$$

where $R_k(x)$ is defined by (2.8) and thus

$$f_k(t) = 1 - \tfrac{1}{2}\sigma_k^2 t^2 + \theta|t|^3\left(\frac{1}{6}\left|\int_{-1/|t|}^{1/|t|} x^3\, dF_k(x)\right| + |t|^{-1}R_k(|t|^{-1}) + \frac{|t|}{12}\int_0^{1/|t|} xR_k(x)\, dx\right). \tag{2.10}$$

From (2.10) and the definition (1.5) of $\varrho_k$ it is easily seen that

$$f_k(t) = 1 - \tfrac{1}{2}\sigma_k^2 t^2 - 2\theta\varrho_k|t|^3$$

and thus
$$f_k\left(\frac{t}{s_n}\right) = 1 - u_k, \tag{2.11}$$

where
$$u_k = \tfrac{1}{2}\frac{\sigma_k^2 t^2}{s_n^2} + 2\theta\frac{\varrho_k}{s_n^3}|t|^3. \tag{2.12}$$

For $|t| \leqslant T_0$ we get from Lemma 2

$$|u_k| \leqslant \frac{1}{18}\frac{\sigma_k^2}{\lambda_k^{2/3}} + \frac{2}{27} < \frac{1}{9} + \frac{2}{27} < \frac{1}{5} \tag{2.13}$$

and
$$|u_k|^2 \leqslant 2\left(\frac{\sigma_k^4}{4s_n^4}t^4 + 4\frac{\varrho_k^2}{s_n^6}t^6\right) \leqslant \frac{|t|^3}{s_n^3}\left(\frac{1}{6}\frac{\sigma_k^4}{\lambda_k^{1/3}} + \frac{8}{27}\varrho_k\right) \leqslant \frac{\varrho_k}{s_n^3}|t|^3.$$

Thus
$$\sum_1^n |u_k|^2 \leqslant \frac{\sum_1^n \varrho_k}{s_n^3}|t|^3. \tag{2.14}$$

From (2.11), (2.12), (2.13), and (2.14) we obtain

$$\log f_k\left(\frac{t}{s_n}\right) = -u_k + \frac{\theta}{2}\frac{|u_k|^2}{1-|u_k|} = -u_k + \frac{5}{8}\theta|u_k|^2$$

and
$$\log \bar{f}_n(t) = \sum_1^n \log f_k\left(\frac{t}{s_n}\right) = -\frac{t^2}{2} + 2\theta\frac{\sum_1^n \varrho_k}{s_n^3}|t|^3 + \frac{5}{8}\theta\frac{\sum_1^n \varrho_k}{s_n^3}|t|^3$$

or
$$\log \bar{f}_n(t) = -\frac{t^2}{2} + \frac{21}{8}\theta\frac{\sum_1^n \varrho_k}{s_n^3}|t|^3 = -\frac{t^2}{2} + A, \tag{2.15}$$

where
$$|A| \leqslant \frac{7}{72} \quad \text{for} \quad |t| \leqslant T_0.$$

From (2.15) we get

$$|\bar{f}_n(t) - e^{-t^2/2}| \leqslant \frac{21}{8}e^{7/72}\frac{\sum_1^n \varrho_k}{s_n^3}|t|^3 e^{-t^2/2} \leqslant 4\frac{\sum_1^n \varrho_k}{s_n^3}|t|^3 e^{-t^2/2} \quad \text{for} \quad |t| \leqslant T_0$$

and the lemma is proved.

The proof of Theorem 1 is now an immediate consequence of the fundamental inequality (1.7), Lemma 4 and Lemma 5. Assume that $T_1 > T_0$ (the case $T_1 \leqslant T_0$ is treated similarly). In (1.7) we choose $T = T_1$ and obtain

$$\sup_x |\bar{F}_n(x) - \Phi(x)| \leqslant C\left(\int_{-T_0}^{T_0} \frac{\sum_1^n \varrho_k}{s_n^3}|t|^2 e^{-t^2/2}dt + \int_{T_0 \leqslant |t| \leqslant T_1} (e^{-t^2/4} + e^{-t^2/2})|t|^{-1}dt + \frac{1}{T_1}\right)$$
$$\leqslant C\frac{\sum_1^n \varrho_k}{s_n^3}.$$

## 3. A generalization of Theorem 1

Various generalizations of the inequality (1.1) have recently been obtained by Katz [3], Petrov [6], Studnev [7], Osipov [4], Osipov and Petrov [5] under weaker conditions than the finiteness of the absolute third order moments. These results are all based on the same method which is in short the following. The random variables are suitably truncated. The deviation of the distribution function of the sum of the original variables from the normal distribution function is estimated by the error term in (1.1) applied to the sum of the truncated variables and by the truncation error. It is not necessary to assume the finiteness of absolute moments of any order. Proceeding in the same way we may easily obtain similar results using this time the inequality (1.6) from Theorem 1 instead of the inequality (1.1).

For the sake of simplicity we shall assume that $E(X_k^2)=\sigma_k^2<\infty$, though this is not at all necessary, see e.g. Osipov–Petrov [5]. Under this condition the following inequality has been given by Studnev [7] and Osipov [4]

$$\sup_x |\overline{F}_n(x)-\Phi(x)| \leqslant C\left(\frac{1}{s_n^3}\sum_1^n \int_{|x|<s_n} |x|^3 dF_k(x)+\frac{1}{s_n^2}\sum_1^n \int_{|x|\geqslant s_n} x^2 dF_k(x)\right). \qquad (3.1)$$

The following analogous inequality contains third order but not absolute third order moments.

**Theorem 2.** *Let* $E(X_k)=0$, $E(X_k^2)=\sigma_k^2<\infty$. *Then*

$$\sup_x |\overline{F}_n(x)-\Phi(x)| \leqslant \frac{C}{s_n^3}\sum_1^n \left[\sup_{0<z\leqslant s_n}\left(\left|\int_{-z}^{z} x^3 dF_k(x)\right| + z\int_{|x|\geqslant z} x^2 dF_k(x)\right)\right]. \qquad (3.2)$$

*Remark.* The inequality (3.1) is clearly a corollary of (3.2).

Sketch of the proof. We define the truncated random variables $X_k^*$ by

$$X_k^*=\begin{cases} X_k & \text{if} \quad |X_k|<s_n, \\ 0 & \text{if} \quad |X_k|\geqslant s_n. \end{cases}$$

Let $F_n^*(x)$ be the distribution function of $\sum_1^n X_k^*$ and

$$\left.\begin{aligned} \alpha_k &= E(X_k^*) = -\int_{|x|\geqslant s_n} x\,dF_k(x), \\ s_n^{*2} &= \operatorname{Var}\left(\sum_1^n X_k^*\right) = s_n^2-\sum_1^n \int_{|x|\geqslant s_n} x^2 dF_k(x)-\sum_1^n \alpha_k^2. \end{aligned}\right\} \qquad (3.3)$$

We distinguish between the two cases $s_n^*\leqslant s_n/2$ and $s_n^*>s_n/2$.

(a) $s_n^*\leqslant s_n/2$. Then it it easily seen from (3.3) that

$$\frac{1}{s_n^2}\sum_1^n \int_{|x|\geqslant s_n} x^2 dF_k(x) \geqslant \frac{3}{8}.$$

Thus the inequality (3.2) is true with $C=8/3$.

(b) $s_n^* > s_n/2$. In the same way as in the paper of Osipov–Petrov [5] we get

$$\sup|\bar{F}_n(x)-\Phi(x)| \leqslant \sum_1^n \int_{|x|\geqslant s_n} dF_k(x) + \sup_x \left| F_n^*(s_n x) - \Phi\left(\frac{s_n x - \sum_1^n \alpha_k}{s_n^*}\right)\right|$$

$$+\frac{1}{\sqrt{2\pi}}\frac{\sum_1^n |\alpha_k|}{s_n^*} + \frac{1}{\sqrt{2\pi e}}\frac{s_n - s_n^*}{s_n^*}. \tag{3.4}$$

In (3.4) the first term to the right is the truncation error; the inequality (1.6) of Theorem 1 is applied to the second term of the right-hand member. The desired inequality (3.2) is then easily obtained by obvious estimations.

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