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Definitions of maximal differential operators

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Let $P(D)$ be a partial differential operator with constant coefficients and Ω an open set in R^ν (for notations and terminology cf. Hörmander [2], particularly pp. 176–177). In [2] the following two operators defined by $P(D)$ in $L^2(\Omega)$ were studied:

(i) The *minimal operator* P_0 defined as the closure of $P(D)$ in $C_0^\infty(\Omega)$.
(ii) The *maximal operator* P_w defined for those $u \in L^2(\Omega)$ such that $P(D)\,u = f$ in Ω in the distribution sense (or, which is the same thing, P_w is the adjoint of the minimal operator $\bar{P}_0$ defined by the formal adjoint $\bar{P}(D)$ of $P(D)$). P_w is also called the weak extension; it was denoted by P in [2].

There is an obvious lack of symmetry between these two definitions, one operator being defined by closure and one by duality. The reason for choosing the definition (ii) of the maximal operator is that it is important in some connections that the adjoint should be easy to study. On the other hand, it would sometimes be important to know that the maximal operator can be obtained by closing the operator $P(D)$ defined in a set of smooth functions. One might thus be interested in the following two "almost maximal" operators also:

(iii) The *strong extension* P_s which is the closure of $P(D)$ defined for those $u \in C^\infty(\Omega)$ such that $u \in L^2(\Omega)$, $P(D)u \in L^2(\Omega)$.
(iv) The *very strong extension* P_S which is the closure of $P(D)$ defined for those u which are restrictions to Ω of functions in $C_0^\infty(R^\nu)$.

It is obvious that we always have

$$P_S \subset P_s \subset P_w \tag{1}$$

and it is natural to expect that they should all be equal if sufficient regularity conditions are imposed on Ω. The following results are known previously:

(i) *For an arbitrary domain* Ω *we have* $P_s = P_w$ *if* $P(D)$ *is an operator of local type* ([2], Theorem 3.12).
(ii) *If* $P(D)$ *is homogeneous and not of local type (i.e. not elliptic modulo its lineality space) there always exists a domain* Ω *such that* $P_s \neq P_w$. (This has been proved by Schwarz [5] by extending an example given by the author.)

Concerning the very strong extension we shall prove here that $P_S = P_s = P_w$ if Ω is a bounded domain with a sufficiently smooth boundary (Theorem 2 below).[1] On the other hand, it is easy to show by examples that one can find to every differential operator $P(D)$ a domain Ω such that $P_S \neq P_w$. (Theorem 4.)

The smoothness condition we need is that of the following definition.

Definition. *A domain Ω is said to have the property T, if Ω is bounded and there exists a finite covering of $\bar{\Omega}$ by open sets O_i, so that for given i and $\varepsilon > 0$ it is possible to find a vector t with $|t| < \varepsilon$ and*

$$\bar{\Omega} \cap O_i + t \subset \Omega.^2 \tag{2}$$

It is obvious that the definition is satisfied by every domain which is situated on only one side of a continuously differentiable boundary. Also note that it follows from the definition that the boundary Γ of Ω is of measure 0 if Ω has the property T. For let Ω_ε be the subdomain of Ω consisting of the centres of the spheres with radius ε contained in Ω. For every $\varepsilon > 0$ we can find a vector t such that (2) is valid and $|t| < \varepsilon$, hence

$$\Gamma \cap 0_i + t \subset \Omega - \Omega_\varepsilon,$$

and $m(\Gamma \cap O_i) \leqslant m(\Omega - \Omega_\varepsilon)$ for every $\varepsilon > 0$. Since $\Omega_\varepsilon \to \Omega$ when $\varepsilon \to 0$, it follows that $m(\Gamma \cap O_i) = 0$ and hence that $m(\Gamma) = 0$.

Theorem 1. *Let Ω be a domain with the property T and $P(D)$ a differential operator with constant coefficients. Then, if $u \in L^2(R^\nu)$, $P(D)u \in L^2(R^\nu)$ in the distribution sense and the support of u is contained in $\bar{\Omega}$, it follows that the restriction of u to Ω is in the domain of the minimal operator P_0 in $L^2(\Omega)$.*

Proof. We take a finite covering O_i of $\bar{\Omega}$ with the properties mentioned in the definition and choose functions $\alpha^i \in C_0^\infty(O_i)$ such that $\sum \alpha^i = 1$ in a neighbourhood of $\bar{\Omega}$. Then we have

$$u = \Sigma u^i, \quad u^i = \alpha^i u. \tag{3}$$

Let Ω' be a bounded domain containing $\bar{\Omega}$. In virtue of Lemma 2.11 in [2], the restriction of u to Ω' is in $\mathcal{D}_{P_0}(\Omega')$, and hence it follows from Theorem 2.10 in [2] that the same is true for u^i. Thus $P(D)u^i \in L^2(R^\nu)$. For every fixed i we now choose a sequence of vectors t_n^i, $n = 1, 2, \ldots$ such that (2) is valid with $t = t_n^i$ for every n and $t_n^i \to 0$ when $n \to \infty$. With these vectors we form the sequences

$$v_n^i(x) = u^i(x - t_n^i). \tag{4}$$

v_n^i vanishes except when $x - t_n^i \in \bar{\Omega} \cap K$, where K is a compact set in O_i, i.e. except when $x \in \bar{\Omega} \cap K + t_n^i$. This is a compact set in Ω so that $v_n^i \in \mathcal{D}_{P_0}(\Omega)$ in virtue of Lemma 2.11 in Hörmander [2].

When $n \to \infty$ we have $v_n^i \to u^i$ and $P(D)v_n^i \to P(D)u^i$, with convergence in $L^2(R^\nu)$, hence also in $L^2(\Omega)$. Since P_0 is closed it follows that $u^i \in \mathcal{D}_{P_0}(\Omega)$, and since $u = \Sigma u^i$ this completes the proof.

We next prove a dual theorem, which was mentioned in the introduction.

[1] This result was given already in 1955, with the same proof, in a mimeographed manuscript with the title "Some results corncerning general partial differential operators".

[2] If A is a set in R^ν we denote by $A + t$ the set of all vectors $x + t$ with $x \in A$.

Theorem 2. *Suppose that Ω has the property T and that $P(D)$ is a differential operator with constant coefficients. Then the weak and the very strong extension of the operator $P(D)$ are equal.*

Proof. We only have to prove that $P_S \supset P_w$. Let u be in the domain of P_S^* and set $P_S^* u = f$. Denoting by $u^\cdot$ and $f^\cdot$ the functions which equal u and f in Ω and 0 elsewhere, we have by the definition of P_S

$$\int_{R^\nu} u^\cdot \overline{P(D)\varphi}\, dx = \int_{R^\nu} f^\cdot \bar{\varphi}\, dx, \quad \varphi \in C_0^\infty(R^\nu).$$

Hence $\bar{P}(D) u^\cdot = f^\cdot$ in the distribution sense, and Theorem 1 shows that $u \in \mathcal{D}_{\bar{P}_0}(\Omega)$. This proves that $P_S^* \subset \bar{P}_0$, hence that $P_S = P_S^{**} \supset \bar{P}_0^* = P_w$. The proof is complete.

Remark. For elliptic second order equations with variable coefficients Theorems 1 and 2 have essentially been announced by Birman [1]. However, the above method combined with the results of Hörmander [3] or Malgrange [4] prove that Theorem 2 holds for every operator with smooth coefficients which is formally hypoelliptic in the sense of [3].

We next give an application to some problems discussed in [2], pp. 200–201. First note that if P_0 and Q_0 are two minimal differential operators in a bounded domain Ω we have

$$(PQ)_0 \subset P_0 Q_0. \tag{5}$$

In fact, let $f \in \mathcal{D}_{(PQ)_0}$. Then there exists a sequence $f_n \in C_0^\infty(\Omega)$ such that $f_n \to f$ and $P(D)Q(D)f_n \to (PQ)_0 f$. Since P_0^{-1} is continuous (Hörmander [2], Theorem 2.1), the sequence $Q(D)f_n$ is convergent. Denoting the limit of $Q(D)f_n$ by g, we have

$$f \in \mathcal{D}_{Q_0}, \quad Q_0 f = g; \qquad g \in \mathcal{D}_{P_0}, \quad P_0 g = (PQ)_0 f.$$

This proves that $f \in \mathcal{D}_{P_0 Q_0}$ and that $P_0 Q_0 f = (PQ)_0 f$.

J. L. Lions has proved by examples, as indicated on p. 201 in [2], that in general we have $P_0^n \neq (P^n)_0$. However, we can prove

Theorem 3. *Let Ω be a domain which has the property T. Then, for arbitrary differential operators P and Q with constant coefficients, we have*

$$P_0 Q_0 = (PQ)_0. \tag{6}$$

In particular, $P_0^n = (P^n)_0$.

Proof. Let $u \in \mathcal{D}_{P_0 Q_0}$ and set $Q_0 u = g$ and $P_0 g = f$. Denoting by $u^\cdot$, $f^\cdot$ and $g^\cdot$ the functions which equal u, f and g in Ω vanish elsewhere, we have $Q(D)u^\cdot = g^\cdot$, $P(D)g^\cdot = f^\cdot$ in the distribution sense. Hence $P(D)Q(D)u^\cdot = f^\cdot$, and Theorem 1 shows that $u \in \mathcal{D}_{(PQ)_0}$.

Finally, we shall provide examples which show that some restriction on Ω is indeed necessary in Theorem 2.

Theorem 4. *Let $\nu > 1$ and $P(D)$ be any differential operator. Then there is a domain Ω such that*

$$P_S \neq P_w. \tag{7}$$

Proof. Let $\Omega^{\cdot}$ be a bounded domain containing the origin and choose the coordinate system such that the degree m of $P(\xi)$ equals its degree with respect to ξ_1,

$$P(\xi) = c\,\xi_1^m + \dots, \; c \neq 0,$$

where the dots indicate terms of order at most $m-1$ with respect to ξ_1. Let γ be a sphere in the plane $x^1 = 0$,

$$x^1 = 0, \; |x| \leqslant \varepsilon, \; \varepsilon > 0,$$

which is contained in $\Omega^{\cdot}$ and set $\Omega = \Omega^{\cdot} - \gamma$. Then we have (7). Indeed, the spaces $L^2(\Omega)$ and $L^2(\Omega^{\cdot})$ can be identified with each other since γ is a null set, and the definition of P_S shows that $P_S(\Omega) = P_S(\Omega^{\cdot})$ if this identification is made. Now let φ be a function in C_0^∞ vanishing for $|x| > \varepsilon/2$ and $= 1$ in a neighbourhood of 0, and set

$$u = \begin{cases} \varphi, & x^1 > 0 \\ 0, & x^1 \leqslant 0 \end{cases}.$$

Then $u \in C^\infty(\Omega)$ and clearly $u \in \mathcal{D}_{P_w}(\Omega)$. However, $u \notin \mathcal{D}_{P_w}(\Omega^{\cdot})$ because the distribution $P(D)u$ in $\Omega^{\cdot}$ contains a multiple layer of order m in γ. More explicitly, if $v \in C_0^\infty(\Omega^{\cdot})$ and vanishes of order $m-1$ in γ we get, if f is the *function* $P(D)u$ in Ω and 0 in γ,

$$\int_{\Omega^{\cdot}} u\,\overline{\bar{P}(D)\,v}\,dx - \int_{\Omega^{\cdot}} f\,\bar{v}\,dx = -c\,i \int_{x^1=0} u\,\overline{D_1^{m-1}\,v}\,dx^2 \dots dx^\nu$$

and thus cannot vanish for all such $v \in C_0^\infty(\Omega^{\cdot})$. Thus we have

$$u \notin \mathcal{D}_{P_S}(\Omega^{\cdot}) = \mathcal{D}_{P_S}(\Omega)$$

and it follows that
$$P_S(\Omega) \neq P_w(\Omega),$$

which should be proved.

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