Distributions of positive mass, which maximize a certain generalized energy integral

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Introduction. The classical theory of the equilibrium distribution for the Newtonian kernel deals with the problem of finding and studying distributions of positive mass with a given total mass, on a compact set F in n-dimensional Euclidean space $(n > 2)$, which minimize the energy integral $\iint\limits_{F\,F} |x - y|^{2-n}\, d\mu(x)\, d\mu(y)$ where $|x - y|$ is the Euclidean distance between the points x and y. M. Riesz and Frostman have generalized this to the case of the "α-potentials" with the energy integral $\iint\limits_{F\,F} |x - y|^{\alpha-n}\, d\mu(x)\, d\mu(y)$, $(0 < \alpha \leq 2)$. If we want to study a similar problem for the "energy integral" $\iint\limits_{F\,F} |x - y|^{\lambda}\, d\mu(x)\, d\mu(y)$ with $\lambda > 0$, we must evidently turn our interest to distributions that *maximize* this integral. Let us call such distributions *maximal distributions*. The purpose of this paper is to investigate for which values of $\lambda > 0$ there always exists a *unique* maximal distribution and to study the properties of maximal distributions.

The value of the "energy integral" with arbitrary λ has been studied by Pólya and Szegö [3]. They studied, in particular, the special case when F is a sphere or a solid sphere.

Frostman [1, p. 39] has proved that, in the case of the α-potentials, that distribution which minimizes the energy integral has certain other extremum properties. For maximal distributions the corresponding properties do not hold, as is shown in theorems 13–15.

Our main results are the following: For any $\lambda > 0$, the potential of a maximal distribution is constant in the support of the distribution and less than or equal to this constant value in F. For any $\lambda > 0$, the mass of a maximal distribution lies on the boundary of F. For $\lambda > 1$, it is in the extreme points of the convex hull of F. For $\lambda < 2$, there is a unique maximal distribution. *For $\lambda > 2$, any maximal distribution consists entirely of point-masses, not more than $(n + 1)$ in number.*

Notation and definitions. Let F be a compact set in the n-dimensional Euclidean space R_n with $n \geq 1$. Denote the points of R_n by x, y, etc. Let subscripts indicate different points and superscripts the coordinates of any one point, e.g. $y_i = (y_i^1, y_i^2, \ldots, y_i^n)$. Denote by $|x - y|$ the Euclidean distance, so that

$|x-y|^2 = \sum_{k=1}^{n} (x^k - y^k)^2$. Let $\mathfrak{M} = \mathfrak{M}_F$ be the class of all non-negative mass distributions on F with total mass equal to unity, that is, the class of all completely additive set-functions μ which take non-negative values on all Borel subsets of R_n and which satisfy $\mu(F) = 1$ and $\mu(R_n - F) = 0$. Let λ be a positive number. For any pair (μ_1, μ_2) of mass distributions on F, and for any mass distribution μ on F, define, as follows, the functionals $I(\mu_1, \mu_2)$ and $I(\mu)$, respectively:

$$I(\mu_1, \mu_2) = \iint\limits_{F\ F} |x - y|^\lambda \, d\mu_1(x) \, d\mu_2(y), \quad I(\mu) = I(\mu, \mu).$$

The *potential* $p_\mu(x)$ of a distribution μ is defined as

$$p_\mu(x) = \int\limits_F |x - y|^\lambda \, d\mu(y).$$

Evidently,

$$I(\mu_1, \mu_2) = \int\limits_F p_{\mu_1}(x) \, d\mu_2(x) = \int\limits_F p_{\mu_2}(x) \, d\mu_1(x).$$

The non-negative number $M = M_F$ is defined as

$$M = \sup_{\mu \in \mathfrak{M}} I(\mu).$$

A *maximal distribution* is defined as a distribution μ, such that

$$\mu \in \mathfrak{M} \quad \text{and} \quad I(\mu) = M.$$

The distribution consisting of the unit mass in the point x is denoted by δ_x.

Theorem 1. *For every F and every $\lambda > 0$ there exists a maximal distribution.*

Proof: Let $\{\mu_n\}$ be a sequence such that $\mu_n \in \mathfrak{M}$ and $I(\mu_n) \to M$. Select from this sequence a sub-sequence $\{\mu_{n_i}\}$, converging weakly to a distribution μ^* (i.e. $\int\limits_F f(y) \, d\mu_{n_i}(y) \to \int\limits_F f(y) \, d\mu^*(y)$ for every continuous f). Since $|x - y|^\lambda$ is continuous in x and in y, we get $I(\mu_{n_i}) \to I(\mu^*)$ (see e.g. FROSTMAN [1, p. 17]). Hence $I(\mu^*) = M$. But μ^*, being a weak limit of elements of $\mathfrak{M}$, must belong to $\mathfrak{M}$. Hence μ^* is maximal, which completes the proof.

Theorem 2. *If μ is a maximal distribution with support F_1, then $p_\mu(y) = M$ for $y \in F_1$, and $p_\mu(y) \leq M$ for $y \in F$.*

Proof: Let ν be a mass distribution on F, satisfying:

$$\left.\begin{array}{c} \nu(F) = 0 \\[4pt] \mu + \varepsilon \cdot \nu \geq 0 \quad \text{for every } \varepsilon \text{ in the interval } 0 \leq \varepsilon \leq 1 \\[4pt] (\mu \text{ is the given maximal distribution}) \end{array}\right\} \cdot \tag{1}$$

It follows that $(\mu + \varepsilon \cdot \nu) \in \mathcal{M}$, and hence

$$M \geq I(\mu + \varepsilon \cdot \nu) = I(\mu) + \varepsilon^2 I(\nu) + 2\,\varepsilon\,I(\mu, \nu) = M + \varepsilon^2 I(\nu) + 2\,\varepsilon\,I(\mu, \nu).$$

Thus, for every ε with $0 \leq \varepsilon \leq 1$,

$$0 \geq \varepsilon \cdot (2\,I(\mu, \nu) + \varepsilon \cdot I(\nu)),$$

which implies

$$I(\mu, \nu) \leq 0. \tag{2}$$

Suppose now that the theorem does not hold, even if M is replaced by some other constant C. There is then a point $y_1^* \in F_1$ and a point $y_2 \in F$, such that

$$a = p_\mu(y_2) > p_\mu(y_1) = b.$$

Let K be a solid sphere[1] with centre y_1 and so small that $y_2 \notin K$ and the oscillation of p_μ in K is less than or equal to $\dfrac{a-b}{2}$. Since $y_1 \in F_1$, $m = \mu(K) > 0$. Define, as follows, the distribution ν which "moves the mass from K to y_2":

$$\nu(e) = m \cdot \delta_{y_2}(e) - \mu(e \cap K).$$

Evidently ν satisfies (1), and hence (2) is true. But

$$I(\mu, \nu) = \int_F p_\mu(x)\,d\nu(x) = p_\mu(y_2) \cdot m - \int_K p_\mu(x)\,d\mu(x)$$

$$\geq a \cdot m - \left(b + \frac{a-b}{2}\right) \cdot m = \frac{a-b}{2} \cdot m > 0,$$

which is contrary to (2). Hence $p_\mu(y)$ is constant $= C$ for $y \in F_1$, and $p_\mu(y) \leq C$ for $y \in F$. Consequently,

$$M = I(\mu) = \int_F p_\mu(x)\,d\mu(x) = \int_{F_1} p_\mu(x)\,d\mu(x) = C \int_{F_1} d\mu(x) = C,$$

which completes the proof.

Corollary. *If* $\lambda = 2$, F_1 *is a subset*[2] *of a sphere.*

We have

$$p_\mu(y) = \int_{F_1} \sum_{i=1}^{n} (x^i - y^i)^2\,d\mu(x) = \sum_{i=1}^{n} (y^i)^2 + \int_{F_1} \sum_{i=1}^{n} (x^i)^2\,d\mu(x) - 2 \sum_{i=1}^{n} y^i \int_{F_1} x^i\,d\mu(x) \tag{3}$$

and if $y \in F_1$, we get $p_\mu(y) = M$, which is the equation of a sphere.

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[1] We call the set of points x which satisfy $|x - y_1| = R$ a *sphere*, and the set of points which satisfy $|x - y_1| \leq R$ a *solid sphere*.

[2] The inclusion sign and the concept of subset are used in the broad sense (equality not excluded).

Theorem 3. *For any $\lambda > 0$ and $n > 1$, the support of any maximal distribution is a subset of the boundary of F.*

Proof: As remarked by PÓLYA and SZEGÖ [3, p. 20], simple calculation shows that for fixed y, $|x - y|^{\lambda}$ is a subharmonic function of x, and we see that it is even strictly subharmonic. Hence $p_{\mu}(x) = \int_{F} |x - y|^{\lambda} d\mu(y)$ is strictly subharmonic. But then if x is an inner point of F, $p_{\mu}(x)$ is strictly less than the mean value of p_{μ} on a sphere in F with centre x. According to theorem 2, this mean value is less than or equal to M. Hence, again according to theorem 2, x cannot belong to the support of μ. This completes the proof.

Theorem 4. *If $0 < \lambda < 2$, the maximal distribution is unique.*

Proof: Suppose that μ and μ_1 are two maximal distributions. Let $\nu = \mu_1 - \mu$. Then ν satisfies the conditions (1). Hence, as in the proof of theorem 2, ν satisfies (2). But

$$M = I(\mu + \nu) = I(\mu) + I(\nu) + 2 \cdot I(\mu, \nu) = M + I(\nu) + 2 \cdot I(\mu, \nu).$$

Hence
$$I(\nu) = -2 I(\mu, \nu) \geq 0.$$

To complete the proof, it is sufficient to prove the following lemma 1. The above ν, in fact, satisfies the assumptions of the lemma, and hence is identical to zero, that is $\mu_1 \equiv \mu$.

Lemma 1. *If $0 < \lambda < 2$ and if ν is a distribution on F with total algebraic mass equal to zero and a non-negative energy integral, that is, if*

$$\left. \begin{array}{c} \int_{F} d\nu = 0 \\[2mm] I(\nu) \geq 0 \end{array} \right\},$$

then ν vanishes identically.

Proof: If $n = 1$, replace n by 2, that is, imbed R_1 in R_2. According to FROSTMAN [2, p. 6], the formula (2) of [2] is valid for $0 < \alpha + \beta < m + 2$, $\alpha + \beta \neq m$. Apply this formula with $m = n$ (modified as indicated), $\alpha = \beta = \dfrac{n + \lambda}{2}$, and $\mu = \nu$. We get

$$\int_{R_n} (U^{\alpha})^2 \, dz = \frac{1}{H_n(n + \lambda)} \cdot I(\nu).$$

Here dz is the element of volume,

$$U^{\alpha} = \text{const.} \times \int_{F} |x - y|^{\frac{\lambda - n}{2}} \, d\nu(y),$$

and

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$$H_n(n+\lambda) = \frac{\pi^{\frac{n}{2}} \cdot 2^{n+\lambda} \cdot \Gamma\left(\dfrac{n+\lambda}{2}\right)}{\Gamma\left(-\dfrac{\lambda}{2}\right)}.$$

But $\Gamma\left(-\dfrac{\lambda}{2}\right) < 0$, and hence $H_n(n+\lambda) < 0$. Since $I(v) \geq 0$, this implies that $U^\alpha = 0$ almost everywhere in R_n. Hence $v \equiv 0$ (see M. Riesz [4, no. 10]). This completes the proof of the lemma and of theorem 4.

Theorem 5. *If* $\lambda = 2$, *and if* F *is a solid sphere with radius* r *and boundary* S, *then* $M = 2r^2$, *and a distribution is maximal if and only if its support is a subset of* S *and its centre of gravity is the centre of* F.

Proof: Let μ be a maximal distribution with support F_1, and let $y \in F_1$. Multiplying (3) by $d\mu(y)$ and integrating over F_1 we obtain

$$I(\mu) = 2 \int_{F_1} \left(\sum_{i=1}^{n} (x^i)^2\right) d\mu(x) - 2 \sum_{i=1}^{n} \left(\int_{F_1} x^i d\mu(x)\right)^2.$$

We suppose now that the coordinate system is chosen with its origin in the centre of F. Then the first term of $I(\mu)$ is maximal $= 2r^2$, if and only if $F_1 \subset S$. The modulus of the second term is minimal $= 0$, if and only if

$$\int_{F_1} x^i d\mu(x) = 0, \quad (i = 1, 2, \ldots, n).$$

Hence $I(\mu)$ is maximal $= 2r^2$ if and only if $F_1 \subset S$ and the origin is the centre of gravity of μ, which completes the proof.

Theorem 6. *If* $\lambda = 2$ *and* F *is contained in a solid sphere* K *with radius* r *and boundary* S, *and if* r *is the minimum radius possible under these conditions, then* $M = 2r^2$, *the support of any maximal distribution is a subset of* $S \cap F$, *and there exists a maximal distribution the support of which consists of not more than* $(n+1)$ *points.*

Proof: Since $F \subset K$, $\mathfrak{M}_F \subset \mathfrak{M}_K$. Hence $M = M_F \leq M_K = 2r^2$. If we can find a distribution $\mu \in \mathfrak{M}_F$ with $I(\mu) = 2r^2$, such that its support consists of not more than $(n+1)$ points, then the theorem is proved. Indeed the existence of such a distribution implies that $M = 2r^2$. Hence every maximal distribution for F is also a maximal distribution for K. According to theorem 5, its support is then a subset of S.

Suppose that the centre c of K is an outer point of the convex hull of $F \cap S$. Then some half-sphere H of S does not meet F. Hence, since F and H are compact sets, they must have a positive distance. It is then possible, by moving c towards F and then shrinking the translated K, to construct a solid sphere containing F and with radius less than r. This is contrary to the assumptions of the theorem. Hence c belongs to the convex hull of $F \cap S$ and consequently is an inner point of a k-dimensional simplex with vertices

$y_0, y_1, \ldots, y_k$ in $F \cap S$, $(1 \leq k \leq n)$. The barycentric coordinates of c in this simplex are consequently positive and are denoted by $m_0, m_1, \ldots, m_k$. We shall now prove that the required distribution is given by

$$\mu = \sum_{i=0}^{k} m_i \, \delta_{y_i}.$$

From the properties of barycentric coordinates we know that $\sum_{i=0}^{k} m_i = 1$, and that c is the centre of gravity of μ. The first of these facts, together with the inequalities $m_i > 0$, shows that $\mu \in \mathfrak{M}_K$. The other, together with the fact that the support of μ is a subset of S, then ensures, by theorem 5, that $I(\mu) = 2 \, r^2$, which completes the proof.

Theorem 7. *If $\lambda > 2$ and if F is a solid sphere with radius r, then $M = 2^{\lambda-1} \, r^\lambda$, and a distribution is maximal if and only if it consists of two point-masses, each with the mass $\frac{1}{2}$, placed in the two end points of a diameter.*

Proof: For any $\mu \in \mathfrak{M}$,

$$I(\mu) = \iint_{F \, F} |x - y|^\lambda \, d\mu(x) \, d\mu(y) = \iint_{F \, F} |x - y|^{\lambda-2} \, |x - y|^2 \, d\mu(x) \, d\mu(y)$$

$$\leq (2 \, r)^{\lambda-2} \iint_{F \, F} |x - y|^2 \, d\mu(x) \, d\mu(y) \leq (2 \, r)^{\lambda-2} \sup_{\mu \in \mathfrak{M}} \iint_{F \, F} |x - y|^2 \, d\mu(x) \, d\mu(y) \qquad (4)$$

$$= (2 \, r)^{\lambda-2} \cdot 2 \, r^2 = 2^{\lambda-1} \, r^\lambda.$$

Hence $$M \leq 2^{\lambda-1} \, r^\lambda.$$

But a distribution μ^* consisting of the mass $\frac{1}{2}$ in each of the two end points of a diameter, gives:

$$I(\mu^*) = 2 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot (2 \, r)^\lambda = 2^{\lambda-1} \, r^\lambda,$$

and consequently,

$$M = 2^{\lambda-1} \, r^\lambda.$$

Either the first inequality sign of (4) gives a strict inequality, or $|x - y| = 2 \, r$ for all pairs of elements of mass that contribute to the integral $\iint_{F \, F} |x - y|^2 \, d\mu(x) \, d\mu(y)$, that is, for all pairs such that $|x - y| \neq 0$. Hence the only type of distribution that gives equality instead of the first inequality of (4) is the μ^*-type. This completes the proof.

Corollary. *Suppose that $\lambda > 2$ and that F has a diameter (that is a segment of maximum length with its end points in F) of length $2 \, r$ such that F is contained in the solid sphere with this diameter. Then $M = 2^{\lambda-1} \, r^\lambda$, and the distribution consisting of the mass $\frac{1}{2}$ in each of the end points of the diameter, is maximal. In particular, this is true for any F that is centrally symmetric. If, in addition, we assume that F has no other diameter, we obtain the result that the maximal distribution is unique.*

Remark 1 (cf. Pólya and Szegö [3, p. 42]). We now know all the maximal distributions for any $\lambda > 0$, when F is a solid sphere (or a sphere). In fact, it follows from theorems 3 and 4, by reasons of symmetry, that, for $0 < \lambda < 2$, the unique maximal distribution is the homogeneous distribution on the boundary. Further, for $\lambda > 2$ the mass of a maximal distribution is concentrated in two diametrically opposite points, and for $\lambda = 2$ both these types of distributions and, in addition, a great variety of intermediate forms are maximal, as specified in theorem 5. This tendency of the mass to concentrate in points when λ increases, together with the last result of theorem 6, suggests a hypothesis, stated and proved in theorem 12.

Remark 2. From theorems 5 and 7 it is clear that, for $\lambda \geq 2$, the maximal distribution is not necessarily unique. In fact, the solid sphere is an example of a set F with an infinity of maximal distributions. However, for $\lambda > 2$ all these are congruent. An example of a set F with two non-congruent maximal distributions is given in remark 2 to theorem 10.

Theorem 8. *If $\lambda > 1$ and if P is an inner point of a k-dimensional simplex $(1 \leq k \leq n)$, then a positive point-mass in P can be replaced by positive point-masses in the vertices of the simplex, with the same total mass, in such a way that the potential strictly increases in every point of R_n.*

Proof: Let the vertices be $x_0, x_1, \ldots, x_k$, and let the barycentric coordinates of P in the simplex be $m_0, m_1, \ldots, m_k$. Since P is an inner point they are all positive. We lose no generality if we assume that the original mass in P is unity, that is, that the original distribution is δ_P. We want to prove that the distribution $\mu = \sum_{i=0}^{k} m_i \delta_{x_i}$ solves the problem, that is, that in every point y of R_n,

$$p_\mu(y) > p_{\delta_P}(y)$$

or

$$\sum_{i=0}^{k} m_i |y - x_i|^\lambda > |y - \sum_{i=0}^{k} m_i x_i|^\lambda.$$

But since $m_i > 0$ and $\sum_{i=0}^{k} m_i = 1$, this follows from the fact that for fixed y, $|x - y|^\lambda$ is a strictly convex function of x. This in turn is a consequence of the facts that $|x - y|$ is a non-negative convex function of x, not constant on any segment, and that t^λ is a monotonously increasing, strictly convex function of the non-negative real variable t $(\lambda > 1)$. This completes the proof.

Theorem 9. *If $\lambda > 1$, the support of any maximal distribution consists only of extreme points of the convex hull of F.* (An extreme point of a convex set is a point which is not an inner point of any segment with end points in the set.)

Proof: Suppose that the theorem does not hold. Then there exists a maximal distribution μ such that the interior S of some k-dimensional simplex $(1 \leq k \leq n)$, with its vertices $x_0, x_1, \ldots, x_k$ in F, carries mass. We then "integrate

the procedure of theorem 8 over S'', that is, we replace the part of μ which lies in S by the distribution $\sum\limits_{i=0}^{k} m_i \, \delta_{x_i}$, where

$$m_i = \int\limits_{S} \xi_i \, (y) \, d\mu \, (y)$$

with $\xi_i(y) =$ the barycentric coordinates of y in S $(i = 0, 1, ..., k)$. If the new distribution obtained from μ in this way is denoted by μ_1, theorem 8 assures that in any point of R_n,

$$p_{\mu_1} > p_\mu .$$

Hence,

$$I \, (\mu_1) = \int p_{\mu_1} (x) \, d\mu_1 \, (x) > \int p_\mu (x) \, d\mu_1 \, (x) = I \, (\mu, \mu_1)$$

$$= \int p_{\mu_1} (x) \, d\mu \, (x) > \int p_\mu (x) \, d\mu \, (x) = I \, (\mu) = M,$$

which is a contradiction, since $\mu_1 \in \mathfrak{M}$. This completes the proof.

Theorem 10. *Suppose that $\lambda > 2$ and that S is a finite set of more than $(n + 1)$ points, all carrying point-masses of the distribution $\mu \in \mathfrak{M}$. Then it is possible, just by moving mass between the points of S, to construct a distribution $\mu' \in \mathfrak{M}$ such that not more than $(n + 1)$ points of S carry masses, and such that $I \, (\mu') > I \, (\mu)$.*

For the proof of this theorem we will need the following two lemmas:

Lemma 2. *Let $y_0, y_1, ..., y_m$ be distinct points of R_n and let $0 < \alpha < 2$. Then the quadratic form*

$$f = \sum_{i, k = 1}^{m} \left(|y_0 - y_i|^\alpha + |y_0 - y_k|^\alpha - |y_i - y_k|^\alpha \right) t_i \, t_k$$

is positive definite.

This was proved by SCHOENBERG [6, theorem 2]. Another proof is obtained from lemma 1 with $\lambda = \alpha$ and $\nu = \sum\limits_{i=0}^{m} t_i \, \delta_{y_i}$, where $\sum\limits_{i=0}^{m} t_i = 0$. Using the latter relation to eliminate t_0, we get:

$$I \, (\nu) = \sum_{i=0}^{m} \sum_{k=0}^{m} t_i \, t_k \, |y_i - y_k|^\alpha = \sum_{i=1}^{m} t_i \left(\sum_{k=0}^{m} t_k \, |y_i - y_k|^\alpha \right) - \sum_{i=1}^{m} t_i \left(\sum_{k=0}^{m} t_k \, |y_0 - y_k|^\alpha \right)$$

$$= \sum_{i=1}^{m} t_i \left(\sum_{k=1}^{m} t_k \, |y_i - y_k|^\alpha - \sum_{k=1}^{m} t_k \, |y_i - y_0|^\alpha - \sum_{k=0}^{m} t_k \, |y_0 - y_k|^\alpha \right) = -f.$$

Hence $f \leq 0$ implies $I \, (\nu) \geq 0$. By lemma 1, this gives $t_i = 0$ $(i = 0, 1, ..., m)$, which completes the proof of lemma 2.

Lemma 3. *Let $a_{ik} = a_{ki}$ $(i \neq k; \ i, k = 0, 1, ..., m)$ be $\frac{1}{2} \, m \, (m + 1)$ given positive numbers. Then a necessary and sufficient condition that there exist in R_r, but not in R_{r-1}, $(m + 1)$ points $x_0, x_1, ..., x_m$, such that $|x_i - x_k| = a_{ik}$, $(i \neq k)$, is that the quadratic form*

$$\sum_{i,k=1}^{m} (a_{0i}^2 + a_{0k}^2 - a_{ik}^2)\, t_i\, t_k \qquad (\text{with } a_{ii}=0)$$

is non-negative and of rank r.

This result was obtained by MENGER. A simple proof is given by SCHOENBERG [5, theorem 1].

Proof of theorem 10: We prove the following statement, which is clearly equivalent to theorem 10: *Suppose that the $(m+1)$ points $y_0, y_1, \ldots, y_m$ carry point-masses of a distribution $\mu_1 \in \mathfrak{M}$. Suppose further that it is not possible, just by moving mass between the y_i, to construct a distribution $\mu_2 \in \mathfrak{M}$ such that at least one of the y_i is free from mass and such that $I(\mu_2) > I(\mu_1)$. Then $m \leq n$.*

Let $t_0, t_1, \ldots, t_m$ be a set of real numbers with sum zero: $\sum_{i=0}^{m} t_i = 0$, and denote by p_i the potential of μ_1 at the point y_i $(i=0, 1, \ldots, m)$: $p_i = p_{\mu_1}(y_i)$. Define the distribution μ_2 as follows:

$$\mu_2 = \mu_1 + \sum_{i=0}^{m} t_i\, \delta_{y_i}.$$

We get

$$I(\mu_2) = I(\mu_1) + 2\sum_{i=0}^{m} t_i\, I(\mu_1, \delta_{y_i}) + \sum_{i=0}^{m}\sum_{k=0}^{m} t_i\, t_k\, I(\delta_{y_i}, \delta_{y_k})$$

$$= I(\mu_1) + 2\sum_{i=0}^{m} t_i\, p_i + \sum_{i=0}^{m}\sum_{k=0}^{m} t_i\, t_k\, |y_i - y_k|^{\lambda}.$$

Hence, using $t_0 = -\sum_{i=1}^{m} t_i$, we get (as in the proof of lemma 2)

$$I(\mu_1) - I(\mu_2) = L + Q,$$

where the linear form L and the quadratic form Q are as follows:

$$L = 2\sum_{i=1}^{m} t_i\, (p_0 - p_i),$$

$$Q = \sum_{i,k=1}^{m} (|y_0 - y_i|^{\lambda} + |y_0 - y_k|^{\lambda} - |y_i - y_k|^{\lambda})\, t_i\, t_k.$$

We shall now prove that Q is non-negative. Suppose that this is not true. Then there exists a set of t-values, $t_i = T_i$ $(i=1, 2, \ldots, m)$, giving Q a negative value. If k is a real number which is not zero, then Q is negative for the set $k \cdot T_i$. By restricting k either to positive or to negative values we can assure that L is non-positive for the considered set. Hence $I(\mu_2) > I(\mu_1)$ for all positive or for all negative values of k. If $|k|$ is sufficiently small, the corresponding $\mu_2 \in \mathfrak{M}$ (μ_2 has no negative masses). If, according to the restriction, k increases or decreases from zero, it will reach a first value for which the mass of μ_2 in one of the points y_i is zero, and still $\mu_2 \in \mathfrak{M}$ and $I(\mu_2) > I(\mu_1)$. This is contrary to our assumptions. Hence we have proved that Q is non-negative.

We now apply lemma 3 to $a_{ik} = |y_i - y_k|^{\frac{\lambda}{2}}$.

As a result, in some R_r there exist $(m+1)$ points $x_0, x_1, \ldots, x_m$, such that

$$|x_i - x_k| = |y_i - y_k|^{\frac{\lambda}{2}}.$$

We then apply lemma 2 to the points $x_0, x_1, \ldots, x_m$ of R_r with $\alpha = \dfrac{4}{\lambda}$.

As a result, the form

$$\sum_{i,k=1}^{m} (|x_0 - x_i|^{\frac{4}{\lambda}} + |x_0 - x_k|^{\frac{4}{\lambda}} - |x_i - x_k|^{\frac{4}{\lambda}}) \, t_i \, t_k =$$

$$= \sum_{i,k=1}^{m} (|y_0 - y_i|^2 + |y_0 - y_k|^2 - |y_i - y_k|^2) \, t_i \, t_k$$

is positive *definite*. Hence its rank is m.

We finally apply lemma 3 again, but this time to $a_{ik} = |y_i - y_k|$.

As a result, there do not exist in R_{m-1}, $(m+1)$ points $z_0, z_1, \ldots, z_m$, such that $|z_i - z_k| = a_{ik}$ $(i \neq k)$. However, in R_n such points do exist, for instance $z_i = y_i$. Hence $n > m-1$, which completes the proof of theorem 10.

Applying theorem 10 to a maximal distribution, we get the following

Corollary. *For $\lambda > 2$, no maximal distribution contains more than $(n+1)$ point-masses.*

Remark 1. Theorem 10 is the best possible in the sense that no smaller number can be substituted for $(n+1)$. This is shown by the following example. Let F be the $(n+1)$ vertices of a regular n-simplex and let μ be a maximal distribution on F. By theorem 2, the potential takes equal values at two points carrying mass. But, due to the symmetry, the contributions from the mass in any third point to the potential at these two points are equal. Hence the masses in the two considered points must be equal. If some vertex should carry no mass, then it would be possible to increase $I(\mu)$, for example by moving a half of one of the point-masses to the empty vertex. But this contradicts the fact that μ is maximal. Hence this special F has, for any $\lambda > 0$, a unique maximal distribution, viz. the mass $1/(n+1)$ in each vertex.

Remark 2. With the aid of theorem 10, we can easily show that theorem 4 fails completely for $\lambda \geq 2$, if $n > 1$. For $\lambda = 2$, this follows from theorem 5, and for $\lambda > 2$ we shall now give an example of a set F with two maximal distributions, one consisting of two point-masses and the other of $(n+1)$ point-masses.

Let $x_0, x_1, \ldots, x_n$ be the vertices of a regular n-simplex with edges of unit length and let c be its centre of gravity. Let y be a point on the straight line $x_0 c$, such that $|x_0 - y| = \left(\dfrac{2n}{n+1}\right)^{\frac{1}{\lambda}}$, and such that c lies between x_0 and y. Let F consist of the $(n+2)$ points $y, x_0, x_1, \ldots, x_n$. According to theorem 10, any maximal distribution μ on F will leave at least one of the points free from mass.

Suppose first that y is free from mass. Then μ is a maximal distribution on the vertices of a regular n-simplex, and hence, by remark 1, each x_i $(i = 0, 1, \ldots, n)$ carries the mass $1/(n+1)$, and

$$I(\mu) = \frac{n(n+1)}{(n+1)^2} = \frac{n}{n+1}.$$

Suppose next that some x_i $(i \neq 0)$ is free from mass. By the same argument of symmetry as in remark 1, every x_i $(i \neq 0)$ is then free from mass. The support of μ thus consists of y and x_0, and hence μ must have the mass $\frac{1}{2}$ in each of these two points, and $I(\mu) = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot |x_0 - y|^\lambda = n/(n+1)$.

Suppose finally that y carries mass but x_0 does not. Then μ is not maximal. In fact, the distance from a vertex of a regular n-simplex with edges of unit length to the opposite $(n-1)$-dimensional face is $= \left(\dfrac{1+n}{2n}\right)^{\frac{1}{2}} > \dfrac{1}{2} \cdot 2^{\frac{1}{\lambda}}$, and $|x_0 - y|$

$= \left(\dfrac{2n}{n+1}\right)^{\frac{1}{\lambda}} < \left(\dfrac{2n}{n+1}\right)^{\frac{1}{2}} < 2^{\frac{1}{\lambda}}$. Hence y is nearer to the face opposite to x_0 than x_0 is, which means that $|y - x_i| < |x_0 - x_i|$ $(i = 1, 2, \ldots, n)$. Then $I(\mu)$ will increase if the mass of y is moved to x_0.

To sum up, we know that a maximal distribution does exist, and we know that the only two distributions which may be maximal, have the same I-value. Hence they are both maximal.

Theorem 11. *For $\lambda > 2$, let μ be a maximal distribution and K a compact subset of F. Then there exists a maximal distribution μ^* which in K has only point-masses and which outside of K is equal to μ.*

Proof: Since K is compact, we can cover it with a finite set of solid spheres $C_1, \ldots, C_p$ with radius ε and with their centres belonging to K. Then we construct p pair-wise disjoint Borel sets $B_1, \ldots, B_p$ as follows:

$$B_1 = C_1 \cap K$$

$$B_i = C_i \cap K - C_i \cap (\bigcup_{j < i} B_j) \qquad (1 < i \leq p).$$

Denote by μ_0 the part of μ in $F - K$, by m_i the mass of μ in B_i, and by δ_i the distribution consisting of the unit mass in the centre of C_i. We then define a distribution $\mu_\varepsilon \in \mathfrak{M}$ as follows:

$$\mu_\varepsilon = \mu_0 + \sum_{i=1}^{p} m_i \, \delta_i,$$

that is, we concentrate into p points of K the mass of μ in K. Since no mass is moved a longer distance than ε, $|x - y|^\lambda$ does not change more than by the amount $k \cdot \varepsilon$, where k is a constant, depending only on F and λ, for since t^λ is an increasing convex function for $t > 0$, the change of $|x - y|^\lambda$ is less than $(d + \varepsilon)^\lambda - (d - \varepsilon)^\lambda < \lambda (2d)^{\lambda-1} \cdot 2\varepsilon$, where d is the diameter of F. Hence the value of I does not change more than by the amount $k \cdot \varepsilon$, that is,

$$I(\mu_\varepsilon) \geq M - k \cdot \varepsilon. \tag{5}$$

We now take as the set S and the distribution μ of theorem 10, the p centres of the solid spheres C_i and the distribution μ_ε, respectively. By theorem 10, there is then a distribution $\mu_\varepsilon' \in \mathfrak{M}$ of the form

$$\mu_\varepsilon' = \mu_0 + \sum_{i=1}^{p} m_i' \, \delta_i$$

with not more than $(n+1)$ of the non-negative numbers m_i' different from zero, and such that

$$I(\mu_\varepsilon') \geq I(\mu_\varepsilon).$$

By (5), this implies

$$M \geq I(\mu_\varepsilon') \geq M - k \cdot \varepsilon. \tag{6}$$

Those m_i' which are positive, together with a sufficient number of zeroes, are now renamed $m_0^{(\varepsilon)}, m_1^{(\varepsilon)}, \ldots, m_n^{(\varepsilon)}$. The corresponding points, and, in the case of the zeroes, some arbitrary points of K, are denoted by $x_0^{(\varepsilon)}, x_1^{(\varepsilon)}, \ldots, x_n^{(\varepsilon)}$.

We now choose a sequence of ε, tending towards zero. For each ε we construct the points $x_0^{(\varepsilon)}, x_1^{(\varepsilon)}, \ldots, x_n^{(\varepsilon)}$ of K and the non-negative numbers $m_0^{(\varepsilon)}, m_1^{(\varepsilon)}, \ldots, m_n^{(\varepsilon)}$ with sum $\mu(K)$. Since both K and the closed interval $[0, \mu(K)]$ are compact, it is possible to select a sub-sequence ε_ν such that the point sequences $\{x_i^{(\varepsilon_\nu)}\}_\nu$ and the number sequences $\{m_i^{(\varepsilon_\nu)}\}_\nu$ all converge. The limit elements are denoted by x_i^* and m_i^* and belong to K and $[0, \mu(K)]$, respectively. We also have $\sum_{i=0}^{n} m_i^* = \mu(K)$. Of course some of the m_i^* may be zero. We define the distribution μ^* as follows:

$$\mu^* = \mu_0 + \sum_{i=0}^{n} m_i^* \, \delta_{x_i^*}.$$

We have just proved that

$$\mu^* \in \mathfrak{M}.$$

But by choosing ν sufficiently large, we can make $|m_i^* - m_i^{(\varepsilon_\nu)}|$ and $|x_i^* - x_i^{(\varepsilon_\nu)}|$, $(i = 0, 1, \ldots, n)$, arbitrarily small. Hence, since $I(\mu_\varepsilon')$ is a continuous function of the variables $x_i^{(\varepsilon)}$ and $m_i^{(\varepsilon)}$, $|I(\mu^*) - I(\mu_{\varepsilon_\nu}')|$ can be made arbitrarily small. By (6), this implies

$$I(\mu^*) = M.$$

This completes the proof that μ^* satisfies the requirements of the theorem.

Theorem 12. *If $\lambda > 2$, every maximal distribution consists entirely of point-masses, not more than $(n+1)$ in number, and all situated in extreme points of the convex hull of F.*

Proof: The last part of the theorem is contained in theorem 9. All we have to prove is that the support of a maximal distribution cannot consist of more than $(n+1)$ points. Suppose that this is not true and that $x_1, x_2, \ldots, x_{n+2}$ are distinct points of the support of a maximal distribution μ. Let C_i $(i = 1, 2, \ldots, n+2)$

be solid spheres with centres x_i and so small that they are pair-wise disjoint. Define the compact sets K_i as

$$K_i = C_i \cap F.$$

Since x_i is in the support of μ, $\mu(K_i) \neq 0$.

We now apply theorem 11 with $K = K_1$. We obtain a maximal distribution μ_1^* which is equal to μ except in K_1 and which has at least one point-mass in K_1.

We then apply theorem 11 to μ_1^* instead of to μ and with $K = K_2$. We obtain a maximal distribution μ_2^*, which, except in K_2, is equal to μ_1^*, and which has at least one point-mass in K_1 and one in K_2.

We go on in this way and finally obtain a maximal distribution μ_{n+2}^*, which contains at least $(n+2)$ point-masses. This contradicts the corollary of theorem 10, which completes the proof.

With theorem 12 we conclude the treatment of our main subject. We now turn our interest to the relations between the maximal distributions and the distributions solving the two other extremum problems mentioned in the introduction, namely the distributions $\mu \in \mathfrak{M}$ for which

$$\max_{x \in F} p_\mu(x) = \min_{\mu \in \mathfrak{M}} \max_{x \in F} p_\mu(x) = M'$$

or

$$\min_{x \in F} p_\mu(x) = \max_{\mu \in \mathfrak{M}} \min_{x \in F} p_\mu(x) = M''.$$

Let us call such distributions *minmax* and *maxmin* distributions, respectively. There evidently always exists at least one maxmin distribution and one minmax distribution, and $M'' \leq M$.

From theorem 8 it follows that, for $\lambda > 1$, a maxmin distribution must have all its mass in the extreme points of the convex hull of F. Otherwise the potential at every point of R_n, and hence also its minimum value on F, could be increased by moving the masses.

We might thus suspect that every maximal distribution is also maxmin, or conversely. However, this is not true for any $\lambda > 0$, as is shown by the following theorems and examples, which also rule out the corresponding hypothesis for the minmax distributions.

Theorem 13. *When F is a sphere, the potential of any minmax or maxmin distribution is constant on F.*

Proof for the maxmin case:

Consider for every maxmin distribution μ the set F_μ of all points $x \in F$ such that $p_\mu(x) = M''$. These F_μ are closed. The intersection of any finite number of sets F_μ is non-empty. In fact, if there is no point where $p_{\mu_1}, \ldots, p_{\mu_n}$ are all equal to their minimum value M'', then the distribution $\dfrac{1}{n} \sum_{i=1}^{n} \mu_i$ has a minimum potential greater than M'', which is impossible. Hence, since F is compact, there is a point belonging to *every* F_μ. Then, by reasons of symmetry, every

point of F belongs to every F_μ, which completes the proof. Hence, when $\lambda > 2$ and F is a sphere, the maximal and the maxmin distributions are different.

To deal with the maxmin distributions for $0 < \lambda \leq 2$, we consider a set F consisting of a sphere S and a point y inside S, but not its centre. A maximal distribution μ must have its minimum potential on F at y (since p_μ is subharmonic), and this minimum is strictly less than the constant value of the potential on S. Hence by moving the masses on S in a direction away from y, we can increase the minimum potential on F, and so μ is not maxmin.

To deal with the minmax distributions for any $\lambda > 0$, we consider as F a solid sphere with centre c. A maximal distribution μ has all its mass on the boundary, and hence $p_\mu(c) = r^\lambda$. But this value is strictly less than the mean value of the potential on the boundary. Hence, $\max_{x \in F} p_\mu(x) > r^\lambda$. On the other hand, for the distribution δ_c we have $\max_{x \in F} p_{\delta_c}(x) = r^\lambda$. Hence the minmax and the maximal distributions are different.

Finally we give a certain property of minmax distributions and of a related class of distributions:

Theorem 14. *When* $\lambda > 1$ *and* F *is convex, any minmax distribution consists of one point-mass.* This follows from theorem 8. In fact, the potential in every point of R_n will strictly decrease, if the distribution is replaced by the unit mass in its centre of gravity.

Let us denote as "free minmax distribution" a distribution such that

$$\max_{x \in F} p_\mu(x) = \min_{\substack{\mu \geq 0 \\ \int_{R_n} \delta\mu = 1}} \max_{x \in F} p_\mu(x),$$

that is, a distribution which solves the minmax problem where μ is not restricted to lie on F. Such distributions are the "Endverteilungen für $R^{(\lambda)}$", considered by PÓLYA and SZEGÖ [3, p. 42]. There, the following result is stated in the special case when F is a sphere or a closed sphere.

Theorem 15. *If* $\lambda > 1$, *any free minmax distribution consists of one point-mass.*

Proof: By the preceding result, it is sufficient to prove that any free minmax distribution μ is a minmax distribution for the convex hull $H(F)$ of F. As remarked by PÓLYA and SZEGÖ [3, footnote 14], the support of μ is a subset of $H(F)$. All we have to prove then, is that there is a point $z \in F$ such that

$$\max_{y \in H(F)} p_\mu(y) = p_\mu(z). \tag{7}$$

But this is true for any distribution $\mu \geq 0$, when $\lambda > 1$. In fact, since $|x - y|^\lambda$ is a convex function of x (theorem 8), so is $p_\mu(x) = \int_{R_n} |x - y|^\lambda d\mu(y)$, and hence there is an *extreme* point z of $H(F)$, such that (7) is true. This completes the proof.

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